We would now like to find all possible matrices $\vec{S}_{ij}$ and hence $D^{(s)}(R)$. We explicitly constructed the matrices for spin $0, \frac{1}{2}, 1$. To determine the general matrix structure for S we can turn to consideration of the eigenvalue determination of a set of commuting observables made from the $\vec{J}$ and whatever other operators may commute with them.

5.3.4. Angular Momentum Commutation Relations and the "Standard Basis"

The angular momentum commutation relations are

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k .$$

Hence $\vec{J}^2 = \vec{J} \cdot \vec{J}$ commutes with J_i since it is a scalar (the dot product of 2 vector operators) operator

$$[\vec{J}^2, J_i] = 0 \quad \text{for } i = 1,2,3 .$$

Hence only $\vec{J}^2$ and one of the J_i can have simultaneous eigenvectors. We choose to find the eigenvectors for $\vec{J}^2$ and $J_3 = J_z$. In addition there generally are other operators which commute with $\vec{J}^2$ and J_3; we include them to obtain a mutually commuting complete set of observables, a CSCO. We can label the eigenvalues of the operators other than $\vec{J}^2$ and J_3 by index k (either continuous or discrete or both) and denote them generically by operator A. Hence we have ~~that~~

$$[A, \vec{J}^2] = 0 = [A, J_3]$$

$$[\vec{J}^2, J_3] = 0 \quad .$$

~~Their~~ simultaneous eigen-kets are given by $|k, \lambda, m\rangle$ with

$$\vec{J}^2 |k, \lambda, m\rangle = \lambda \hbar^2 |k, \lambda, m\rangle$$

$$J_z |k, \lambda, m\rangle = m\hbar |k, \lambda, m\rangle$$

$$A |k, \lambda, m\rangle = a_k |k, \lambda, m\rangle ,$$

where since J_i is Hermitian, $J_i = J_i^\dagger$, the λ and m are real (as is a_k). The eigenstates are chosen to be orthonormal

$$\langle k', \lambda', m' | k, \lambda, m \rangle = \delta_{kk'} \delta_{\lambda\lambda'} \delta_{mm'}$$

and complete

$$\sum_k{}' \sum_\lambda{}' \sum_m{}' |k, \lambda, m\rangle\langle k, \lambda, m| = \mathbb{1},$$

(we are writing k as a discrete index for convenience, it may be either discrete or continuous).

Since $\langle \psi|\psi\rangle \geq 0$ we have for

$$|\psi\rangle = J^i |k, \lambda, m\rangle \text{ that}$$

$$\sum_{i=1}^{3}{}' \langle k, \lambda, m | J^{i\dagger} J^i | k, \lambda, m\rangle \geq 0$$

$$= \sum_{i=1}^{3}{}' \langle k, \lambda, m | J^i J^i | k, \lambda, m\rangle$$

$$= \langle k, \lambda, m | \vec{J}^2 | k, \lambda, m\rangle = \lambda\hbar^2 \geq 0$$

Thus $\lambda \geq 0$, the eigenvalues of $\vec{J}^2$ are non-negative.

We begin by determining the eigenvalue spectrum of $\vec{J}^2$ and J_3. As in the SHO case, we exploit the commutation relations for $\vec{J}$; $[J_i, J_j] = i\hbar\epsilon_{ijk} J_k$.

Define the raising operator

$$J_+ \equiv J_x + iJ_y = J_1 + iJ_2$$

and the lowering operator

$$J_- \equiv J_x - iJ_y = J_1 - iJ_2$$

and label $J_3 = J_0 = J_z$. Note $J_+^\dagger = J_-$. The commutation relations become

$$[J_0, J_+] = \hbar J_+$$

$$[J_0, J_-] = -\hbar J_-$$

$$[J_+, J_-] = 2\hbar J_0 \quad .$$

(Inverse: $J_x = \frac{1}{2}(J_- + J_+)$, $J_y = \frac{i}{2}(J_- - J_+)$, $J_z = J_0$) ($J_i^\dagger = J_i$)

Since
$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$
$$= \tfrac{1}{2}(J_+ J_- + J_- J_+) + J_0^2$$

we clearly have, due to $[\vec{J}^2, J_i] = 0$,
$$[\vec{J}^2, J_{\pm, 0}] = 0.$$

Again we can only diagonalize $\vec{J}^2$ and one of the ~~operators~~ operators $J_{\pm, 0}$; we choose J_0.

