except for the first 3 exponentials, all the others collapse to the identity; so we find

$$\boxed{U(R(\theta,\varphi,\psi)) = e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_x}\, e^{-\frac{i}{\hbar}\psi J_z}}$$

5.3.3. Physical Description of Spin

Under rotation the position vector $\vec{r}$ becomes $R\vec{r} = \vec{r}'$ that is if the components of the position vector $\vec{r}$ in a Cartesian coordinate system are x_i before the rotation the position vector, $\vec{r}'$ after rotation has components in this coordinate system given by $x_i' = R_{ij} x_j$ and we write $\vec{r}' = R\vec{r}$ as short hand for this. Hence the quantum mechanical position eigenstates $|\vec{r}\rangle$ and $|\vec{r}' = R\vec{r}\rangle$ are related by the unitary operator $U(R(\vec{\theta}))$ representing the rotation $R(\vec{\theta})$

$$|\vec{r}'\rangle = U(R(\vec{\theta}))|\vec{r}\rangle = |R(\vec{\theta})\vec{r}\rangle$$

Consequently we can determine the rotation transformation properties of the position operator $\vec{R}$, since

$$U^{-1}(R)|\vec{r}\rangle = |R^{-1}\vec{r}\rangle$$

we have

$$X_i U^{-1}(R)|\vec{r}\rangle = X_i|R^{-1}\vec{r}\rangle$$
$$= (R^{-1}_{ij}x_j)|R^{-1}\vec{r}\rangle$$
$$= (R^{-1}_{ij}x_j)U^{-1}(R)|\vec{r}\rangle.$$

$(R^{-1}\vec{r})$ is a c-number, so $U^{-1}(R)$ does not operate on it and the RHS above is

$$= U^{-1}(R)(R^{-1}_{ij}x_j|\vec{r}\rangle)$$

Multiplying by $U(R)$ we have

$$U(R(\vec{\theta}))X_i U^{-1}(R(\vec{\theta}))|\vec{r}\rangle$$
$$= R^{-1}_{ij}(\vec{\theta})x_j|\vec{r}\rangle$$
$$= R^{-1}_{ij}(\vec{\theta})X_j|\vec{r}\rangle.$$

This implies the operator relation

$$X_i' = \boxed{U(R(\vec\theta))\, X_i\, U^{-1}(R(\vec\theta)) = R^{-1}_{ij}(\vec\theta)\, X_j},$$

hence rotating the system one way is equivalent to rotating the coordinate axes the inverse or opposite way.

Any operator which transforms like $\vec R$ under rotation transformations we call a vector operator i.e. if

$$V_i' = U(R(\vec\theta))\, V_i\, U^{-1}(R(\vec\theta)) = R^{-1}_{ij}(\vec\theta)\, V_j$$

then V_i is a vector operator.

As we have seen $\vec J$, $\vec P$ and $\vec K$ are all vector operators.

These rotation properties allow us to classify operators according to how they transform under rotations, this results in an operator tensor analysis.

1) A scalar operator S is invariant under rotations

$$S' = U(R(\vec\theta))\, S\, U^{-1}(R(\vec\theta)) \equiv S$$

Such operators commute with $U(R(\vec{\theta}))$ as indicated by their definition $US = SU$ and hence commute with $\vec{J}$; $[S, \vec{J}] = 0$.

2) $\vec{V}$ is a vector operator if it transforms as $\vec{R}$ under rotations

$$V_i' = U(R(\vec{\theta}))\, V_i\, U^{-1}(R(\vec{\theta})) \equiv R_{ij}^{-1}(\vec{\theta})\, V_j$$

$$\Rightarrow \quad [J_i, V_j] = i\hbar \epsilon_{ijk} V_k \, .$$

3) Rank 2 tensor operators T_{ij}, transform as the product $X_i X_j$ transforms

$$T_{ij}' = U(R(\vec{\theta}))\, T_{ij}\, U^{-1}(R(\vec{\theta}))$$

$$\equiv R^{-1}_{ii'}(\vec{\theta})\, R^{-1}_{jj'}(\vec{\theta})\, T_{i'j'} \, ,$$

