\begin{align*}
\langle x' | \hat{J} | x \rangle &= \sum_{m,m',r} (x')^*(J^z)_{m,m'} (x)^r \\
&\text{or in matrix notation} \\
&= x'^T \hat{J} (x) \quad x
\end{align*}

\begin{align*}
&= \begin{pmatrix} x' \star x' \star \\
&x'^T \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} x' \\
x'' \end{pmatrix} \quad \text{in the} \\
&\hat{J} = \frac{\sqrt{3}}{2} \quad \text{case.}
\end{align*}

5.3.6, Spin and Orbital Angular Momentum Revisited

Finally, suppose we recall again our physical discussion of angular momentum in the preceding Section. We can use the definition of the orbital angular momentum operator \( \hat{L} \in \mathbb{R} \times \mathbb{R} \) to define the spin operator \( \hat{S} \) by

\[ \hat{J} = \hat{L} + \hat{S} \]

Since \( \hat{R}, \hat{P} \) as well as \( \hat{J} \) are vector operators

\begin{align*}
[\hat{J}^i, \hat{J}^j] &= i \hbar \varepsilon^{ijk} \hat{J}^k \\
[\hat{J}^i, \hat{S}^j] &= i \hbar \varepsilon^{ijk} \hat{S}^k \\
[\hat{J}^i, \hat{P}^j] &= i \hbar \varepsilon^{ijk} \hat{P}^k
\end{align*}
we have that \( \hat{L} \) is a vector operator

\[
[\hat{L}^i, \hat{L}^j] = ihe_{ijh} \hat{L}^k.
\]

Hence

\[
[\hat{S}^i, \hat{S}^j] = ihe_{ijh} \hat{S}^k \quad \text{which follows from the} \quad [\hat{L}^i, \hat{L}^j] = ihe_{ijh} \hat{J}^k \quad \text{and} \quad [\hat{J}^i, \hat{J}^j] \text{commutators.}
\]

In addition we can calculate the commutator of \( \hat{L} \) with \( \hat{R}(\hat{P}) \) directly to find

\[
[\hat{L}^i, \hat{X}^j] = ihe_{ijh} \hat{X}^k,
\]

\[
[\hat{L}^i, \hat{P}^j] = ihe_{ijh} \hat{P}^k,
\]

hence

\[
[\hat{S}^i, \hat{X}^j] = 0 = [\hat{S}^i, \hat{P}^j] \quad \text{from the}
\]

\( \hat{J} - \hat{R}(\hat{P}) \) commutators. Then we have that

\[
[\hat{L}^i, \hat{S}^j] = 0. \quad \text{Further the} \quad \hat{J} - \hat{J}
\]

commutator then yields

\[
[\hat{S}^i, \hat{S}^j] = ihe_{ijh} \hat{S}^k,
\]

\[
[\hat{L}^i + \hat{S}^i, \hat{L}^j + \hat{S}^j] = [\hat{L}^i, \hat{L}^j] + [\hat{S}^i, \hat{L}^j] + [\hat{L}^i, \hat{S}^j] + [\hat{S}^i, \hat{S}^j]
\]

\[
= ihe_{ijh} (\hat{L}^k + \hat{S}^k)
\]

\[
\Rightarrow [\hat{S}^i, \hat{S}^j] = ihe_{ijh} \hat{S}^k.
\]
The spin angular momentum operator obeys the SU(2) algebra. Hence we can construct the eigenstates of the \( \hat{S}^2 \) and \( \hat{S}_z \) operators, \( \mid S, m_s \rangle \), with
\[
\hat{S}^2 \mid S, m_s \rangle = S(S+1) \hbar^2 \mid S, m_s \rangle
\]
\[
\hat{S}_z \mid S, m_s \rangle = m_s \hbar \mid S, m_s \rangle
\]
where \( S = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \) and \( m_s = -S, ..., +S \).

The spin raising, \( \hat{S}^+ \) and lowering, \( \hat{S}^- \) operators can be defined as in the total angular momentum base
\[
\hat{S}^\pm = \hat{S}_x \pm i \hat{S}_y
\]
and the orthonormal standard spin basis can be constructed using them. Hence we have
\[
\hat{S}^+ \mid S, m_s \rangle = \sqrt{S(S+1) - m_s(m_s+1)} \mid S, m_s+1 \rangle
\]
\[
\hat{S}^- \mid S, m_s \rangle = \sqrt{S(S+1) - m_s(m_s-1)} \mid S, m_s-1 \rangle
\]
\[
\hat{S}_z \mid S, m_s \rangle = m_s \hbar \mid S, m_s \rangle
\]
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Hence the spin operators in the standard spin basis are represented by spin matrices

\[ \hat{S} | s, m_s \rangle = \sum_{m_s' = -s}^{+s} (\hat{S}^{(s)})_{m_sm_{s'}} | s, m_s \rangle \]

i.e. \( \langle s, m_s | \hat{S} | s, m_s' \rangle = (\hat{S}^{(s)})_{m_s m_{s'}} \).

Clearly the spin basis matrix elements of the spin operator are the same as the standard basis matrix elements of the angular momentum operator:

\[ (\hat{S}^{(s)})_{m_s m_{s'}} = (\hat{J}^{(s)})_{m_s m_{s'}} \].

For example for \( s = \frac{1}{2} \), \( \hat{S}^{(s)} = \frac{\hbar}{2} \hat{J} \) the Pauli matrices. As well the spin basis can be represented by column vectors as discussed for the standard basis.

For \( s = \frac{1}{2} \) we have the spin-up, \( e^\uparrow = (1) \), and spin-down, \( e^\downarrow = (i) \), basis vectors.
As an illustration of the use of the spin basis, consider the interaction of a spin $\frac{1}{2}$ particle with an external magnetic field. Classically, a spinning charge has a magnetic moment $\vec{\mu}$ which interacts with the external magnetic field $\vec{B}$. The classical Hamiltonian describing the interaction is simply $H = -\vec{\mu} \cdot \vec{B}$.