Often we find that the products J_+J_- and J_-J_+ appear. They can be written as

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y)$$
$$= J_x^2 + J_y^2 - i[J_x, J_y]$$
$$= J_x^2 + J_y^2 + \hbar J_0 .$$

Similarly
$$J_-J_+ = J_x^2 + J_y^2 - \hbar J_0 .$$

Thus
$$J_+ J_- = \vec{J}^2 - J_0^2 + \hbar J_0$$
$$J_- J_+ = \vec{J}^2 - J_0^2 - \hbar J_0 .$$

As we have seen $\vec{J}^2$ is a non-negative operator, for any state

$$\langle\psi|\vec{J}^2|\psi\rangle = \langle\psi|J_x^2|\psi\rangle + \langle\psi|J_y^2|\psi\rangle + \langle\psi|J_z^2|\psi\rangle$$
$$= \langle\psi|J_x^\dagger J_x|\psi\rangle + \langle\psi|J_y^\dagger J_y|\psi\rangle + \langle\psi|J_z^\dagger J_z|\psi\rangle$$
$$= \| J_x|\psi\rangle\|^2 + \|J_y|\psi\rangle\|^2 + \|J_z|\psi\rangle\|^2$$
$$\geq 0 .$$

If $|\psi\rangle$ is an eigenvector of $\vec{J}^2$; $\vec{J}^2|\psi\rangle = \lambda\hbar^2 |\psi\rangle$,

then $\langle\psi|\vec{J}^2|\psi\rangle = \lambda\hbar^2 \||\psi\rangle\|^2 \geq 0$, but

$$\||\psi\rangle\|^2 > 0 \quad \Rightarrow \quad \lambda \geq 0 .$$

This is analogous to the case of the number operator $a^\dagger a$ for the SHO; it too was a non-negative operator.

By convention we write $\lambda \equiv j(j+1)$ with $j \geq 0$. Since both $\lambda, j \geq 0$, the equation

$$\lambda = j(j+1) \quad \text{has only one}$$

positive or zero root $\left(j = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\lambda}\right)$.

Thus given j we know λ, given λ we know j uniquely. Similarly, $J_z^\dagger = J_z$, thus $m \in \mathbb{R}$.

Hence the $\vec{J}^2$, J_z eigenvalue equations and eigenvectors ($|k, \lambda, m\rangle = |k, j, m\rangle$) are

$$\vec{J}^2 |k, j, m\rangle = j(j+1)\hbar^2 |k, j, m\rangle$$

$$J_z |k, j, m\rangle = m\hbar \, |k, j, m\rangle$$

with $j \geq 0$, $m \in \mathbb{R}$.

<u>Lemma 1</u>: Properties of $\vec{J}^2$ and J_z eigenvalues.

With j, m defined above and associated with the same eigenket $|k, j, m\rangle$, then

$$-j \leq m \leq +j .$$

Proof: Since $J_+^\dagger = J_-$ we have

$$\| J_+ |k,j,m\rangle \|^2 = \langle k,j,m | J_- J_+ | k,j,m\rangle \geq 0$$

and

$$\| J_- |k,j,m\rangle \|^2 = \langle k,j,m | J_+ J_- | k,j,m\rangle \geq 0.$$

Since
$$J_+ J_- = \vec{J}^2 - J_z^2 + \hbar J_z$$
$$J_- J_+ = \vec{J}^2 - J_z^2 - \hbar J_z$$
we can evaluate the norms

1) $$\langle k,j,m | J_- J_+ | k,j,m\rangle = \langle k,j,m | \vec{J}^2 - J_z^2 - \hbar J_z | k,j,m\rangle$$
$$= [j(j+1) - m^2 - m]\hbar^2 \geq 0$$

2) $$\langle k,j,m | J_+ J_- | k,j,m\rangle = \langle k,j,m | \vec{J}^2 - J_z^2 + \hbar J_z | k,j,m\rangle$$
$$= [j(j+1) - m^2 + m]\hbar^2 \geq 0$$