(since $R^T = R^{-1}$ this can be written

$$T' = R^{-1} T R \, .)$$

4) Hence in general a rank n tensor operator $T_{i_1 i_2 \cdots i_n}$ transforms under rotations as the product

$X_{i_1} X_{i_2} \cdots X_{i_n}$;

$$T'_{i_1 i_2 \cdots i_n} = U(R(\vec{\theta}))\, T_{i_1 i_2 \cdots i_n}\, U^{-1}(R(\vec{\theta}))$$

$$= R^{-1}_{i_1 j_1}(\vec{\theta})\, R^{-1}_{i_2 j_2}(\vec{\theta}) \cdots R^{-1}_{i_n j_n}(\vec{\theta})\, T_{j_1 \cdots j_n} .$$

Since any rotation can be built up by making successive infinitesimal rotations, we can equivalently define the tensor classification of operators according to the above formulae for $\vec{\theta} = \vec{\omega} \equiv$ infinitesimal, that is by their commutation relations with $\vec{J}$. So we have

$$R_{ij}(\vec{\omega}) = \delta_{ij} + \omega_{ij}$$

and hence $R^{-1}_{ij}(\vec{\omega}) = \delta_{ij} - \omega_{ij}$.

Further

$$U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}$$

So for infinitesimal angles $U(R(\vec{\omega})) = 1 - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}$

and $U^{-1}(R(\vec{\omega})) = 1 + \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}$

Then
$$T' = U(R(\vec{\omega}))\,T\,U^{-1}(R(\vec{\omega}))$$
$$= \left(1 - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right) T \left(1 + \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right)$$
which to first order in $\vec{\omega}$ yields
$$= T - \frac{i}{\hbar}[\vec{\omega}\cdot\vec{J}, T]$$
The above classification of operators are equivalent to

1) Scalar operators S obey
$$[J_i, S] = 0$$

2) Vector operators V_j obey
$$[J_i, V_j] = i\hbar\epsilon_{ijk}V_k$$

3) Rank 2 Tensor operators T_{ij} obey
$$[J_i, T_{mn}] = i\hbar\epsilon_{imm'}T_{m'n} + i\hbar\epsilon_{inn'}T_{mn'}$$

4) Rank n Tensor operators $T_{i_1\cdots i_n}$ obey

$$[J_i, T_{i_1 \cdots i_n}] = i\hbar \epsilon_{i i_1 j_1} T_{j_1 i_2 \cdots i_n} + \cdots + i\hbar \epsilon_{i i_n j_n} T_{i_1 i_2 \cdots i_{n-1} j_n}.$$

As we know from classical physics, expressing the laws of physics in terms of tensor quantities insures that the laws are covariant under a transformation of the coordinates, in this case by rotations.

Since the structure of the Hilbert space is determined by the eigenstates of a CSCO, it is reasonable that if the operators have a tensor classification so do the states. Indeed we have been discussing in wave mechanics particles whose wavefunctions are invariant under rotation transformations,

$$\psi'(\vec{r}\,') = \psi(\vec{r}),$$

The function O' uses in his frame is the same as O uses in his frame. Since $|\psi\rangle$ and $|\psi'\rangle$ are related by $U(R(\vec{\theta}))$

$$|\psi'\rangle = U(R(\vec{\theta}))\,|\psi\rangle$$

we have that

$$\psi'(\vec{r}) \equiv \langle \vec{r} | \psi' \rangle = \langle \vec{r} | U(R(\vec{\theta})) | \psi \rangle$$

$$= \langle R^{-1}(\vec{\theta})\vec{r} | \psi \rangle$$

$$= \psi(R^{-1}(\vec{\theta})\vec{r})$$

, as stated above. Such a <u>scalar</u> wavefunction is said to describe systems with <u>zero spin</u>. This becomes clearer if we consider infinitesimal rotations.