If the spinning charge has angular momentum $\vec{S}$ due to its spin then the magnetic moment is proportional to it,

$$\vec{\mu} = g \gamma \vec{S} = \frac{e \hbar}{2mc} \vec{S}$$

$$\equiv g \left( \frac{e}{\hbar} \right) \mu_B \vec{S}$$

where $\mu_B = \frac{e \hbar}{2mc}$ is the Bohr magneton and $e$ is the magnitude of the electron charge, $g$ is the effective charge of the particle, $g \gamma$ depends on the particle being considered. (For neutral particles like neutron $\gamma = g \mu \frac{e \hbar}{2mc} = g \mu_B$. $g$ is called the Landé g-factor.
The interaction of the spin $\mathbf{s}$ with the external magnetic field $\mathbf{B}$ is again described by the Hamiltonian

$$H = -\mu_B \cdot \mathbf{B} = -\gamma \mathbf{s} \cdot \mathbf{B}$$

$$= -g \frac{\hbar}{2} \frac{\mu_B}{\hbar} \mathbf{s} \cdot \mathbf{B}.$$ 

For $\mathbf{B} = B \mathbf{\hat{z}}$ this yields

$$H = -g \frac{\hbar}{2} \frac{\mu_B}{\hbar} S_z$$

$$\equiv \omega_B S_z$$

with the Bohr frequency $\omega_B = -g \frac{\hbar}{2} \frac{\mu_B}{\hbar}$. The spin basis vectors are the eigenstates of $H$

$$H |s = \frac{1}{2}, m_s \rangle = m_s \hbar \omega_B |s = \frac{1}{2}, m_s \rangle$$

with $m_s = \pm \frac{1}{2}$. Hence the energy eigenvalues are

$$E_{m_s} = m_s \hbar \omega_B = m_s (-g \frac{\hbar}{2} \frac{\mu_B}{\hbar})$$

For example, for an electron $\frac{g}{2} = -1$ and $\hbar \omega_B = g \mu_B B$. Experimentally, we find

$$g - 2 = 0.002319312 \ (\text{i.e., } g = 2.002319312)$$

which is known as the anomalous magnetic
moment of the electron.

Hence

\[
\begin{align*}
\text{E} & \arrow{\downarrow} \\
-716 & \quad m_s = +\frac{1}{2} \\
\leftarrow E_1 = \frac{g}{2} \mu_B B & \approx \mu_B B \\
\rightarrow E_2 = -\frac{1}{2} \mu_B B & \approx -\mu_B B \\
\text{m}_s = -\frac{1}{2}
\end{align*}
\]

Suppose at time \( t = 0 \), the spin is in the state

\[ |\psi(0)\rangle = \cos \frac{\theta}{2} e^{-i\frac{\theta}{2}} \left| \frac{1}{2}, \frac{1}{2} \rightangle + \sin \frac{\theta}{2} e^{i\frac{\theta}{2}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \]

The state at time \( t \) is simply

\[ |\psi(t)\rangle = \cos \frac{\theta}{2} e^{-i\frac{\theta}{2}} e^{-\frac{i}{\hbar} E_1 t} \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sin \frac{\theta}{2} e^{i\frac{\theta}{2}} e^{-\frac{i}{\hbar} E_2 t} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \]

Since \( \left| \frac{1}{2}, m_s \right\rangle \) are energy eigenstates and

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \]
So

\[ |\psi(t)\rangle = \cos \frac{\theta}{2} e^{-i(\frac{\theta}{2} + \omega Bt)} |\frac{1}{2}, \frac{1}{2}\rangle + \sin \frac{\theta}{2} e^{-i(\frac{\theta}{2})} |\frac{1}{2}, -\frac{1}{2}\rangle. \]

The magnetic field \( B \) produces a phase shift \( \theta \) between the coefficients of the spin eigenstates that depend on time.

Since \[ [H, S_z] = 0 \]
the observable \( S_z \) is a constant of the motion. The probability to measure spin \( \pm \frac{1}{2} \) at \( t \) is

\[ P_{\pm \frac{1}{2}} = |\langle \frac{1}{2}, \pm \frac{1}{2} | \psi(t) \rangle|^2 = \begin{cases} \cos^2 \frac{\theta}{2} & \text{for } +\frac{1}{2} \\ \sin^2 \frac{\theta}{2} & \text{for } -\frac{1}{2} \end{cases} \]

independent of \( t \). However \( S_x \) and \( S_y \) do not commute with \( H \); \([H, S_x, y] \neq 0\) and hence are not constants of the motion. Their expectation values are

\[ \langle \hat{S}_x | \hat{\psi}(t) \rangle = \langle \hat{S}_y | \hat{\psi}(t) \rangle = 0 \]

and

\[ \langle \hat{S}_z | \hat{\psi}(t) \rangle = -\frac{1}{2}. \]
\[ -718 - \]

\[ = \left[ \cos \frac{\theta}{2} e^{\frac{-i (\omega + \omega_B t)}{2}} - i \frac{\theta}{2} \right] x \]

\[ \times \left( \begin{array}{c} \cos \frac{\theta}{2} e^{\frac{i (\omega + \omega_B t)}{2}} \\ \sin \frac{\theta}{2} e^{\frac{i (\omega + \omega_B t)}{2}} \end{array} \right) \]

\[ \Rightarrow \]

\[ \langle \hat{S}_x | 2(t) \rangle = \frac{i}{2} \sin \theta \cos (\omega + \omega_B t) \]

\[ \langle \hat{S}_y | 2(t) \rangle = \frac{i}{2} \sin \theta \sin (\omega + \omega_B t) \]

\[ \langle \hat{S}_z | 2(t) \rangle = \frac{1}{2} \cos \theta \]

The expectation values of \( \hat{S} \) behave like classical angular momentum of magnitude \( \frac{1}{2} \) undergoing Larmor precession with angular velocity \( \omega_B \).

\[ \frac{1}{2} \cos \theta \]

\[ \frac{1}{2} \sin \theta \]
The standard basis consisted of the simultaneous eigenvectors of the CSCO \( \mathcal{E} A, J^2, \mathcal{S}^2 \) and \( \mathcal{E} H, J^m, \mathcal{S}^m \). Since \( \mathcal{R} \) and \( \mathcal{P} \) commute with \( \mathcal{S} \), we may also use the sets \( \mathcal{R}, \mathcal{S}^2, \mathcal{S}^m \) or \( \mathcal{P}, \mathcal{S}^2, \mathcal{S}^m \) as CSCO. Since the \( [\mathcal{S}^2, \mathcal{S}^m] = 0 \), the eigenstates \( | F, s, m_s \rangle \) can be written as

\[
| F, s, m_s \rangle = | F \rangle \otimes | s, m_s \rangle.
\]