Thus equation 1) yields

1') $$j(j+1) - m(m+1) = (j-m)(j+m+1) \geq 0$$

while equation 2) yields

2') $$j(j+1) - m(m-1) = (j-m+1)(j+m) \geq 0.$$

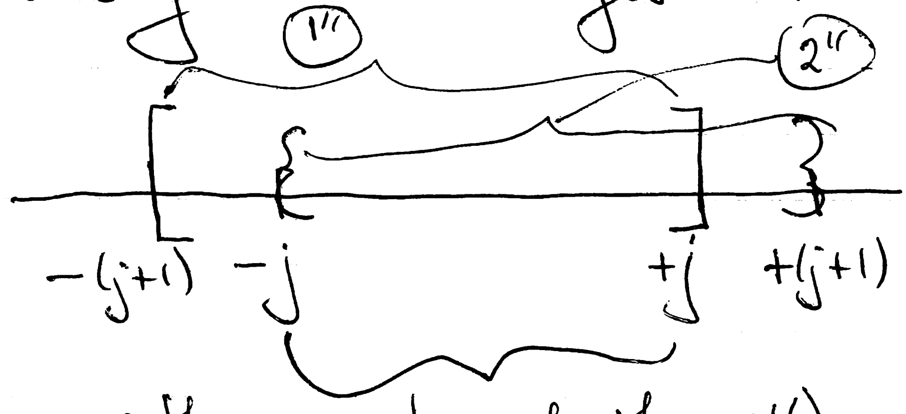
In order for $(j-m)(j+m+1) \geq 0$ we must have that

$$j \geq m \quad \text{and} \quad m \geq -(j+1)$$

That is 1″) $\boxed{-(j+1) \leq m \leq j}$. We also have that 2′) implies that $m \leq j+1$ and $m \geq -j$

That is 2″) $\boxed{-j \leq m \leq j+1}$

Pictorially those ranges are



the overlap of the 1″) and 2″) regions imply

$$-j \leq m \leq +j \quad , \text{as required.}$$

Lemma 2: Properties of $J_-|k,j,m\rangle$.

Let $|k,j,m\rangle$ be an eigenvector of $\vec{J}^2$ and J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, respectively.

a.) Iff $m=-j$, then $J_-|k,j,m\rangle = 0$

b.) If $m>-j$, then $J_-|k,j,m\rangle$ is

a non-zero eigenvector of $\vec{J}^2$ and J_z with the eigenvalues $j(j+1)\hbar^2$ and $(m-1)\hbar$, respectively.

Proof: a.) From above

$$\|J_-|k,j,m\rangle\|^2 = [j(j+1) - m(m-1)]\hbar^2,$$

hence if $m=-j$, the RHS $=0 \Rightarrow$

$$\|J_-|k,j,-j\rangle\|^2 = 0 \Rightarrow$$

$$J_-|k,j,-j\rangle = 0, \text{ the zero vector.}$$

Further, if $J_-|k,j,m\rangle = 0 \Rightarrow m=-j$ since we can operate with J_+ to find

$$J_+ J_- |k,j,m\rangle = 0$$

$$= \hbar^2 [j(j+1) - m^2 + m] |k,j,m\rangle$$

$$= \hbar^2 (j+m)(j-m+1) |k,j,m\rangle = 0$$

$\Rightarrow (j+m)(j-m+1) = 0$; But $-j \le m \le j$,

so there is only one solution $\Rightarrow \boxed{m = -j.}$

b) if $m > -j$, then from above

$$\| J_- |k,j,m\rangle \|^2 = [j(j+1) - m(m-1)] \hbar^2 \neq 0.$$

Thus $J_- |k,j,m\rangle$ is <u>not</u> the zero vector.

Using the angular momentum algebra we ~~have~~ that

$$[\vec{J}^2, J_-] = 0 \quad ; \text{ thus}$$

$$[\vec{J}^2, J_-] |k,j,m\rangle = 0$$

$$\Rightarrow \vec{J}^2 (J_- |k,j,m\rangle) = J_- \underbrace{(\vec{J}^2 |k,j,m\rangle)}_{= j(j+1)\hbar^2 |k,j,m\rangle}$$

$$= j(j+1)\hbar^2 (J_- |k,j, \ \rangle)$$

Hence $J_-|k,j,m\rangle$ is an eigenvector of $\vec{J}^2$ with eigenvalue $j(j+1)\hbar^2$.

Now the angular momentum algebra also has

$$[J_z, J_-] = -\hbar J_- \text{ ; so}$$

$$J_z(J_-|k,j,m\rangle) = J_-\underbrace{(J_z|k,j,m\rangle)}_{=m\hbar|k,j,m\rangle} - \hbar J_-|k,j,m\rangle$$

$$= (m-1)\hbar(J_-|k,j,m\rangle).$$

Thus $J_-|k,j,m\rangle$ is also an eigenvector of J_z with the eigenvalue $(m-1)\hbar$.

