(The inverse transformation $R^{-1} = R^T$ gives the transformation in the active view.)

5.3.2. <u>The Angular momentum operators and their commutation relations.</u>

Isotropy of space

If the state of the system is given by $|\psi\rangle$ before the system and the measuring apparatus is rotated, it is given by $|\psi'\rangle$ after it is rotated by $R(\vec{\theta})$ where

$$|\psi'\rangle = U(R(\vec{\theta}))\,|\psi\rangle .$$

Since any proper rotation can be written as a square $R(\vec{\theta}) = R^2(\tfrac{1}{2}\vec{\theta})$ we have

$$U(R(\vec{\theta})) = U^2(R(\tfrac{1}{2}\vec{\theta})),$$

hence $U(R)$ is unitary because the product of two unitary or two anti-unitary operators is unitary. So

$$U^\dagger(R(\vec{\theta})) = U^{-1}(R(\vec{\theta})) .$$

Since equivalence of physical states only requires the vectors to be equal up to a phase (if they are normalized the same) we have that the unitary operators $U(R)$ obey the rotation group multiplication law only up to a phase

$$\vec{r} \xrightarrow{R(\vec{\theta}_1)} \vec{r}\,' \xrightarrow{R(\vec{\theta}_2)} \vec{r}\,''$$

$$R(\vec{\theta}_3) = R(\vec{\theta}_2) R(\vec{\theta}_1)$$

$$|\psi\rangle \xrightarrow{U(R(\vec{\theta}_1))} |\psi'\rangle \xrightarrow{U(R(\vec{\theta}_2))} e^{+i\alpha(\vec{\theta}_2,\vec{\theta}_1)} |\psi''\rangle$$

$$\to |\psi''\rangle$$

$$U(R(\vec{\theta}_3)) = e^{-i\alpha(\vec{\theta}_2,\vec{\theta}_1)} U(R(\vec{\theta}_2)) U(R(\vec{\theta}_1))$$

So

$$\boxed{U(R(\vec{\theta}_3)) = e^{-i\alpha(\vec{\theta}_2,\vec{\theta}_1)} U(R(\vec{\theta}_2)) U(R(\vec{\theta}_1))}$$

(By convention we choose $U(R(\vec{0}) = 1) = \mathbb{1}$ so

$$U(R(\vec{\theta})) = e^{-i\alpha(\vec{\theta},\vec{0})} U(R(\vec{\theta})) U(1)$$

$$\Rightarrow \alpha(\vec{\theta},0) = 0 .$$

And since $U(R)$ is unitary

$$U(R)\,U^{\dagger}(R) = \mathbb{1} \Rightarrow$$

$$\alpha(R(\vec{\theta}), R^{-1}(\vec{\theta})) = 0 = \alpha(\vec{\theta}, -\vec{\theta}),$$

Assuming that U depends continuously on $\vec{\theta}$ so that

$$U(1+\omega) \to \mathbb{1} \text{ as } \omega_{ij} \to 0$$

we have

$$\mathbb{1} = U^{\dagger}U = [1+(U-1)]^{\dagger}[1+(U-1)]$$

$$= \mathbb{1} + (U-\mathbb{1}) + (U-\mathbb{1})^{\dagger} \text{ to first order}$$

$$\Rightarrow U-\mathbb{1} = i(\text{Hermitian operator})$$

So $U(1+\omega) = \mathbb{1} + \frac{i}{2}\omega_{ij}\frac{J_{ij}}{\hbar}$

Since $\omega_{ij} = -\omega_{ji}$ we have $J_{ij} = -J_{ji}$ and $J_{ij} = J_{ij}^{\dagger}$. The J_{ij} are 3-independent Hermitian operators, the generators of spatial rotations.

Now recall that $\omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$
and inverse

$$\omega_{ij} = \epsilon_{ijk}\,\omega_k$$

$$\Rightarrow \quad \boxed{\omega_{ij} = -\omega_{ji}}$$

anti-symmetric 3x3 real matrix
$\Leftrightarrow$ vector in 3 space

Hence we have $J_{ij} = -J_{ji}$

and J_{ij} is Hermitian $J_{ij} = J_{ij}^\dagger$
as an operator in Hilbert space.