Since these are complete, any state \( | 12 \rangle \) may be expanded in terms of them

\[
| 12 \rangle = \int d^3r \sum_{m_s = -s}^{+s} 2^{l(s)} (s^l_{m_s}) | F \rangle | F, s, m_s \rangle
\]

for a particular subspace \( H(s) \) with \( s \) fixed and we have the multi-component wavefunction

\[
2^{l(s)} (s^l_{m_s}) = \langle F, s, m_s | 12 \rangle.
\]

Since \( \mathcal{J} = \mathcal{L} + \mathcal{S} \), we have

\[
\langle F, s, m_s | \mathcal{J} | 12 \rangle = \langle F, s, m_s | \mathcal{L} | 12 \rangle + \langle F, s, m_s | \mathcal{S} | 12 \rangle.
\]
Since \( L = \mathbf{R} \times \mathbf{P} \) we have

\[
\langle \mathbf{P}, s, m_s | \mathbf{F} | \mathbf{F} \rangle = (\mathbf{P} \times \frac{\mathbf{F}}{c}) \langle \mathbf{P}, s, m_s | \mathbf{F} \rangle + \frac{s}{2} \left( \mathbf{S}(s) \right)_{m_s m_s} \langle \mathbf{F}, s, m_s | \mathbf{F} \rangle
\]

\[
= \left( \mathbf{P} \times \frac{\mathbf{F}}{c} \right) \delta_{m_s m_s} + \left( \mathbf{S}(s) \right)_{m_s m_s} \mathbf{J} \mathbf{F}(s)_{m_s}
\]

as we found in our physical discussion of angular momentum. Since

\[
| \mathbf{F} \rangle = U(R(s)) | \mathbf{F} \rangle
\]

\[
= e^{-\frac{i}{\hbar} \mathbf{F} \cdot \mathbf{J}} | \mathbf{F} \rangle
\]

we have 

\[
21(s)(\mathbf{F}) = \langle \mathbf{P}, s, m_s | 21 \rangle
\]

\[
= \langle \mathbf{P}, s, m_s | 1 - \frac{i}{\hbar} \mathbf{F} \cdot \mathbf{J} | \mathbf{F} \rangle
\]

\[
= 21(s)(\mathbf{F}) - \frac{i}{\hbar} \mathbf{F} \cdot \mathbf{J} \langle \mathbf{P}, s, m_s | \mathbf{F} \rangle
\]

\[
211(s)(\mathbf{F}) = 21(s)(\mathbf{F}) - \frac{i}{\hbar} \mathbf{F} \cdot \left( \mathbf{P} \times \frac{\mathbf{F}}{c} \right) \delta_{m_s m_s}
\]

\[
+ \left( \mathbf{S}(s) \right)_{m_s m_s} \mathbf{J} \mathbf{F}(s)_{m_s}
\].
and for finite rotations

\[ Q^{(s)}_{\text{ms}} (\hat{R}) = D^{(s)} (R(\theta)) \bar{Q}^{(s)}_{\text{ms}} (R(\theta) \hat{R}) \].

Since any basis is as good as any other, we can imagine expanding the \( |F, J, m_J \rangle \) states in terms of the \( |F, S, m_S \rangle \) states. Since \( J = L + S \), the relationship between the eigenstates of \( \vec{J}^2 \), \( J_z \) and the orbital angular momentum and spin angular momentum that add up to be the total angular momentum we desire is complicated. We must first determine how to add angular momentum.

Before doing this, however, we can consider the case of spin-zero particles. In this situation, the total angular momentum is just the orbital angular momentum; \( \vec{J} = \vec{L} \) (i.e., \( J = L + 0 J \)). So in the \( 3J \times 3J \) basis

\[ \langle \tilde{F} | \tilde{J} | 12 \rangle = \langle \tilde{F} | \tilde{L} | 12 \rangle = \langle \tilde{F} | \tilde{R} x \tilde{P} | 12 \rangle = (\tilde{R} \times \frac{\tilde{P}}{c}) \tilde{2}(\tilde{F}) \]
When the particle has spin 0, the total 4-momentum is just the orbital 4-momentum.

Now for orbital angular momentum

\[ \mathbf{J} = \mathbf{L} = \mathbf{R} \times \mathbf{P} \]

we found in wave mechanics that \( L_x, L_y, \) and \( L_z \) were integer valued only. This follows from the general properties of \( \mathbf{L} \) that is unlike spin \( \mathbf{S} \).

Aside: In particular \( L_z = X P_y - Y P_x \)

and \( [X_i, P_j] = i \hbar \delta_{ij} \).

Now define 4 operators that are hermitian:

\[
\begin{align*}
P_1 &= \frac{X + P_y \frac{a^2}{\hbar^2}}{\sqrt{2}} \frac{a^2}{\hbar^2}, \\
P_2 &= \frac{X - P_y \frac{a^2}{\hbar^2}}{\sqrt{2}}, \\
\mathcal{P}_1 &= \left[ \frac{\frac{a^2}{\hbar^2} P_y - X}{\sqrt{2}} \right] \hbar^{-1/2}, \\
\mathcal{P}_2 &= \left[ \frac{\frac{a^2}{\hbar^2} P_y + X}{\sqrt{2}} \right] \hbar^{-1/2}.
\end{align*}
\]

\( a = \text{const. with} \, \hbar \) 

\( \text{dim of} \, \mathcal{P} \rightarrow \text{length} \)

\( \text{d} = \frac{4\pi \hbar^2}{me^2} \)

Bohr radius
\[ [g_i, p_j] = \frac{\hbar}{2m} \{ x_j p_i - x_i p_j \} = i \hbar \delta_{ij} \]

Further \[ [q_i, q_j] = 0 = [p_i, p_j] \]

\[ [q_i, p_j] = \hbar \delta_{ij} \]

CCR.

