5.3.5 Angular Momentum Operator Matrix Representation In the Standard Basis

As seen above, we only need to construct the matrix elements of $\vec{J}$ within each finite dimensional $\mathcal{H}(k,j)$ subspace. Also, each finite dimensional submatrix is independent of k, it depends only of j.

Specifically, we have

$$J_z|k,j,m\rangle = m\hbar|k,j,m\rangle$$

$$J_+|k,j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|k,j,m+1\rangle$$

$$J_-|k,j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|k,j,m-1\rangle .$$

Since the basis vectors are orthonormal, we find

$$\langle k,j,m|J_z|k',j',m'\rangle = m\hbar\,\delta_{kk'}\delta_{jj'}\delta_{mm'}$$

$$\langle k,j,m|J_\pm|k',j',m'\rangle = \hbar\sqrt{j(j+1)-m'(m'\pm 1)}\,\times$$

$$\times\,\delta_{kk'}\delta_{jj'}\delta_{m(m'\pm 1)}.$$

The matrix elements depend only on j and m.

Hence to find all matrix representations of $\vec{J}$ we need only to calculate the $(2j+1)\times(2j+1)$ dimensional matrices above for each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. These matrix elements are defined by

$$\langle k, j, m | \vec{J} | k', j', m' \rangle \equiv (\vec{J}^{(j)})_{mm'} \delta_{kk'} \delta_{jj'}.$$

Since the $[J^a, J^b] = i\hbar \epsilon_{abc} J^c$ we have for the matrices that

$$\langle k, j, m | [J^a, J^b] | k', j', m' \rangle = i\hbar \epsilon_{abc} \langle k, j, m | J^c | k', j', m' \rangle$$

Inserting a complete set of states on the LHS,

$$\sum_{k''} \sum_{j''} \sum_{m''=-j''}^{+j''} \Big\{ \underbrace{\langle k, j, m | J^a | k'', j'', m'' \rangle}_{= (J^{(j)a})_{mm''} \delta_{kk''} \delta_{jj''}} \overbrace{\langle k'', j'', m'' | J^b | k', j', m' \rangle}^{= (J^{(j)b})_{m''m'} \delta_{k''k'} \delta_{j''j'}}$$

$$- \langle k, j, m | J^b | k'', j'', m'' \rangle \langle k'', j'', m'' | J^a | k', j', m' \rangle \Big\}$$

$$= i\hbar \epsilon_{abc} (J^{(j)c})_{mm'} \delta_{kk'} \delta_{jj'}$$

performing the k'', j'' sums $\Rightarrow$

$$\left\{ \sum_{m''=-j}^{+j} \left(J^{(j)a}\right)_{mm''} \left(J^{(j)b}\right)_{m''m'} - \sum_{m''=-j}^{+j} \left(J^{(j)b}\right)_{mm''} \left(J^{(j)a}\right)_{m''m'} \right\} \delta_{kk'} \delta_{jj'}$$

$$= i\hbar\, \epsilon_{abc} \left(J^{(j)c}\right)_{mm'} \delta_{kk'} \delta_{jj'} .$$

Thus we obtain that the same angular momentum algebra is obeyed by each $(2j+1)\times(2j+1)$ matrix representation of $\vec{J}$

$$\left(\left[J^{(j)a}, J^{(j)b}\right]\right)_{mm'} = i\hbar\epsilon_{abc}\left(J^{(j)c}\right)_{mm'} .$$

Hence we have found by determining the $(\vec{J}^{(j)})_{mm'}$ all finite dimensional representations of the SU(2) rotation group commutation laws, and, after exponentiation, all finite dimensional matrix representations of the group multiplication laws.

With this in mind, recall that the spin matrices $\vec{S}$ which occurred in our intuitive discussion of angular momentum were finite dimensional matrix representations of the SU(2) algebra. Thus, by finding all such matrices $(\vec{J}^{(j)})_{mm'}$, we have found all possible spin matrices $\vec{S}$ also. For each $\vec{J}^{(j)}$ we have an $\vec{S}$, thus with $s=j$ we have $(\vec{S}^{(s)})_{m_s m_s'} = (\vec{J}^{(s)})_{m_s m_s'}$ for

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m_s = -s, -s+1, \ldots, s-1, s$.