Lemma 3: Properties of $J_+|k,j,m\rangle$.

Let $|k,j,m\rangle$ be an eigenvector of $\vec{J}^2$ & J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$ respectively.

a.) Iff $m=j$, then $J_+|k,j,m\rangle = 0$.

b.) If $m<j$, then $J_+|k,j,m\rangle$ is a non-zero eigenvector of $\vec{J}^2$ and J_z with the eigenvalues $j(j+1)\hbar^2$ and $(m+1)\hbar$, respectively.

Proof: a) As in Lemma 2,

$$\| J_+ |k,j,m\rangle \|^2 = \hbar^2 \big(j(j+1) - m(m+1) \big),$$

so $m = +j \Rightarrow \| J_+ |k,j,j\rangle \|^2 = 0 \Rightarrow \boxed{J_+ |k,j,j\rangle = 0}$

Conversely if $J_+ |k,j,m\rangle = 0$, then act again with J_- to find

$$J_- J_+ |k,j,m\rangle = \hbar^2 [j(j+1) - m^2 - m] |k,j,m\rangle$$

$$= \hbar^2 [(j-m)(j+m+1)] |k,j,m\rangle = 0.$$

But $-j \leq m \leq j$, hence the only solution to $(j-m)(j+m+1) = 0 \Rightarrow m = j$.

b) If $m < j$ we have from above that

$$\| J_+ |k,j,m\rangle \|^2 = \hbar^2 [j(j+1) - m(m+1)] \neq 0,$$

hence $J_+ |k,j,m\rangle$ is <u>not</u> the zero vector.

As in Lemma 2, the angular momentum algebra implies $[\vec{J}^2, J_+] = 0$ so that

$$\vec{J}^2\left(J_+|k,j,m\rangle\right) = J_+\left(\vec{J}^2|k,j,m\rangle\right)$$
$$= j(j+1)\hbar^2\left(J_+|k,j,m\rangle\right)$$

and using the algebra again, $[J_z, J_+] = +\hbar J_+$ yields

$$J_z\left(J_+|k,j,m\rangle\right) = J_+\left(J_z|k,j,m\rangle\right) + \hbar\left(J_+|k,j,m\rangle\right)$$
$$= (m+1)\hbar\left(J_+|k,j,m\rangle\right).$$

Hence $J_+|k,j,m\rangle$ has $\vec{J}^2$, J_z eigenvalues $\hbar^2 j(j+1)$ and $(m+1)\hbar$, respectively.

We are now in a position to determine the spectrum of $\{\vec{J}^2, J_z\}$ eigenvalues. (The analysis is similar to that used in the SHO case.)

Suppose we have an (non-null) eigenvector $|k,j,m\rangle \neq 0$ with

$$\vec{J}^2|k,j,m\rangle = j(j+1)\hbar^2|k,j,m\rangle$$
$$J_z|k,j,m\rangle = m\hbar|k,j,m\rangle \text{ and}$$

by Lemma 1, $-j \le m \le +j$.

Consider the set of vectors

$$J_+|k,j,m\rangle, J_+^2|k,j,m\rangle, \ldots, J_+^p|k,j,m\rangle, \ldots.$$

Since $-j \le m \le j$, if $m=j$, $J_+|k,j,j\rangle = 0$ by Lemma 3. If $m<j$, $J_+|j,m\rangle$ is non-zero and has $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m+1)\hbar\}$, in particular $m+1 \le j$. If $m+1=j$ then $J_+^2|k,j,m\rangle = 0$ by Lemma 3. If $m+1<j$, $J_+^2|k,j,m\rangle$ is a non-zero eigenvector with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m+2)\hbar\}$. This process can be continued. However, the series must terminate at some point or else we will make non-zero eigenvectors with J_z eigenvalue $> j$; in contradiction with Lemma 1. Therefore there exists an integer $p \ge 0$ such that $J_+^p|k,j,m\rangle$ is a non-zero eigenvector of $\{\vec{J}^2, J_z\}$ with eigenvalues $\{j(j+1)\hbar^2, (m+p)\hbar\}$ such that $J_+(J_+^p|k,j,m\rangle)=0$

that is $J_+^{p+1}|k,j,m\rangle = 0$ is the zero vector.