Further,

\[ L_z = \frac{1}{2} (q_1 + q_2) \]

\[ Y = \frac{1}{2} \hbar (p_1 + p_2) \]

\[ P_x = \frac{1}{2} (p_1 - p_2) \]

\[ a^2 \gamma_j = \frac{\hbar}{2} (q_1 - q_2) \]

Set \( \alpha = 1 \)

\[ L_z = \frac{1}{2} \hbar (p_1^2 + \frac{\hbar}{2} x_j q_j^2) - \frac{\hbar}{2} (p_1^2 + \frac{\hbar}{2} x_j q_j^2) \]

This is just Hamiltonian for 2 independent 1-d OSHO with \( M = \frac{\hbar}{\omega}, \quad m = 1 \)

\[ \alpha^2 = \frac{1}{m} \]
Hence their individual ev. are

\[ L_2 = h_1 \cdot h_2 \quad ; \quad h_1 \rightarrow E_1 = \hbar (n_1 + \frac{1}{2}) \]
\[ h_2 \rightarrow E_2 = \hbar (n_2 + \frac{1}{2}) \]
\[ n_{1,2} = 0, 1, 2, \ldots \]

\[ \Rightarrow \quad L_2 \rightarrow m = E_1 - E_2 \]
\[ = \hbar (n_1 + \frac{1}{2}) - \hbar (n_2 + \frac{1}{2}) \]
\[ = \hbar (n_1 - n_2) \]

\[ \Rightarrow \quad m = n_1 - n_2 = 0, \pm 1, \pm 2, \ldots \Rightarrow \text{integer} \]
\[ \text{(not \( \frac{1}{2} \) integer!)} \]

Indeed this is what we found explicitly in the \( \Xi(\mathbf{r}) \) basis

\[ \langle \Psi | \mathbf{L} | \Delta \rangle = \langle \Psi | \mathbf{L} | \Delta \rangle = \langle \Psi | \mathbf{R} \times \overleftrightarrow{\mathbf{L}} | \Delta \rangle \]
\[ = (\mathbf{r} \times \frac{\hbar}{i} \overleftrightarrow{\mathbf{R}}) \Delta (\mathbf{r}) \]
That is \( \hat{L} = \hat{L} \mathcal{L} = (\hat{r} \times \frac{\hat{\theta}}{r} \frac{\hat{\phi}}{r}) \mathcal{L} \) for the spin 0 subspace.

In spherical polar coordinates we have as found in wave mechanics

\[
\begin{align*}
\hat{L}_x &= \frac{\hbar}{i} \left[ -\sin \theta \frac{\partial}{\partial \phi} - \cot \theta \cos \phi \frac{\partial}{\partial \theta} \right] \mathcal{L} \mathbf{r}, \theta, \phi \\
\hat{L}_y &= \frac{\hbar}{i} \left[ \cos \theta \frac{\partial}{\partial \phi} - \cot \theta \sin \phi \frac{\partial}{\partial \theta} \right] \mathcal{L} \mathbf{r}, \theta, \phi \\
\hat{L}_z &= \frac{\hbar}{i} \frac{\partial}{\partial \theta} \mathcal{L} \mathbf{r}, \theta, \phi
\end{align*}
\]

Thus,

\[
\hat{L}_\pm = \hbar e^{\pm i \phi} \left[ \pm \frac{\partial}{\partial \phi} + i \cot \theta \frac{\partial}{\partial \theta} \right] \mathcal{L} \mathbf{r}, \theta, \phi
\]

and

\[
\hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{\partial^2}{\partial \phi^2} \right] \mathcal{L} \mathbf{r}, \theta, \phi
\]

Since these operators are independent of the radius \( r \), they act only on the spherical angle variables. We can introduce a spherical angle basis \( | \theta, \phi \rangle \). The action of \( \hat{L} \) in this basis is given above. \( | \theta, \phi \rangle \) is not complete, we need also the radial dependence of the wavefunction, but since it is not relevant for the following discussion we suppress it, as the \( \mathcal{L} \) in the \( | \theta, \phi, j, m \rangle \) basis.
Now since \( J_z = l \) we know that the

eigenstates \( |l, m\rangle \) of \( L^2 \) and \( L_z \) are
given by

\[
L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle \\
L_z |l, m\rangle = m\hbar |l, m\rangle
\]

with \( l = 0, 1, 2, \ldots \) and \( m = -l, -l+1, \ldots, l-1, l \).

Hence the \( |l, m\rangle \) wavefunctions obey

the angular differential equation

\[
\langle \theta, \phi | L^2 |l, m\rangle = l(l+1)\hbar^2 \langle \theta, \phi | l, m\rangle \\
= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d}{d\theta} \right) + \frac{1}{\sin^2 \theta} \frac{d^2}{d\phi^2} \right] \langle \theta, \phi | l, m\rangle
\]

and

\[
\langle \theta, \phi | L_z |l, m\rangle = m\hbar \langle \theta, \phi | l, m\rangle \\
= \frac{\hbar}{i} \frac{d}{d\phi} \langle \theta, \phi | l, m\rangle .
\]

The solutions to these differential
equations are the spherical harmonics.
\[ \langle \theta, \phi | l, m \rangle = Y^m_l (\theta, \phi) \]

with \( l = 0, 1, 2, \ldots \) and \( m = -l, -l+1, \ldots, l-1, l \).

Since \( \langle l, m | l, m \rangle = 1 \) we have

\[
\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \left| Y^m_l (\theta, \phi) \right|^2 = 1.
\]

Hence we have transformed from one basis \( |l, m \rangle \) to another \( \langle \theta, \phi \rangle \) by means of the spherical harmonics

\[ Y^m_l (\theta, \phi) = \langle \theta, \phi | l, m \rangle \]

Finally, any wavefunction for a spin 0 particle has the expansion in terms of the standard basis wavefunctions

\[ 2 \Lambda_{k,l,m} (\vec{r}) = \langle \vec{r} | k, l, m \rangle \]

\[ = R_{k,l,m} (\vec{r}) Y^m_l (\theta, \phi) \]

Since \( \vec{r}^2 \) and \( \vec{L}^2 \) only act on the angular variables, \( R_{k,l,m} (\vec{r}) \) acts as a constant factor.
Using $L^+$ we find $R_{k_1,\ell_1, m_1}(r) = R_{k_1, \ell_1, m_1}(r)$ and hence $R_{k_2, \ell_2, m_2}(r) = R_{k_2, \ell_2}(r)$ is independent of $m$. So

$$2 R_{k_2, \ell_2, m_2}(r) = R_{k_2, \ell_2}(r) Y_{\ell_2}^m(\Theta, \Phi)$$

form a complete set of spin 0 wave functions which are orthonormal (by convention).