The group multiplication law representatives are the $D^{(s)}(R(\vec{\theta}))$ matrices. So we have on the other hand that, for finite rotations, for each $j = 0, \frac{1}{2}, 1, \ldots$

$$\left(D^{(j)}(R(\vec{\theta}))\right)_{mm'} = \left(e^{\frac{-i}{\hbar}\vec{\theta}\cdot\vec{J}^{(j)}}\right)_{mm'}.$$

Returning to the problem at hand, we can, using the matrix elements for J_z, $J_\pm$ above to find the first few angular momentum or spin matrices.

1) $j=0$: The $\mathcal{H}(k, j=0)$ space has dimension $2j+1=1$. Since $j=0 \Rightarrow m=0$ we have only one row and one column, the matrix is just a number. Explicitly we have

$$\langle k, j=0, m=0 | \vec{J} | k', j=0, m=0 \rangle = 0$$

from above. Thus $\vec{J}^{(0)} = 0$

and $D^{(0)}(R(\vec{\theta})) = 1$.

This is just the case of ordinary wave mechanics with a spin zero particle.

2) $j=\frac{1}{2}$: $\mathcal{H}(k, j=\frac{1}{2})$ has dimension $2j+1 = 2(\frac{1}{2})+1 = 2$ with $m=\pm\frac{1}{2}$.

Thus we have 2×2 matrices with rows and columns labelled by $m=+\frac{1}{2}, -\frac{1}{2}$, in that order.

From our matrix element formulae we find

$$J_z^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_+^{(\frac{1}{2})} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$J_-^{(\frac{1}{2})} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Recalling that the raising & lowering operators are

$$J_\pm \equiv J_x \pm i J_y$$

so inverting yields the $J_{\substack{x \\ y}}$ operators

$$J_x = \frac{1}{2}(J_- + J_+)$$

$$J_y = \frac{i}{2}(J_- - J_+).$$

Hence we find the $j=\frac{1}{2}$ matrix representations for $J_{x,y}$

$$J_x^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}.$$

Combining the results, we see we obtain the earlier result that spin $\frac{1}{2}$ particles have angular momentum matrices given by the Pauli matrices

$$\boxed{\vec{J}^{(\frac{1}{2})} = \frac{\hbar}{2}\vec{\sigma}\,.}$$

(Note: $\vec{J}^{(\frac{1}{2})2} = \frac{1}{2}(\frac{1}{2}+1)\hbar^2 \times \begin{pmatrix}1&0\\0&1\end{pmatrix}$)

3) $j=1$: $\mathcal{H}(k, j=1)$ has dimension

$$2j+1 = 2(1)+1 = 3 \text{ with } m=+1,0,-1.$$

Thus we have the 3×3 dimensional matrix representation of $\vec{J}$. The rows and columns are labelled by $m=+1,0,-1$, in that order.

Applying the matrix element formulas, we find

$$J_z^{(1)} = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$J_+^{(1)} = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad J_-^{(1)} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

which implies

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad J_y^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

with $(\vec{J}^{(1)})^2 = 1(1+1)\hbar^2 \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$.

The $J_{x,y,z}^{(1)}$ above are just equivalent to our previous spin 1 matrices after a change of basis vectors.

Given the matrix representation of $\vec{J}$ in the standard basis, we know also how the basis vectors transform under the action of $\vec{J}$ since $\vec{J}\,\mathcal{H}(k,j) = \mathcal{H}(k,j)$ we have

$$\vec{J}\,|k,j,m\rangle = \overbrace{\sum_{m'=-j}^{+j} |k,j,m'\rangle\langle k,j,m'|}^{=\mathbb{1}_{(k,j)}}\underbrace{\vec{J}\,|k,j,m\rangle}$$

$$= \sum_{m'=-j}^{+j} (\vec{J}^{(j)})_{m'm}\,|k,j,m'\rangle$$

where the underbraced matrix element $\langle k,j,m'|\vec{J}|k,j,m\rangle = (\vec{J}^{(j)})_{m'm}$.