Thus by Lemma 3, $m+p=j$. Hence

we have that $j-m=p$ is an integer ≥ 0. Further the p-vectors

$$J_+|k,j,m\rangle, J_+^2|k,j,m\rangle, \ldots, J_+^p|k,j,m\rangle$$

are non-zero eigenstates of $\vec{J}^2$ with all the same eigenvalue $j(j+1)\hbar^2$ of $\vec{J}^2$ and are ~~also~~ eigenstates of J_z with eigenvalues $(m+1)\hbar, (m+2)\hbar, \ldots, (m+p)\hbar = j\hbar$ of J_z, respectively.

Analogously we can consider the set of vectors

$$J_-|k,j,m\rangle, J_-^2|k,j,m\rangle, \ldots, J_-^q|k,j,m\rangle, \ldots$$

Since $-j \leq m \leq j$, if $m=-j$, $J_-|k,j,-j\rangle = 0$ by Lemma 2. If $m>-j$, $J_-|k,j,m\rangle$ is non-zero with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m-1)\hbar\}$ with $(m-1) \geq j$. If $m-1=j$ then $J_-^2|k,j,m\rangle = 0$ by Lemma 2. If $m-1>j$, then $J_-^2|k,j,m\rangle \neq 0$ with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m-2)\hbar\}$. And so on, as before this process must terminate or there will be non-zero eigenvectors with J_z eigenvalue $<-j$, in contradiction with Lemma 1. Therefore there exists an integer $q \geq 0$ such that $J_-^q|k,j,m\rangle$ is a non-zero eigenvector with $\{\vec{J}, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m-q)\hbar\}$ and $J_-(J_-^q|k,j,m\rangle) = 0$.

By Lemma 2, $m-q=-j$. Thus $j+m=q \geq 0$ is a non-negative integer. Hence the vectors

$$J_-|k,j,m\rangle, J_-^2|k,j,m\rangle, \ldots, J_-^q|k,j,m\rangle,$$

all have $\vec{J}^2$ eigenvalue $j(j+1)\hbar^2$ and have J_z eigenvalues $(m-1)\hbar, (m-2)\hbar, \ldots, (m-q)\hbar = -j\hbar$, respectively.

Since p and q are non-negative integers, there sum is a non-negative integer also and

$$n = p+q = (j-m)+(j+m)$$

integer ≥ 0

$$= 2j$$

Thus $\boxed{j = \frac{n}{2} \text{ with } n = 0, 1, 2, \ldots}$

Hence the total angular momentum $\vec{J}^2$ eigenvalue, $j(j+1)\hbar^2$, is given by

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots$$

j can have integer as well as half-odd integer (half-integer), non-negative values. Furthermore, if $|k,j,m\rangle$ is a non-zero eigenvector of $\{\vec{J}^2, J_z\}$, then we can use $J_\pm$ to form the series of all non-zero eigenvectors of $\vec{J}^2$ with eigenvalue $j(j+1)\hbar^2$ and J_z with eigenvalues

$$-j\hbar, (-j+1)\hbar, (-j+2)\hbar, \ldots, (j-2)\hbar, (j-1)\hbar, j\hbar;$$

there are $(2j+1)$ eigenvectors of J_z with the same $\vec{J}^2$ eigenvalue $j(j+1)\hbar^2$, given by $m = -j, -j+1, \ldots, j-1, j$. Hence we have found the complete spectrum of $\{\vec{J}^2, J_z\}$.

<u>Theorem:</u> Let $\vec{J}$ be an operator (the angular momentum operator) obeying the commutation relations

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k \quad .$$

If $j(j+1)\hbar^2$ and $m\hbar$ denote the eigenvalues of $\vec{J}^2$ and J_z, respectively, then

1) The only possible values for j are

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots$$

2) For a given j, m can take on any of the $2j+1$ values only

$$-j, -j+1, -j+2, \ldots, j-2, j-1, +j.$$

If j is integral, m is integral; if j is half-integral, m is half-integral.

We next exploit the lowering, J_-, and raising, J_+, operators to construct the eigenvectors of $\{\vec{J}^2, J_z\}$ and hence to form the "standard basis" for $\mathcal{H}$, since $\{A, \vec{J}^2, J_z\}$ are a CSCO, $\{|k, j, m\rangle\}$. In doing this we will just bring together the results we already have shown above.