$$\int d^3 r \, 2 R_{k_2, \ell_2, m_2}(r) \overline{R_{k, \ell, m}}(r)$$

$$= \int_0^\infty dr \, r^2 R^*_k(r) R_{k_2, \ell_2}(r) \times$$

$$\times \int d\Omega \, Y_{\ell_2}^m(\Theta, \Phi) Y_{\ell_2}^m(\Theta, \Phi)$$

$$= \langle k_2, \ell_2, m_2 | k, \ell, m \rangle = \delta_{k_2 k} \delta_{\ell_2 \ell} \delta_{m_2 m}$$

Thus $R_{k_2, \ell_2}$ must be normalized to

$$\int d^3 r \, r^2 R^*_k(r) R_{k_2, \ell_2}(r) = \delta_{k_2 k}.$$
Hence any spin 0 wavefunction has the expansion

\[ \psi(\vec{r}) = \sum_{k, l, m} C_{klm} \langle \psi | k, l, m \rangle \langle k, l, m | \psi \rangle \]

\[ = \sum_{k, l, m} C_{klm} \langle \psi | k, l, m \rangle \]

\[ = \sum_{k, l, m} C_{klm} \psi_{klm}(\vec{r}) \]

\[ = \sum_{k, l, m} C_{klm} \psi_{klm}(\vec{r}) \chi_{klm}(\Theta, \Phi) \]

In particular, we can apply this to the case of the free particle with 0 spin. The Hamiltonian is

\[ H = \frac{1}{2m} \vec{\mathbf{p}}^2 \]

On the one hand, we can choose eigenstates of \( \mathbf{H} \) and \( \mathbf{P} \) as a basis

\[ \psi_{klm} \rightarrow \psi_{klm} \]

\[ \psi_{klm} \rightarrow \phi_{klm} \]

and in the coordinate representation...
This becomes
\[ \langle \hat{F} | \hat{F} | \psi \rangle = \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \mathbf{r} \cdot \partial \mathbf{r}} \right) \langle \hat{F} | \psi \rangle \]
\[ = \langle \hat{F} | \psi \rangle \]
\[ \Rightarrow \langle \hat{F} | \psi \rangle = c e^{\frac{\hat{F}^2}{2m^2}} \]
and \[ \langle \hat{F} | H | \psi \rangle = -\frac{\hbar^2}{2m} \nabla^2 \langle \hat{F} | \psi \rangle = E \langle \hat{F} | \psi \rangle \]
\[ = \frac{\hbar^2}{2m} \langle \hat{F} | \psi \rangle \]
\[ \Rightarrow E \langle \phi | \psi \rangle = \frac{\hbar^2}{2m} \]
Choosing the normalization factor \( c = 1 \)
we have
\[ \langle \psi | \hat{F} | \psi \rangle = \int d^3r \ c \ e^{\frac{\hat{F}^2}{2m^2}} \]
\[ = (2\pi \hbar)^3 \delta^3(\mathbf{\hat{r}} - \mathbf{\hat{p}}) \]
Continuum orthogonality. Completeness is then given by,
\[ 1 = \int \frac{d^3p}{(2\pi \hbar)^3} \langle \hat{F} | \psi \rangle \langle \hat{F} | \psi \rangle \]
\[ \Rightarrow \]
\[ \langle \hat{F} | \hat{F} \rangle = \int \frac{d^3p}{(2\pi \hbar)^3} \langle \hat{F} | \hat{F} | \psi \rangle \langle \hat{F} | \psi \rangle \]
\[ = \int \frac{d^3p}{(2\pi \hbar)^3} e^{\frac{\hat{F}^2}{2m^2}} \]
\[ = c^3(\mathbf{\hat{F}} - \mathbf{\hat{p}}) \]
As just discussed, we have that $\mathcal{F} = \mathcal{L}$ and 

$$[\mathcal{L}, \phi^2] = 0 \Rightarrow [\mathcal{L}, \mathcal{H}] = 0$$

so we can equally well consider $[\mathcal{L}, \phi^2 \mathcal{L}]$ as our CSCO and expand our states in terms of their mutual eigenfunctions.

$$\mathcal{H} |E, l, m\rangle = E |E, l, m\rangle = \frac{\phi^2}{2m} |E, l, m\rangle$$

$$\mathcal{L}^2 |E, l, m\rangle = (l(l+1)) |E, l, m\rangle$$

$$\mathcal{L}_z |E, l, m\rangle = m |E, l, m\rangle$$

where $E \geq 0$, $l = 0, 1, 2, \ldots$, $m = -l, -l+1, \ldots, +l$.

Then

$$|\Psi\rangle = \int \frac{dE}{\sqrt{2E}} \sum_l \sum_m \langle E, l, m | \Psi \rangle |E, l, m\rangle$$

Now if $|\Psi\rangle$ is also an eigenstate of $\mathcal{H}$ with eigenvalue $E$ we have

$$|\Psi_E\rangle = \sum_l \sum_m \langle E, l, m | \Psi \rangle |E, l, m\rangle$$

In particular

$$|\Phi\rangle = \sum_l \sum_m \langle E(\Psi), l, m | \Phi \rangle |E(\Psi), l, m\rangle$$
and

\[ \left\langle \Psi_{\ell} \right| \phi \right\rangle = e^{\frac{i}{\hbar} \mathcal{P} \cdot \mathbf{r}} \]

\[ \begin{align*}
= & \frac{\alpha}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\langle k, \ell, m \left| \phi \right> \right\rangle \mathcal{A}_{k, \ell, m}(r) \mathcal{R}_{\ell l}(r) Y_{\ell m}(\Theta, \Phi) \\
= & \frac{\alpha}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \text{Chern \text{ R}_{\ell l}(r) Y_{\ell m}(\Theta, \Phi)}
\end{align*} \]

where we have written

\[ E_{\ell m} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k^2}{2m} \quad ; \quad k \geq 0. \]

Let's find the free particle energy eigenstates in the angular momentum basis

\[ \mathcal{A}_{k, \ell, m}(r) \] with completeness

\[ \int_{0}^{\infty} \frac{dE}{(2\pi \hbar)^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\langle E, k, m \left| E', k', m' \right\rangle = 1 \]

and orthonormality

\[ \int d^3r \, \mathcal{A}_{k, \ell, m}(r) \mathcal{A}^{*}_{k', \ell', m'}(r) = \delta(k-k') \delta(\ell - \ell') \delta(m - m') \]
\[ \delta(E-E') = \delta \left( \frac{\hbar^2}{2m} (k^2 - k'^2) \right) \\
= \delta \left( \frac{\hbar^2}{2m} (k + k')(k - k') \right) \text{ with } k, k' \geq 0 \]