Thus in the $j=\frac{1}{2}$ case for example we have eigenvectors for $\mathcal{H}(k, j=\frac{1}{2})$ $|k,\frac{1}{2},+\frac{1}{2}\rangle$ and $|k,\frac{1}{2},-\frac{1}{2}\rangle$. Any vector in this subspace can be written as a linear combination of the 2 basis vectors,

$$|\chi\rangle = \chi_\uparrow |k,\tfrac{1}{2},+\tfrac{1}{2}\rangle + \chi_\downarrow |k,\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

where

$$\chi_\uparrow = \langle k,\tfrac{1}{2},+\tfrac{1}{2}|\chi\rangle$$

$$\chi_\downarrow = \langle k,\tfrac{1}{2},-\tfrac{1}{2}|\chi\rangle .$$

The probability for finding state $|\chi\rangle$ in the J_z-eigenstate with $+\frac{\hbar}{2}$ eigenvalue (spin up) is just $|\chi_\uparrow|^2$ while the probability to be in the J_z eigenstate with $-\frac{\hbar}{2}$ eigenvalue (spin down) is $|\chi_\downarrow|^2$. And of course $|\chi_\uparrow|^2+|\chi_\downarrow|^2=1$.

The state vectors can be represented by column vectors just as the angular momentum operators were represented by matrices in the standard basis. In $\mathcal{H}_{k,j}(j,m)$ the basis vectors are

$$|k,j,m\rangle = \sum_{m'=-j}^{+j} (e^m)_{m'} \, |k,j,m'\rangle$$

with $(e^m)_{m'} = \delta_{mm'}$; thus the $|k,j,m\rangle$ vector has components $(e^m)_{m'}$ in this basis

$$|k,j,m\rangle \longrightarrow e^m = \begin{array}{c} \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} m' \\ j \\ \\ \\ \leftarrow m'=m \\ \\ \\ -j \end{array}$$

For $j=\frac{1}{2}$ the basis vectors can be represented by their 2-component column vectors

$$|k,\tfrac{1}{2},+\tfrac{1}{2}\rangle \longrightarrow e^{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|k,\tfrac{1}{2},-\tfrac{1}{2}\rangle \longrightarrow e^{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And hence any vector can be represented by its expansion coefficient column vector

$$|\chi\rangle \longrightarrow (\chi)_m = \langle k, j, m|\chi\rangle$$
$$= \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}_m .$$

Thus, the state column vector is given by a sum over the basis column vectors

$$\chi = \chi_\uparrow e^\uparrow + \chi_\downarrow e^\downarrow .$$

The angular momentum operators acting on the states can then be represented by the matrix multiplication of their representative and the state column vectors

$$\langle k, j, m|\vec{J}|\chi\rangle = \sum_{m'=-j}^{+j} \langle k, j, m|\vec{J}|k, j, m'\rangle\langle k, j, m'|\chi\rangle$$

$$\left(\vec{J}|\chi\rangle\right)_m = \sum_{m'=-j}^{+j} \left(\vec{J}^{(j)}\right)_{mm'} (\chi)_{m'}$$

Hence expectation values can be calculated by matrix manipulations

$$\langle\chi'|\vec{J}|\chi\rangle = \sum_{m=-j}^{+j} \langle\chi'|k, j, m\rangle\left(\vec{J}|\chi\rangle\right)_m$$

$$\langle \chi' | \vec{J} | \chi \rangle = \sum_{m,m'} (\chi')^*_m (\vec{J}^{(j)})_{mm'} (\chi)_{m'}$$

or in matrix notation

$$= \chi'^{\dagger} \vec{J}^{(j)} \chi$$

$$= \begin{pmatrix} \chi'^*_{\uparrow} & \chi'^*_{\downarrow} \end{pmatrix} \frac{\hbar}{2} \vec{\sigma} \begin{pmatrix} \chi_{\uparrow} \\ \chi_{\downarrow} \end{pmatrix}$$ in the $j = \frac{1}{2}$ case.

5.3.6. Spin and Orbital Angular Momentum Revisited

Finally suppose we recall again our physical discussion of angular momentum in the preceding section. We can use the definition of the orbital angular momentum operator $\vec{L} = \vec{R} \times \vec{P}$ to define the spin operator $\vec{S}$ by

$$\vec{J} = \vec{L} + \vec{S} \quad .$$

Since $\vec{R}$, $\vec{P}$ as well as $\vec{J}$ are vector operators

$$[J^i, J^j] = i\hbar \epsilon_{ijk} J^k$$

$$[J^i, X^j] = i\hbar \epsilon_{ijk} X^k$$

$$[J^i, P^j] = i\hbar \epsilon_{ijk} P^k$$