$$\psi'(\vec{r}) = \psi(R^{-1}(\vec{\omega})\vec{r}) = \psi(\vec{r} - \vec{\omega} \times \vec{r})$$

$$= \psi(\vec{r}) - (\vec{\omega} \times \vec{r}) \cdot \vec{\nabla}_{\vec{r}} \psi(\vec{r})$$

$$= \psi(\vec{r}) - \epsilon_{ijk}\, \omega_j\, x_k \frac{\partial}{\partial x_i} \psi(\vec{r})$$

$$= \psi(\vec{r}) - \vec{\omega} \cdot (\vec{r} \times \vec{\nabla}_{\vec{r}}) \psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar} \vec{\omega} \cdot \left(\vec{r} \times \frac{\hbar}{i} \vec{\nabla}_{\vec{r}}\right) \psi(\vec{r})$$

On the other hand

$$\psi'(\vec{r}) = \langle \vec{r} | \psi' \rangle = \langle \vec{r} | U(\vec{R}(\vec{\omega})) | \psi \rangle$$

$$= \langle \vec{r} | \mathbb{1} - \frac{i}{\hbar} \vec{\omega} \cdot \vec{J} | \psi \rangle$$

$$\psi'(\vec{r}) = \psi(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\langle\vec{r}|\vec{J}|\psi\rangle$$
$$= \psi(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\psi(\vec{r})$$

Hence on the space of spin zero wave-functions we have

$$\langle\vec{r}|\vec{J}|\psi\rangle = \left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\langle\vec{r}|\psi\rangle$$
$$= \langle\vec{r}|\vec{R}\times\vec{P}|\psi\rangle$$
$$= \langle\vec{r}|\vec{L}|\psi\rangle$$

So the total angular momentum $\vec{J}=\vec{L}$ is just the orbital angular momentum when acting on the space of scalar states that is spin zero states.

Hence if $\psi'(\vec{r}) = \psi(R^{-1}(\vec{\theta})\vec{r})$ (which is equivalent to $\vec{J}=\vec{L}$ for these states) the states of the system have spin zero.

As with tensor operators, we can introduce multi-component wavefunctions that describe states with spin > 0. These functions are not invariant but mimic the transformation properties of products of position operators X_i! For instance a vector wavefunction has 3 components at each value of space $\psi_i(\vec{r})$. Under rotations not only do the position components rotate into each other, but also to the wave-function components.

Thus we define a vector, or spin 1 wavefunction to transform as

$$\psi'_i(\vec{r}) \equiv R_{ij}(\vec{\theta})\,\psi_j(R^{-1}(\vec{\theta})\vec{r}) \ .$$

Hence we have that the state vector consists of 3 ordinary state vectors in direct product with the basis vectors for the three dimensional spin space

$$|\psi\rangle = \sum_{i=1}^{3} |\psi_i\rangle \otimes |e_i\rangle$$

where

$$\Psi(\vec{r}) \equiv \langle \vec{r} | \psi \rangle$$

$$= \sum_{i=1}^{3} \langle \vec{r} | \psi_i \rangle \; |e_i\rangle$$

$$= \sum_{i=1}^{3} \psi_i(\vec{r}) |e_i\rangle \quad .$$

that is the 3 wavefunctions $\psi_i(\vec{r})$ are the component wavefunctions of $|\psi\rangle$ in the $\{|\vec{r}\rangle \otimes |e_i\rangle\}$ basis. The $\{|e_i\rangle\}$ basis spans a 3 dimensional spin Hilbert space with inner product

$$\langle e_i | e_j \rangle = \delta_{ij} \quad \text{and}$$

completeness,

$$\sum_{i=1}^{3} |e_i\rangle\langle e_i| = 1 \; ,$$

in the 3-dimensional spin space.

The rotation operators are now the direct product of operators acting on the $|\psi_i\rangle$ kets and the $|e_i\rangle$ kets

$$|\psi'\rangle = U^{orbital}(R(\vec{\theta}))|\psi_i\rangle \otimes U^{spin}(R(\vec{\theta}))|e_i\rangle$$

where $U^{orbital}(R(\vec{\theta}))$ only acts on the spatial degrees of freedom while $U^{spin}(R(\vec{\theta}))$ only acts to rotate the 3-dimensional spin space basis vectors. That is

$$\langle \vec{r} | U^{orbital}(R(\vec{\theta})) | \psi_i \rangle = \langle R^{-1}(\vec{\theta}) \vec{r} | \psi_i \rangle$$

while

$$U^{spin}(R(\vec{\theta})) | e_i \rangle = R^{-1}_{ij}(\vec{\theta}) | e_j \rangle.$$