So

\[ \frac{1}{\frac{\hbar^2 k}{m}} \delta(k-k') \]

\[ \frac{1}{\left( \frac{2E\hbar^2}{m} \right)^{\frac{1}{2}}} \delta(k-k') ; \text{ i.e. } \text{d}E = \left( \frac{2E\hbar^2}{m} \right)^{\frac{1}{2}} \text{ d}k \]

So completeness becomes

\[ \int_0^\infty \text{d}k \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |E(k), l, m \rangle \langle E(k), l, m| = \mathbf{1} \]

\[ \equiv |k, l, m \rangle \langle k, l, m| \]

Now in spherical polar coordinates then we have that

\[ H |k, l, m \rangle = \frac{\hbar^2 k^2}{2m} |k, l, m \rangle \]

\[ \Rightarrow -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 h_{klm}(r^2)) - \frac{1}{r^4} \frac{\partial^2}{\partial \theta^2} h_{klm}(r^2) \right] \]

\[ = \frac{\hbar^2 k^2}{2m} h_{klm}(r^2) \]
\[
\frac{1}{r} \frac{d^2}{dr^2} \left( r R_{ke}(r) \right) - \frac{\ell(\ell+1)}{r^2} R_{ke}(r) = -\hbar^2 R_{ke}(r).
\]

This is just Bessel's equation \( \Rightarrow \)
\[
R_{ke}(r) = N_{ke} J_\ell(\hbar r)
\]

where \( J_\ell(\kappa r) \) are the spherical Bessel functions. The normalization factor \( N_{ke} \) is determined by
\
\int_0^{\infty} 4\pi r^2 J^2_\ell(\hbar r) \, dr = \frac{\pi}{\hbar^2} \delta(\kappa - \kappa')
\
\Rightarrow

\[
N_{ke} = \frac{2\pi}{\hbar^2} \delta(\kappa - \kappa')
\]

\[
\Rightarrow \quad |N_{ke}|^2 = \frac{2\hbar^2}{\pi} \quad \text{independent of } \ell.
\]

Choosing \( N_{ke} \) as real and positive \( \Rightarrow \)
\[ N_{k\ell m} = \sqrt{\frac{2}{\pi^3}} \ell \bar{k}, \quad \text{and} \quad R_{k\ell m}(r) = \sqrt{\frac{2}{\pi^3}} k^\ell \bar{k} J_{\ell}(kr) \]

Thus, the complete set of free particle wavefunctions that are \(K\), \(L^2\), \(L^2\) eigenfunctions are

\[ 2_{k\ell m}(\vec{r}) = \sqrt{\frac{2}{\pi^3}} k^\ell \bar{k} J_{\ell}(kr) Y^m_l(\theta, \phi) \]

\[ = \langle \vec{r} | k, \ell, m \rangle. \]

Thus we have 2-sets of basis vectors. The plane wave eigenstates of \(E\), \(\vec{p}\)

\[ H |\vec{p}\rangle = \frac{\hbar^2 k^2}{2m} |\vec{p}\rangle = \frac{\hbar^2}{2m} |\vec{p}\rangle \]

\[ \vec{p} |\vec{p}\rangle = \vec{p} |\vec{p}\rangle \quad \text{and} \quad k^2 \geq 0 \quad \vec{p} \in \mathbb{R}^3 \]

with wavefunctions

\[ 2_{\vec{p}}(\vec{r}) = \langle \vec{r} | |\vec{p}\rangle = e^{i\vec{p} \cdot \vec{r}} \]

On the other hand, the eigenstates of \(E\), \(\vec{p}\)
ave \(1k, \ell, m\)
\[ \Psi_{klm}(\vec{r}) = \langle \vec{r} | k, l, m \rangle = \sqrt{\frac{2}{\pi}} k j_l(kr) Y_l^m(\theta, \phi). \]

Either set is a basis, so we can expand one in terms of the other:

\[ |\tilde{\phi}\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle k, l, m | \tilde{\phi} \rangle | k, l, m \rangle \]

where \( \vec{p}^2 = -\hbar^2 \nabla^2 \) and so

\[ \langle \vec{q} | \tilde{\phi} \rangle = e^{i \frac{\vec{q} \cdot \vec{p} \cdot \vec{r}}{\hbar}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle k, l, m | \tilde{\phi} \rangle 2 \Psi_{klm}(\vec{r}) \]
So \( e^{\frac{i}{\hbar} \hat{P} \tau} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{l,m \nu} \sqrt{\frac{2l+1}{4\pi}} Y_l^m(\theta, \phi) \)

and we are left to determine the coefficients \( C_{l,m} \). The result is given by

\[
e^{\frac{i}{\hbar} \hat{P} \tau} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l Y_l^m \left( \Theta_p, \Phi_p \right) J_l \left( \kappa r \theta \right) Y_l^m \left( \Theta, \Phi \right)
\]

where \((\Theta_p, \Phi_p)\) are the spherical angular coordinates of \( \hat{P} \). We can obtain this general result by rotating the \( \hat{P} \)-vector so that it lies along the \( z \)-axis. In this new frame, only \( m=0 \) states contribute since \( \left< F | \hat{P} \right> = e^{\frac{i}{\hbar} \tau} \). Then, when \( \kappa = \frac{\hbar}{2} \) we have

\[
e^{\frac{i}{\hbar} \tau} = \sum_{l=0}^{\infty} i^l \sqrt{\frac{2l+1}{4\pi}} J_l \left( \frac{\hbar}{2} \right) Y_l^0 \left( \Theta, \Phi \right)
\]

This can be rotated back to the original frame using the addition formula for spherical harmonics. In turn, this
Consider the proof of the expansion

\[ e^{ikz} = e^{ikr \cos \theta} \]

\[ = \sum_{l=0}^{\infty} \frac{2^l 2^l \Gamma(l+1) j_l(kr) P_l(\cos \theta)}{l!} \]

In general we have

\[ e^{ikr \cos \theta} = \sum_{l=0}^{\infty} C_{lm=0} \sqrt{\frac{2l^2}{\pi}} j_l(kr) Y_l^m(\theta, \phi) \]

\[ = \sum_{l=0}^{\infty} C_{lm=0} \sqrt{\frac{2l^2}{\pi}} j_l(kr) Y_l^0(\theta, \phi) \]