Consider $\{|k, j, m\rangle\}$ for fixed j and m. The subspace of $\mathcal{H}$ that these vectors span we denote as $\mathcal{H}(j,m) \subseteq \mathcal{H}$. The dimension of $\mathcal{H}(j,m)$ can be >1 due to the $\{\vec{J}^2, J_z\}$ degeneracy of eigenvalues (i.e. in general the $\{\vec{J}^2, J_z\}$ is not a CSCO, we also need A eigenvalues to uniquely specify each eigenvector). Suppose the dimension of $\mathcal{H}(j,m)$ is $g(j,m)$. Then $\mathcal{H}(j,m)$ has as an orthonormal basis

$$\{|k,j,m\rangle \mid k=1,\dots,g(j,m)\}$$

(note g can be ∞ or even continuously infinite).

If $m \neq j$, then there exists another subspace $\mathcal{H}(j,m+1) \subseteq \mathcal{H}$ composed of eigenvectors with $\{\vec{J}^2, J_z\}$ eigenvalues $j(j+1)\hbar^2$ and $(m+1)\hbar$, respectively.

Likewise if $m \neq -j$, then there exists another subspace $\mathcal{H}(j,m-1) \subseteq \mathcal{H}$ composed of eigenvectors with $\{\vec{J}^2, J_z\}$ eigenvalues $j(j+1)\hbar^2$ and $(m-1)\hbar$, respectively.

We can use the raising and lowering operators J_+, J_- to construct $\mathcal{H}(j,m+1), \mathcal{H}(j,m-1)$ from $\mathcal{H}(j,m)$.

For $\mathcal{H}(j, m\pm 1)$ $(m \neq \pm j)$, if $k_1 \neq k_2$ then $J_\pm |k_1, j, m\rangle$ and $J_\pm |k_2, j, m\rangle$ are orthonormal. To see this consider

$$\langle k_2, j, m | J_\mp J_\pm | k_1, j, m\rangle$$

$$= \langle k_2, j, m | (\vec{J}^2 - J_z^2 \mp \hbar J_z) | k_1, j, m\rangle$$

$$= [j(j+1) - m(m\pm 1)]\hbar^2 \langle k_2, j, m | k_1, j, m\rangle .$$

Since $\{ |k, j, m\rangle \mid k = 1, 2, \dots, g(j,m)\}$ are an orthonormal set, the RHS becomes

$$= [j(j+1) - m(m\pm 1)]\hbar^2 \delta_{k_1 k_2} .$$

So we can define the $g(j,m)$ vectors

$$|k, j, m+1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1) - m(m+1)}} J_+ |k, j, m\rangle$$

(we choose the phase $e^{i\varphi(j,m+1)} = 1$, above) for $k=1,2,\ldots,g(j,m)$. Thus we have

$$\langle k,j,m+1|k',j,m+1\rangle = \delta_{kk'} .$$

Further, $\{|k,j,m+1\rangle \mid k=1,\ldots,g(j,m)\}$ are a basis for $\mathcal{H}(j,m+1)$. To see this assume there exists a vector $|\chi,j,m+1\rangle \in \mathcal{H}(j,m+1)$ orthogonal to all $\{|k,j,m+1\rangle\}$. Since $m+1 \neq -j$, $J_-|\chi,j,m+1\rangle$ is non-zero, and has J_z eigenvalue $m\hbar$.
Further $J_-|\chi,j,m+1\rangle \in \mathcal{H}(j,m)$ and

$$\begin{aligned}&\langle \chi,j,m+1|J_+J_-|k,j,m+1\rangle \\ &= [j(j+1)-m(m+1)]\hbar^2 \underbrace{\langle \chi,j,m+1|k,j,m+1\rangle}_{=0 \text{ by assumption}} \\ &= 0 .\end{aligned}$$

This implies that $J_-|\chi,j,m+1\rangle$ is orthogonal to all $J_-|k,j,m+1\rangle$. But

$$= \vec{J}^2 - J_z^2 - \hbar J_z$$

$$J_- |k,j,m+1\rangle = \frac{1}{\hbar\sqrt{j(j+1)-m(m+1)}} \underbrace{J_- J_+}_{} |k,j,m\rangle$$

$$= \hbar\sqrt{j(j+1)-m(m+1)}\, |k,j,m\rangle$$

Hence $J_-|k,j,m+1\rangle$ is non-zero and an element of $\mathcal{H}(j,m)$ but it is orthogonal to all $\{|k,j,m\rangle\}$, the basis vectors of $\mathcal{H}(j,m)$. Hence a contradiction.