The $U(R(\vec{\theta})) = U^{orbital}(R(\vec{\theta})) \otimes U^{spin}(R(\vec{\theta}))$ are tensor operators on the space $\{|\vec{r}\rangle\} \otimes E^3 = \mathcal{H}_1 \otimes E^3$. (another use of the word tensor operator)

$$\Psi'(\vec{r}) = \langle \vec{r} | \psi' \rangle$$

$$= \langle \vec{r} | U^{orbital}(R(\vec{\theta})) | \psi_i \rangle \, U^{spin}(R(\vec{\theta})) | e_i \rangle$$

$$= \langle R^{-1}(\vec{\theta}) \vec{r} | \psi_i \rangle \, R^{-1}_{ij}(\vec{\theta}) | e_j \rangle$$

$$= R_{ji}(\vec{\theta}) \psi_i (R^{-1}(\vec{\theta}) \vec{r}) | e_j \rangle$$

Recalling that

$$|\psi'\rangle = |\psi'_j\rangle \otimes | e_j \rangle$$

we have that

$$\Psi'(\vec{r}) = \langle \vec{r} | \Psi' \rangle = \langle \vec{r} | \psi'_j \rangle |e_j\rangle = \psi'_j(\vec{r}) |e_j\rangle$$
$$= R_{ji}(\vec{\theta})\, \psi_i(R^{-1}(\vec{\theta})\vec{r}) |e_j\rangle$$

$$\Rightarrow \quad \boxed{\psi'_i(\vec{r}) = R_{ij}(\vec{\theta})\psi_j(R^{-1}(\vec{\theta})\vec{r})}$$

as we stated earlier.

(Recall the single component scalar field, spin zero wavefunction, transformed as
$$\psi'(\vec{r}) = \psi(R^{-1}(\vec{\theta})\vec{r})\,)$$

The relation of these multi-component wavefunction transformation laws to spin 1 is most easily seen by considering infinitesimal rotations $R(\vec{\omega})$

$$\psi'_i(\vec{r}) = R_{ij}(\vec{\omega})\psi_j(R^{-1}(\vec{\omega})\vec{r})$$
$$= (\delta_{ij} - \epsilon_{ijk}\omega_k)\psi_j(\vec{r} - \vec{\omega}\times\vec{r})$$

This becomes, to first order in $\vec{\omega}$

$$\psi_i'(\vec{r}) = \psi_i(\vec{r}-\vec{\omega}\times\vec{r}) - \epsilon_{ijk}\omega_k\psi_j(\vec{r})$$

now Taylor expanding the first term on the RHS

$$= \psi_i(\vec{r}) - (\vec{\omega}\times\vec{r})\cdot\vec{\nabla}_{\vec{r}}\,\psi_i(\vec{r})$$
$$- \epsilon_{ijk}\omega_k\psi_j(\vec{r})$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\delta_{ij} + \vec{S}_{ij}\right]\times$$
$$\times\psi_j(\vec{r})$$

where the "spin vector" $(\vec{S})_{ij}$ has components

$$(S^k)_{ij} \equiv \frac{\hbar}{i}\epsilon_{kij} = -\frac{\hbar}{i}(I^k)_{ij}$$

that is $\vec{S}_{ij} = -\frac{\hbar}{i}\vec{I}_{ij}$ where

the $\vec{I}_{ij}$ matrices were defined on page -615-616-.

$$\boxed{(I_k)_{ij} \equiv \epsilon_{ikj}}$$

$(\vec{I})_{ij}$ are 3 matrices

$$(\vec{I})_{ij} = (I^1)_{ij}\,\hat{i} + (I^2)_{ij}\,\hat{j} + (I^3)_{ij}\,\hat{k}.$$

So

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (= I_x)$$

$$I_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (= I_y)$$

$$I_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (= I_z)$$

Since ϵ_{ijk} obeys the Jacobi identity

$$0 = \overbrace{\epsilon_{ijk}\,\epsilon_{mnk}}^{\text{sum}} + \overbrace{\epsilon_{jnk}\,\epsilon_{mik}}^{\text{sum}} + \overbrace{\epsilon_{nik}\,\epsilon_{mjk}}^{\text{sum}}$$

fixed fixed fixed

Cyclic permutation of (i,j,n)