Now recall the recursion relation

\[ Y_{l+1}^m(\theta, \phi) = \sqrt{\frac{(l+1)!}{(2l+1)!}} (\frac{L}{\hbar})^{l-m} Y_l^m(\theta, \phi) \]

where

\[ L_+ = L_x + i L_y \]

\[ = \hbar e^{\pm i \phi} \left( \pm \frac{\partial}{\partial \theta} + i \left( \pm \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right) \]

\[ \Rightarrow Y_{l+1}^m(\theta, \phi) = \sqrt{\frac{1}{(2l+1)!}} \left( \frac{L}{\hbar} \right)^l Y_l^m(\theta, \phi) \]
Since $y^o$ are orthonormal we have

\[
Ce^{-jklr} \sqrt{\frac{2k^2}{\pi}} = \int d\Omega \ y^o_\ell^*(\theta, \phi) e^{-ikr \cos \theta}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int d\Omega \left( \left( \frac{L-1}{\ell} \right) y_\ell^*(\theta, \phi) \right) e^{-ikr \cos \theta}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int d\Omega \ y_\ell^*(\theta, \phi) \left( \frac{L+1}{\ell} \right) e^{-ikr \cos \theta}
\]

by definition of adjoint operator and $L^+ = -L = L^*$. Now

\[
\left( \frac{L+1}{\ell} \right) e^{-ikr \cos \theta} = e^{jlp} \left( \frac{1}{\sin \theta} \frac{d}{d\cos \theta} \right) e^{-ikr \cos \theta}
\]

\[
= -i \ell e^{jlp} \sin \theta \, l! \, k^l \, l! \, e^{-ikr \cos \theta}
\]

But

\[
y_\ell^*(\theta, \phi) = \frac{-i \ell e^{jlp}}{2^\ell \ell! \sqrt{\frac{2\ell+1}{4\pi}}} e^{jlp} (\sin \theta)^l
\]

So

\[
\left( \frac{L+1}{\ell} \right) e^{-ikr \cos \theta} = 2^\ell \sqrt{\frac{4\pi}{(2\ell+1)}} \ y_\ell^*(\theta, \phi) \, l! \, k^l \, l! \, e^{-ikr \cos \theta}
\]
and so

\[
C_l e^{jkr} \sqrt{\frac{2k^2}{\pi}} = \frac{2^{l+1}}{\sqrt{(2l+1)!}} \sqrt{\frac{4\pi}{(2l+1)!}} (ihr)^l \times
\]

\[
\int d\Omega \ y_l^* (\theta, \phi) y_l (\theta, \phi) e^{i hr \cos \Theta}
\]

Now for \( kr \to 0 \) \( j \to (kr)^l \frac{(kr)^l}{(2l+1)!} \)

So with \( e^{i hr \cos \Theta} \to 1 \) on the RHS due to the \((kr)^l \) factor already and the normalization of \( Y_l^m \), we have

\[
C_l \sqrt{\frac{2k^2}{\pi}} \frac{(k^2)^l}{(2l+1)!} = \frac{2^{l+1}}{\sqrt{(2l+1)!}} \sqrt{\frac{4\pi}{(2l+1)!}} i^l (k^2)^l
\]

\[
\Rightarrow \sqrt{\frac{2k^2}{\pi}} C_l = i^l \sqrt{\frac{4\pi}{(2l+1)!}} (k^2)^l
\]

hence

\[
e^{i hr \cos \Theta} = \sum_{l=0}^{\infty} i^l \sqrt{\frac{4\pi}{(2l+1)!}} (k^2)^l \ y_l^* (\theta, \phi) y_l (\theta, \phi)
\]

\[
= \sum_{l=0}^{\infty} i^l (2l+1)! \ j_l (kr) \ P_l (\cos \Theta)
\]
Now recall it could be in any direction in general instead of along the z-axis, but the above derivation will still be valid except that $\theta$ on the RHS is replaced by $\phi$, the $\phi$ between $0$ and $\pi$. Thus

$$\psi = \sum_{l=0}^{\infty} \frac{1}{2l+1} j_{l+1/2}(kr) R_l(\cos \theta).$$

Next, the addition theorem for spherical harmonics allows us to re-write $R_l(\cos \theta)$ in terms of the angles $(\Theta, \Phi)$ as above.

Thus, the plane wave state, the state of well defined momentum, involves a sum over all possible orbital $\Phi$ momentum. Likewise, we can expand a state of well defined orbital angular momentum in terms of states with arbitrary direction of momentum (both states have fixed energy $E_{l \Phi}$)

Using the orthonormality of the spherical harmonics we have
\[ j_{l}(kr) \ Y_{l}^{m}(\theta, \phi) = (-1)^{l} \ i^{l} \ \frac{4 \pi}{r^{l}} \ \int d\Omega_{p} \ \ Y_{l}^{m}(\theta_{p}, \phi_{p}) e^{i \ h \ \alpha} \]

\[ \frac{1}{\sqrt{2 \ h^{2}}} \ \mathcal{A}_{\ell m}(\vec{r}) = \frac{1}{\sqrt{2 \ h^{2}}} \ \left\{ \vec{r} \ | \ \ell, \ m \ \right\}. \]

So a state of well-defined \emph{X} momentum involves all possible \emph{directions} of the linear momentum.
As another example of the importance of symmetry groups, consider the $O(4)$ symmetry of the Coulomb potential:

Recall the Hamiltonian for the hydrogen atom for the relative particle is given by

$$H = -\frac{\hat{p}^2}{2m} + V(r/l)$$

where $V(r/l) = -\frac{e^2}{r/l}$, the Coulomb potential.