Therefore

$\{|k,j,m+1\rangle \mid k=1,2,\ldots,g(j,m)\}$ is a orthonormal basis for $\mathcal{H}(j,m+1)$.

Similarly we define

$$|k,j,m-1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1)-m(m-1)}} J_-|k,j,m\rangle$$

(with the convention that the phase factor $e^{i\varphi(j,m-1)} = 1$).

And by analogous arguments the

$\{|k,j,m-1\rangle \mid k=1,2,\dots,g(j,m)\}$ is

an orthonormal set of basis vectors for $\mathcal{H}(j,m-1)$;

$$\langle k,j,m-1|k',j,m-1\rangle = \delta_{kk'}.$$

As we see, $\mathcal{H}(j,m\pm1)$ has the same dimension as $\mathcal{H}(j,m)$. Thus $g(j,m) = g(j,m\pm1)$ is independent of m and we denote it $g(j) \equiv g(j,m)$; the dimensionality depends only on the total angular momentum eigenvalue j. Hence we can form this type of construction for each allowed value of j.

To summarize the construction of the standard basis for $\mathcal{H}$.

For each allowed value of j for the system under study, choose an arbitrary orthonormal basis for $\mathcal{H}(j,j)$ (i.e. $m=j$), $\{|k,j,j\rangle \mid k=1,\dots,g(j)\}$. Then

define the set

$$|k,j,j-1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1)-(j-1)(j-2)}}\, J_-|k,j,j\rangle.$$

This is a basis for $\mathcal{H}(j,j-1)$. We then act again with J_- on the $|k,j,j-1\rangle$ to construct the basis vectors for $\mathcal{H}(j,j-2)$

$\{|k,j,j-2\rangle \mid k=1,2,\ldots,g(j)\}$ with

$$|k,j,j-2\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1)-(j-2)(j-3)}}\, J_-|k,j,j-1\rangle.$$

We continue this process, the orthonormal basis for $\mathcal{H}(j,m)$ is given by

$\{|k,j,m\rangle \mid k=1,\ldots,g(j)\}$ with

$$|k,j,m\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1)-m(m+1)}}\, J_-|k,j,m+1\rangle.$$

The procedure stops when we finally construct the basis for $\mathcal{H}(j,-j)$;

$\{|k,j,-j\rangle \mid k=1,2,\ldots,g(j)\}$.

Pictorially we have

$$
\left.
\begin{array}{ccc}
 & \{|k,j,j\rangle\} = \mathcal{H}(j,j) & \\
J_+ \uparrow & \downarrow J_- & \\
 & \{|k,j,j-1\rangle\} = \mathcal{H}(j,j-1) & \\
J_+ \uparrow & \downarrow J_- & \\
 & \{|k,j,j-2\rangle\} = \mathcal{H}(j,j-2) & \\
J_+ \uparrow & \downarrow J_- & \\
\vdots & \vdots & \\
J_+ \uparrow & \downarrow J_- & \\
 & \{|k,j,-j+1\rangle\} = \mathcal{H}(j,-j+1) & \\
J_+ \uparrow & \downarrow J_- & \\
 & \{|k,j,-j\rangle\} = \mathcal{H}(j,-j) &
\end{array}
\right\}
\begin{array}{l}
(2j+1)\text{-} \\
\text{subspaces} \\
\text{each with} \\
\text{dimension} \\
g(j).
\end{array}
$$

The basis vectors we constructed to be orthonormal

$$\langle k,j,m|k',j',m'\rangle = \delta_{kk'}\delta_{jj'}\delta_{mm'}$$

and their totality forms a basis for $\mathcal{H}$

$$\sum_{j}\sum_{m=-j}^{+j}\sum_{k=1}^{g(j)} |k,j,m\rangle\langle k,j,m| = \underline{1}$$

identity operator in $\mathcal{H}$.

Note: the set $\{|k,j,m\rangle \mid k=1\ldots g(j)\}$ forms an orthonormal basis for each subspace $\mathcal{H}(j,m)$:

orthonormality $\langle k,j,m|k',j,m\rangle = \delta_{kk'}$

completeness in $\mathcal{H}(j,m)$ $\displaystyle\sum_{k=1}^{g(j)} |k,j,m\rangle\langle k,j,m| = \underline{1}_{(j,m)}$

identity operator in $\mathcal{H}(j,m)$.

Clearly, we may start this construction with any $\mathcal{H}(j,m)$ not just $\mathcal{H}(j,j)$ since $J_\pm$ move us from one subspace to another.