This is

$$0 = -\epsilon_{ijk}(\epsilon_{mkn}) + (\epsilon_{mik})(\epsilon_{kjn}) - (\epsilon_{mjk})(\epsilon_{kin})$$

Which is $\boxed{([I_i, I_j])_{mn} = \epsilon_{ijk}(I_k)_{mn}}$

So

$$[i\hbar I_i, i\hbar I_j] = i\hbar \epsilon_{ijk}(i\hbar I_k)$$

$$\Rightarrow \quad [S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

The angular momentum commutation relations are obeyed by the spin vector $(\vec{S})_{ij}$.

(Aside: Define matrix $\Theta_{ij} \equiv \hat{\theta} \cdot (\vec{I})_{ij}$

$$= \begin{bmatrix} 0 & -\hat{\theta}_3 & \hat{\theta}_2 \\ \hat{\theta}_3 & 0 & -\hat{\theta}_1 \\ -\hat{\theta}_2 & \hat{\theta}_1 & 0 \end{bmatrix}$$

One can show that

$$R_{ij}(\vec{\theta}) = \delta_{ij} + (\Theta^2)_{ij}(1-\cos\theta) + (\Theta)_{ij}\sin\theta$$

$$= \left(e^{\theta\Theta}\right)_{ij}$$)

Further we have $U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}$

So

$$\psi_i'(\vec{r}) = (\langle\vec{r}| \otimes \langle e_i|)|\psi'\rangle$$

$$= (\langle\vec{r}| \otimes \langle e_i|)\, e^{-\frac{i}{\hbar}\vec{\omega}\cdot\vec{J}}\, |\psi\rangle$$

$$= (\langle\vec{r}| \otimes \langle e_i|)\left[\mathbb{1} - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right]|\psi\rangle$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\Big(\langle\vec{r}|\vec{J}^{\,orbital}|\psi_i\rangle$$

$$+ \langle\vec{r}|\psi_j\rangle\langle e_i|\vec{J}^{\,spin}|e_j\rangle\Big)$$

Which equals

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\delta_{ij} + \vec{S}_{ij}\right)\psi_j(\vec{r})$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\vec{L}\,\delta_{ij} + \vec{S}_{ij}\right]\psi_j(\vec{r})$$

Thus on states with spin one the angular momentum operator $\vec{J}$ is represented by the sum of orbital angular momentum $\vec{L}$ and spin angular momentum $\vec{S}$,

$$\vec{J} = \vec{L} + \vec{S}$$

where the $\vec{L}$ acts only on the spatial variables, $\vec{L} = (\vec{r} \times \frac{\hbar}{i}\vec{\nabla}_r)$, as usual, while $\vec{S}$ acts only on the spin space variables

Thus

$$\psi_i'(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega} \cdot (\vec{L}\delta_{ij} + \vec{S}_{ij})\psi_j(\vec{r})$$

The $\vec{L}$ and $\vec{S}$ operators commute and, as we know, the

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

and we have shown that

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k \, .$$

The $\vec{L}$ and $\vec{S}$ operators obey the same algebra as the $\vec{J}$ operators, hence they generate the same group of operators.

We have that the eigenvalues of $\vec{L}^2$ are $\hbar^2 l(l+1)$; $l = 0, 1, 2, \dots$ and L_3 are $\hbar m$; $m = -l, -l+1, \dots, +l$

At the same time the commuting set of matrices $\vec{S}^2$ and S_3 which we can make out of S_i are given by

$(S^k)_{ij} = +i\hbar\,\epsilon_{ikj}$; thus

$(\vec{S}^2)_{ij} = (i\hbar)^2 \epsilon_{ik\ell}\epsilon_{\ell kj} = -\hbar^2(-2\delta_{ij})$