As we know, the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \hat{p}$ commutes with the Hamiltonian

$$[H, L_i] = 0.$$ 

As first pointed out by W. Pauli ($Z. Physik. 36 (1926) 336$), there is a larger symmetry that this Hamiltonian has than just the $SU(2)$ of angular momentum, and this larger symmetry can be exploited to find the energy eigenvalues of hydrogen.
Suppose we introduce the Runge-Lenz vector operator

\[ \vec{W} = \sqrt{-\frac{m}{2\hbar}} \left[ \frac{1}{2m} (\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \frac{e^2}{r} \vec{r} \right] \]

That is,

\[ W_i = \sqrt{-\frac{m}{2\hbar}} \left[ \frac{1}{2m} \epsilon_{ijk} (P_j L_k - L_j P_k) - \frac{e^2}{r} x_i \right] \]

Using the commutation relations of \( [x_i, p_j] = i\hbar \delta_{ij} \) we find the following properties of \( \vec{W} \):

1. \( [W_i, H] = 0 \)
2. \( [W_i, W_j] = i\hbar \epsilon_{ijk} L_k \)
3. \( [L_i, W_j] = i\hbar \epsilon_{ijk} W_k \)
4. \( \vec{L} \cdot \vec{W} = \vec{W} \cdot \vec{L} = 0 \)
5. \( \vec{W}^2 = -\frac{m}{2\hbar} \left[ \frac{2\hbar}{m} (L^2 + \hbar^2) + (e^2)^2 \right] \)

\[ = -\vec{L}^2 - \hbar^2 - \frac{m}{2\hbar} (e^2)^2 \]

\[ \Rightarrow \vec{W}^2 + \vec{L}^2 = -\hbar^2 - \frac{m}{2\hbar} (e^2)^2 \]
6) \([L_i, L_j] = i \hbar \varepsilon_{ijk} L_k\). 

Then we find

\[
[I_i, I_j] = i \hbar \varepsilon_{ijk} I_k \\
[F_i, F_j] = i \hbar \varepsilon_{ijk} F_k \\
[I_i, F_j] = 0
\]
\[ -234 - i \]

Since \([H, \hat{I}] = 0 = [H, \hat{W}]\) we also have \([\hat{I}_i, H] = 0 = [\hat{F}_i, H]\).

Thus the hydrogen Hamiltonian actually has an \(O(4)\) symmetry.

Now any vector \(\hat{J}\) that obeys the \(SU(2)\) algebra

\[ [\hat{J}_i, \hat{J}_j] = i(\hat{e}_i \epsilon_{ijk} \hat{J}_k) \]

has

1) \(\hat{J}^2\) with eigenvalues \(\hbar^2 j(j+1)\)
   where \(j = 0, \frac{1}{2}, 1, \ldots\)

2) \(\hat{J}_z\) with eigenvalues for a given \(j\)
   \(m = -j, -j+1, \ldots, j-1, j\).

Thus

\(\hat{I}^2\) has eigenvalues \(i(i+1)\hbar^2\), \(i = 0, \frac{1}{2}, \frac{3}{2}, \ldots\)

\(\hat{F}^2\) has eigenvalues \(f(f+1)\hbar^2\), \(f = 0, \frac{1}{2}, \frac{3}{2}, \ldots\).

Since \([\hat{I}, H] = 0 \Rightarrow [\hat{I}^2, H] = 0\)

\([\hat{F}, H] = 0 \Rightarrow [\hat{F}^2, H] = 0\)

\([\hat{I}_i, \hat{F}_j] = 0 \Rightarrow [\hat{I}^2, \hat{F}^2] = 0\)
Hence \( E, F, F^2 \) are a CSCO.

Their simultaneous eigenstates are defined by \( | E, i, f \rangle \) with

\[
H|E, i, f \rangle = E|E, i, f \rangle \\
F^2 |E, i, f \rangle = i(i+1) \hbar^2 |E, i, f \rangle \quad ; i = 0, \frac{1}{2}, ... \\
\bar{F}^2 |E, i, f \rangle = f(f+1) \hbar^2 |E, i, f \rangle \quad ; f = 0, \frac{1}{2}, ...
\]

Next consider the operators

\[
\hat{I}^2 = \frac{1}{4} (\hat{L}^2 + \hat{W}^2) = \frac{1}{4} (\hat{L}^2 + \hat{W}^2 + \hat{L} \hat{W} + \hat{W} \hat{L}) \\
\bar{F}^2 = \frac{1}{4} (\hat{L} - \hat{W})^2 = \frac{1}{4} (\hat{L}^2 + \hat{W}^2 - 2 \hat{L} \hat{W} + \hat{W} \hat{L})
\]

But \( \hat{L} \hat{W} = \hat{W} \hat{L} = 0 \)

\( \Rightarrow \) \( \hat{I}^2 = \bar{F}^2 = \frac{1}{4} (\hat{L}^2 + \hat{W}^2) \)

\( \Rightarrow \hat{I}^2 |E, i, f \rangle = \bar{F}^2 |E, i, f \rangle \\
\hat{I} (i(i+1) \hbar^2 |E, i, f \rangle = f(f+1) \hbar^2 |E, i, f \rangle \)

\( \hat{I} \)}
\[ i = f \]

Further,

\[
\hat{I}^2 |E_i; f\rangle = \frac{1}{4} (\hat{L}^2 + \hat{W}^2) |E_i; f\rangle
\]

\[
\mathbb{P} = -\frac{1}{4} \left( \frac{1}{h^2} + \frac{m}{2\hbar^2} |E_i; f\rangle \right)
\]

\[
i(i+1)\frac{h^2}{\mathbb{P}} |E_i; f\rangle = -\frac{1}{4} \left( \frac{1}{h^2} + \frac{m(e^2)^2}{2E} \right) |E_i; f\rangle
\]

\[
\Rightarrow \quad \hbar^2 i(i+1) = -\frac{1}{4} \left( \frac{1}{h^2} + \frac{m e^4}{2E} \right)
\]

\[
\Rightarrow \quad E = -\frac{m e^4}{2 \left( 4i(i+1)+1 \right) h^2} ; i = 0, \frac{1}{2}, 1, \ldots
\]

\[
\Rightarrow \qquad E = -\frac{m e^4}{2(2i+1)^2 h^2} ; i = 0, \frac{1}{2}, 1, \ldots
\]
Now let
\[ N = 2i + 1 \quad \text{so for } \iota = 0, \frac{1}{2}, 1, \ldots \]
and \[ N = 1, 2, 3, \ldots, \] a positive integer.

\[ E_n = -\frac{mc^2 e^r}{2\hbar^2 n^2}, \quad n = 1, 2, 3, \ldots \]

and defining the fine-structure constant\[ \alpha = \frac{e^2}{\hbar c} \implies \]

\[ E_n = -\frac{mc^2 \alpha}{2n^2}; \quad n = 1, 2, 3, \ldots \]
as we found earlier.