The standard basis vectors can be re-grouped so that instead of belonging to subspaces $\mathcal{H}(j,m)$ with (j,m) fixed, they belong to subspaces $\mathcal{H}(k,j)$ consisting of the basis vectors with (k,j) fixed. For a fixed j the space has the structure

$$
\left.\begin{array}{l}
\text{(2j+1)-}\\
\mathcal{H}(j,m)\\
\text{spaces}\\
\text{(each}\\
\text{of}\\
\text{dimension}\\
g(j))
\end{array}\right\{
\begin{array}{ccccc}
\mathcal{H}(j,j) = & |1,j,j\rangle, & |2,j,j\rangle, & \ldots, & |g(j),j,j\rangle \\
\mathcal{H}(j,j-1) = & |1,j,j-1\rangle, & |2,j,j-1\rangle, & \ldots, & |g(j),j,j-1\rangle \\
\vdots & \vdots & & & \\
\mathcal{H}(j,m) = & |1,j,m\rangle, & |2,j,m\rangle, & \ldots, & |g(j),j,m\rangle \\
\vdots & \vdots & & & \\
\mathcal{H}(j,-j) = & |1,j,-j\rangle, & |2,j,-j\rangle & \ldots, & |g(j),j,-j\rangle \\
 & \| & \| & & \| \\
 & \multicolumn{4}{l}{\underbrace{\mathcal{H}(1,j) \quad\quad \mathcal{H}(2,j) \quad - - - \quad \mathcal{H}(g(j),j)}_{\substack{g(j)-\mathcal{H}(k,j)\ \text{spaces} \\ \text{(each of dimension } (2j+1))}}}
\end{array}
$$

Each $\mathcal{H}(k,j)$ space has dimension $(2j+1)$, further the whole space $\mathcal{H}$ is the sum of these $\mathcal{H}(k,j)$ subspaces

$$\mathcal{H} = \bigoplus_{j} \bigoplus_{k=1}^{g(j)} \mathcal{H}(k,j) \quad .$$

The advantage of the $\mathcal{H}(k,j)$ subspaces over the $\mathcal{H}(j,m)$ subspaces is that they are invariant under operation by $\vec{J}$. That is any vector in $\mathcal{H}(k,j)$ is mapped by $\vec{J}$ into the same $\mathcal{H}(k,j)$ subspace

$$\vec{J} : \mathcal{H}(k,j) \longrightarrow \mathcal{H}(k,j)$$

(or we simply write $\vec{J}\,\mathcal{H}(k,j) = \mathcal{H}(k,j)$).

We see this directly by recalling that

$$J_z |k,j,m\rangle = m\hbar\, |k,j,m\rangle \quad \in \mathcal{H}(k,j)$$

$$J_\pm |k,j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\, |k,j,m\pm1\rangle \quad \in \mathcal{H}(k,j)$$

(however we see that $J_\pm |k,j,m\rangle \notin \mathcal{H}(j,m)$
So $\vec{J}\,\mathcal{H}(j,m) \neq \mathcal{H}(j,m)$, $\mathcal{H}(j,m)$ is not invariant under $\vec{J}$.)

The utility of the $\mathcal{H}(k,j)$ is in determining the matrix elements of $\vec{J}$ in the standard basis. Since $\vec{J}\,\mathcal{H}(k,j) = \mathcal{H}(k,j)$ and the different subspaces $\mathcal{H}(k,j)$ are orthogonal, the matrix elements of $\vec{J}$ between vectors of different subspaces are zero. Hence the matrix representation of $\vec{J}$ in the standard basis with basis vectors grouped according to which $\mathcal{H}(k,j)$ subspace they belong has a block diagonal form.

Rows \ Columns	$\mathcal{H}(k'',j'')$	$\mathcal{H}(k,j)$	$\mathcal{H}(k',j)$	$\mathcal{H}(k',j')$	...
$\mathcal{H}(k'',j'')$	$(2j''+1)\times(2j''+1)$ matrix	0	0	0	...
$\mathcal{H}(k,j)$	0	$(2j+1)\times(2j+1)$ matrix	0	0	...
$\mathcal{H}(k',j)$	0	0	$(2j+1)\times(2j+1)$ matrix	0	...
$\mathcal{H}(k',j')$	0	0	0	$(2j'+1)\times(2j'+1)$ matrix	0
$\vdots$	0	0	0	0	$\ddots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	