$= 2\hbar^2\delta_{ij} = 1(1+1)\hbar^2\delta_{ij}$

$(\vec{S}^2)_{ij} = \hbar^2 s(s+1)\delta_{ij}$ with $s=1$

the total spin eigenvalue is $s=1$

$(S^3)_{ij} = i\hbar\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$ and this can be

diagonalized to $(\tilde{S}^3)_{ij} = \hbar\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}_{ij}$

or finding the eigenvectors of $(S^3)_{ij}$ we have (in components)

$$e_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \; ; \; e_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \; ; \; e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So

$$(S^3)e_+ = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar e_+$$

$$(S^3)e_- = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar\begin{pmatrix} -1 \\ +i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = -\hbar e_-$$

$$(S^3)e_0 = 0\hbar\, e_0$$

Then the eigenvalues of $\frac{1}{\hbar}S^3$ are $+s, s-1, \dots, -s$ for $s=1$ this is $+1, 0, -1$.

The projection of the spin onto the z-axis has discrete values, just like the orbital angular momentum, ranging from $-s\hbar, \dots, +s\hbar$.

For finite transformations we can exponentiate the angular momentum operator to find

$$|\psi'\rangle = U(R(\vec{\theta}))|\psi\rangle = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}|\psi\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot(\vec{L}+\vec{S})}|\psi_i\rangle \otimes |e_i\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}|e_i\rangle$$

Now in general $e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}|e_i\rangle = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ji}|e_j\rangle$

and this matrix is defined as

$$D^{(s)}_{ij}(R(\vec{\theta})) \equiv \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ij}$$

For $s=1$ we have $\vec{S} = -i\hbar\vec{I}$ and

$$D^{(1)}_{ij}(R(\vec{\theta})) = R_{ij}(\vec{\theta})$$ the vector

representation matrix. We can ask if there are other matrices for different spin. Certainly we could put several spin 1 states together in a direct product to obtain higher integer spins like we did for higher rank tensor operators.
So by this we obtain all integer spin states. have we missed any states? As we shall show shortly the eigenvalue spectrum of $\vec{J}^2$ and J_z are $\hbar^2 j(j+1)$ and $-j, \ldots, +j$ respectively with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
There is also the possibility of odd-half integer spin. This suggests for $s=\frac{1}{2}$ we can find a 2x2 spin matrix $\vec{S}$. That is we desire a 2 component wavefunction that transforms under rotations as

$$\psi'_i(\vec{r}) = D^{(s=\frac{1}{2})}_{ij}(R(\vec{\theta}))\,\psi_j(R^{-1}(\vec{\theta})\vec{r})$$

where $i,j = 1,2$ now.

That is in state space we have

$$|\psi'\rangle = U(R(\vec{\theta}))|\psi\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes U^{spin}(R(\vec{\theta}))|e_i\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes \underbrace{\left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ji}}_{=D^{(1/2)}_{ji}(R(\vec{\theta}))}|e_j\rangle$$

Since $U(R(\vec{\theta}))$ is unitary we always have

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$$

$$\Rightarrow \quad D^{(s)*}_{ji}(R(\vec{\theta}))\,D^{(s)}_{jk}(R(\vec{\theta})) = \delta_{ik}$$

$\Rightarrow$ $D^{(s)}_{ij}(R(\vec{\theta}))$ are unitary matrices

$$D^{*(s)}_{ji}(R) = D^{-1(s)}_{ij}(R).$$

Further $\det D^{(s)}(R) = \pm 1$ and we only include ~~proper~~ proper Rotations so $\det D^{(s)}(R) = +1$ we have ~~that~~ since

$$D^{(s)}(R) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}$$

$$\det D^{(s)}(R(\vec{\theta})) = e^{\mathrm{Tr}\ln D^{(s)}(R(\vec{\theta}))}$$
$$= e^{\mathrm{Tr}\left(-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}\right)}$$

Since $\det D^{(s)} = 1 \Rightarrow \boxed{\mathrm{Tr}\,\vec{S} = 0}$

Further $D^{(s)}$ is unitary, so $\boxed{\vec{S} \text{ is Hermitian}}$

All 2×2 Hermitian matrices that are traceless can be written as linear combinations of the Pauli-matrices $\vec{\sigma}$

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(= \sigma_x\right)$$

$$\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \left(= \sigma_y\right)$$