The J_{ij} are 3-independent Hermitian operators

They are the generators of spatial rotations in Hilbert space.

Note: $J_{ij} = -J_{ji} \Rightarrow J_i \equiv \frac{1}{2}\epsilon_{ijk}J_{jk}$

$$\left(\begin{array}{ll} J_1 \equiv J_{23} & x \\ J_2 \equiv J_{31} & y \\ J_3 \equiv J_{12} & z \end{array}\right)$$

Rotation about

invert:

$$J_{ij} = \epsilon_{ijk}J_k.$$

Since $R(\vec{\theta}_3) = R(\vec{\theta}_2)R(\vec{\theta}_1)$ according to the rotation group multiplication law we have that ~~the~~ unitary operators represent this group up to a phase. As we have seen $\vec{\theta}_3$ is determined by $\vec{\theta}_1$ and $\vec{\theta}_2$, as well $U(R(\vec{\theta}_3))$ is determined by $U(R(\vec{\theta}_1))$ and $U(R(\vec{\theta}_2))$ (up to a phase).

This group multiplication law will imply algebraic relations (commutation relations) for the generators J_{ij}. Consider again the group multiplication law as embodied in the successive transformations

$$\begin{tikzcd} \vec{r} \arrow[r, "R^{-1}(\vec{\theta})"] \arrow[d, "(1+\omega')"'] & \vec{r}\,' \arrow[r, "(1+\omega)"] & \vec{r}\,'' \arrow[dll, bend left, "R(\vec{\theta})"] \\ \vec{r}\,''' & & \end{tikzcd}$$

The group multiplication law

$$R(\vec{\omega}\,') = R(\vec{\theta})R(\vec{\omega})R^{-1}(\vec{\theta})$$

$$\Rightarrow \quad \omega'_{ij} = (R(\vec{\theta})\,\omega\, R^{-1}(\vec{\theta}))_{ij}$$

Quantum mechanically we have that

$$\begin{tikzcd} |\psi\rangle \arrow[r, "U(R^{-1})"] \arrow[d, "U(1+\omega')"'] & |\psi'\rangle \arrow[r, "U(1+\omega)"] & |\psi''\rangle \arrow[dll, bend left, "U(R)"] \\ |\psi'''\rangle = e^{+i\alpha(\omega,\vec{\theta})}|\psi'''\rangle & & \end{tikzcd}$$

Thus

$$\boxed{\begin{aligned}&\mathcal{U}(\vec{R}(\vec{\theta}))\,\mathcal{U}(1+\omega)\,\mathcal{U}(R^{-1}(\vec{\theta}))\\&\quad = e^{i\hat{\alpha}(\omega,\vec{\theta})}\,\mathcal{U}(1+\omega')\end{aligned}}$$

Now $\mathcal{U}(R^{-1}) = \mathcal{U}^{-1}(R)$ and

$\vec{\theta}$ for Rotation $R+R\omega$

$$\hat{\alpha}(\omega,\vec{\theta}) = \alpha(\vec{\theta},\omega) + \alpha(\vec{\theta}(R+R\omega), -\vec{\theta})$$

Note: $\hat{\alpha}(0,\vec{\theta}) = \underbrace{\alpha(\vec{\theta},0)}_{=0} + \underbrace{\alpha(\vec{\theta},-\vec{\theta})}_{=0} = 0$

hence we can taylor expand $\hat{\alpha}(\omega,\vec{\theta})$

$$\hat{\alpha}(\omega,\vec{\theta}) = \tfrac{1}{2}\omega_{ij} f_{ij}(\vec{\theta}) \quad \text{with } f_{ij} = -f_{ji}.$$

So $\boxed{e^{i\hat{\alpha}(\omega,\vec{\theta})} = 1 + \tfrac{i}{2}\omega_{ij} f_{ij}(\vec{\theta})}$