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(= \sigma_z\right) .$$

The Pauli matrices have the properties that

$$\det M = \det \underbrace{S^{-1} M S}_{= M_{diag.}}$$

$$= \det \begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_N \end{pmatrix} = \prod_{i=1}^{N} m_i$$

$$= e^{\ln m_1} e^{\ln m_2} \cdots e^{\ln m_N} = e^{\sum_{i=1}^{N} \ln m_i}$$

$$= e^{\ln m_1 + \ln m_2 + \cdots + \ln m_N}$$

Now

$$\operatorname{Tr} \ln M_{diag.} = \operatorname{Tr} \ln [1 - (1 - M_{diag})]$$

$$= \operatorname{Tr} \left(\sum_{n=1}^{\infty} \frac{-1}{n} (1 - M_{diag})^n \right)$$

$$= \operatorname{Tr} \left[\sum_{n=1}^{\infty} \frac{-1}{n} \begin{pmatrix} 1-m_1 & & & 0 \\ & 1-m_2 & & \\ & & \ddots & \\ 0 & & & 1-m_N \end{pmatrix}^n \right]$$

$$= \operatorname{Tr} \left[\sum_{n=1}^{\infty} \frac{-1}{n} \begin{pmatrix} (1-m_1)^n & & & 0 \\ & (1-m_2)^n & & \\ & & \ddots & \\ 0 & & & (1-m_N)^n \end{pmatrix} \right]$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} \operatorname{Tr} \begin{pmatrix} (1-m_1)^n & & 0 \\ & \ddots & \\ 0 & & (1-m_N)^n \end{pmatrix}$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} \left[(1-m_1)^n + (1-m_2)^n + \cdots + (1-m_N)^n \right]$$

$$= \ln m_1 + \ln m_2 + \cdots + \ln m_3$$

$$\det M = e^{\mathrm{Tr}\ln M_{diag}}$$

but $\ln SS^{-1} = 0 = \ln S + \ln S^{-1}$

$$\det M = e^{\mathrm{Tr}\ln M_{diag} + \mathrm{Tr}\ln S + \mathrm{Tr}\ln S^{-1}}$$

$$= e^{\mathrm{Tr}\ln(S M_{diag} S^{-1})}$$

$$\boxed{\begin{aligned} &= e^{\mathrm{Tr}\ln M} \\ &= \det M \end{aligned}}$$

$$\det \sigma_i = -1$$

$$\mathrm{Tr}\, \sigma_i = 0$$

$$\left.\begin{array}{l} \sigma_i^2 = 1 \\ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \end{array}\right\} \sigma_i\sigma_j = \delta_{ij}1 + i\epsilon_{ijk}\sigma_k$$

Hence the matrices $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ obey the angular momentum algebra

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$$

so that $D^{(1/2)}$ and hence U^{spin} will obey the rotation group multiplication law for

$$D_{ij}^{(1/2)}(R(\vec{\theta})) = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ij} .$$

For infinitesimal transformations we find

$$\psi'_i(\vec{r}) = \left(1 - \frac{i}{\hbar}\vec{\omega}\cdot\vec{S}\right)_{ij}\psi_j(\vec{r}) + \psi_i(\vec{r} - \vec{\omega}\times\vec{r})$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\delta_{ij} + \vec{S}_{ij}\right]\psi_j(\vec{r})$$

As usual

$$\psi'_i(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\Big[\langle\vec{r}|\vec{J}^{\text{orbital}}|\psi_i\rangle + \langle\vec{r}|\psi_j\rangle\langle e_i|\vec{J}^{\text{spin}}|e_j\rangle\Big]$$

From above

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\big[\vec{L}\,\delta_{ij} + \vec{S}_{ij}\big]\psi_j(\vec{r})$$

Then $\vec{J} = \vec{L} + \vec{S}$ again where on the space of spin $\frac{1}{2}$ wavefunctions, called Spinor wavefunctions,

$$(\vec{S})_{ij} = \frac{\hbar}{2}\vec{\sigma}_{ij}$$, the Pauli-matrices $\vec{\sigma}$.

As before

$$(\vec{S}^2)_{ij} = \left(\frac{\hbar}{2}\right)^2(\vec{\sigma}^2)_{ij} = \frac{1}{4}\hbar^2\delta_{ij}\cdot 3$$
$$= \hbar^2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\delta_{ij} = \hbar^2 s(s+1)\delta_{ij}$$

with $s=\frac{1}{2}$; the total spin eigenvalue is $s=\frac{1}{2}$. Since the $S_i = \frac{\hbar}{2}\sigma_i$ do not commute, we can only simultaneously diagonalize one of them with $\vec{S}^2$; we choose $S_3 = \frac{\hbar}{2}\sigma_3$

$(S_3) = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ it has eigenvectors (in components) $e_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$ & $e_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$ so that

$$S_3 e_\uparrow = +\tfrac{1}{2}\hbar e_\uparrow \; ; \; S_3 e_\downarrow = -\tfrac{1}{2}\hbar e_\downarrow$$

$|e_\uparrow\rangle$ has spin $+\frac{1}{2}\hbar$ when projected onto the z-axis, it is said to have "spin up", while $|e_\downarrow\rangle$ has spin $-\frac{1}{2}\hbar$ when projected onto the z-axis, it is said to have "spin down".

The projected spin eigenvalues are $+s, \dots, s$ in this case $+\frac{1}{2}$ and $-\frac{1}{2}$.

The finite transformation matrix is

$$D^{(\frac{1}{2})}(R(\vec{\theta})) = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right).$$

$$= \left(e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\right)$$

$$= \cos\frac{\theta}{2}\,\mathbb{1} - i\hat{\theta}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$

Note that
$$D^{(\frac{1}{2})}(R(2\pi)) = \cos\frac{2\pi}{2} = -1$$

The representation $D^{(1/2)}$ is said to be double-valued $D^{(1/2)}(R(0)) = -D^{(1/2)}(R(2\pi))$

Thus on the spin $\frac{1}{2}$ states, the identity rotation, $\theta = 0$ or 2π, is represented by $+1$ or -1. Since $\pm|e_i\rangle$ is in the same Ray, it corresponds to the same physical state.

Of course we can imagine combining multiple spin $\frac{1}{2}$ states to obtain all integer and odd-half integer spin states. The spin-$\frac{1}{2}$ states form the fundamental representation of the rotation group $(O(3))$ $SU(2)$.

To be sure we have not missed any states, we next determine the eigenvalue spectrum of J^2, J_3 and construct the spin matrices $(S)_{ij}$ in their eigenvector basis.

Thus in general we have spin states s that transform as

$$\psi'_i(\vec{r}) \equiv D^{(s)}_{ij}(R(\vec{\theta}))\,\psi_j(R^{-1}(\vec{\theta})\vec{r})$$

which for infinitesimal rotations we write

$$\psi'_i(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\vec{L}\delta_{ij} + \vec{S}_{ij}\right]\psi_j(\vec{r})$$

where

$$D^{(s)}(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}.$$

The group multiplication law implies for $R(\vec{\theta}_3) = R(\vec{\theta}_2)R(\vec{\theta}_1)$ that

$$D^{(s)}(R(\vec{\theta}_3)) = D^{(s)}(R(\vec{\theta}_2))\,D^{(s)}(R(\vec{\theta}_1)),$$

that is the $D^{(s)}(R)$ matrices form a matrix representation of the SU(2) rotation group. Equivalently the group multiplication law implies the $\vec{S}$ matrices obey the SU(2) Lie algebra of rotations

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k.$$

We would now like to find all possible matrices $\vec{S}_{ij}$ and hence $D^{(s)}(R)$. We explicitly constructed the matrices for spin $0, \frac{1}{2}, 1$. To determine the general matrix structure for S we can turn to consideration of the eigenvalue determination of a set of commuting observables made from the $\vec{J}$ and whatever other operators may commute with them.

5.3.4. Angular Momentum Commutation Relations and the "Standard Basis"

The angular momentum commutation relations are

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k .$$

Hence $\vec{J}^2 = \vec{J}\cdot\vec{J}$ commutes with J_i since it is a scalar (the dot product of 2 vector operators) operator

$$[\vec{J}^2, J_i] = 0 \quad \text{for } i = 1,2,3 .$$