Using that $\omega' = R(\vec{\theta})\,\omega\,R^{-1}(\vec{\theta})$, we have

$$\mathcal{U}(1+\omega') = \mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\left(R(\vec{\theta})\,\omega\,R^{-1}(\vec{\theta})\right)_{kl} J_{kl}$$

and $\mathcal{U}(1+\omega) = \mathbb{1} + \frac{i}{2}\frac{\omega_{ij}}{\hbar} J_{ij}$

Thus the group multiplication law becomes

$$U(R(\vec{\theta}))\left[\mathbb{1} + \frac{i}{2}\frac{\omega_{ij}}{\hbar} J_{ij}\right] U^{-1}(R(\vec{\theta}))$$

$$= \left(1 + \frac{i}{2}\omega_{ij} f_{ij}(\vec{\theta})\right)\left(\mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\left(R(\vec{\theta})\omega R^{-1}(\vec{\theta})\right)_{kl} J_{kl}\right)$$

$\Rightarrow$

$$\frac{i}{2}\frac{\omega_{ij}}{\hbar}\, U(R(\vec{\theta}))\, J_{ij}\, U^{-1}(R(\vec{\theta}))$$

$$= \frac{i}{2}\frac{\omega_{ij}}{\hbar}\left[R_{ki}(\vec{\theta}) R^{-1}_{jl}(\vec{\theta}) J_{kl} + \hbar f_{ij}(\vec{\theta})\right]$$

Since ω_{ij} is arbitrary $\Rightarrow$

$$\boxed{U(R(\vec{\theta}))\, J_{ij}\, U^{-1}(R(\vec{\theta})) = R_{ki}(\vec{\theta}) R^{-1}_{jl}(\vec{\theta})\, J_{kl} + \hbar f_{ij}(\vec{\theta})}$$

Now $\vec{\theta}$ is completely arbitrary; choose it to be infinitesimal; Let $\vec{\theta} = \vec{\omega}$ again expanding we have

$$R_{ij}(\vec{\omega}) = \delta_{ij} + \omega_{ij} \quad ; R^{-1}_{ij}(\vec{\omega}) = \delta_{ij} - \omega_{ij}$$

$$f_{ij}(\vec{\omega}) = \tfrac{1}{2}\omega_{mn} f_{ijmn} \quad \text{with}$$

$$f_{ijmn} = -f_{ijnm} = -f_{jimn}$$

So

$$\left(\mathbb{1} + \frac{i}{2}\frac{\omega_{mn}}{\hbar} J_{mn}\right) J_{ij} \left(\mathbb{1} - \frac{i}{2}\frac{\omega_{mn}}{2} J_{mn}\right)$$

$$= J_{kl}(\delta_{ki} + \omega_{ki})(\delta_{jl} - \omega_{jl}) + \tfrac{1}{2}\omega_{mn}\hbar f_{ijmn}$$

$\Rightarrow$

$$\frac{i}{2}\frac{\omega_{mn}}{\hbar}[J_{mn}, J_{ij}]$$

$$= \omega_{mn}\left[\delta_{in} J_{mj} + \delta_{jn} J_{im} + \frac{\hbar}{2} f_{ijmn}\right]$$

Since $\omega_{mn} = -\omega_{nm}$, we anti-symmetrize the first term on the RHS

$$\frac{i}{2}\frac{\omega_{mn}}{\hbar}[J_{mn}, J_{ij}]$$
$$= \frac{\omega_{mn}}{2}[\delta_{in}J_{mj} - \delta_{im}J_{nj} + \delta_{jn}J_{im} - \delta_{jm}J_{in} + \hbar f_{ijmn}]$$

equating coefficients of $\frac{\omega_{mn}}{2}$ $\Rightarrow$

$$\boxed{[J_{mn}, J_{ij}] = -i\hbar[\delta_{in}J_{mj} - \delta_{im}J_{nj} + \delta_{jn}J_{im} - \delta_{jm}J_{in}] - i\hbar^2 f_{ijmn}}$$

Since $J_{ij} = -J_{ji}$ define the 3 independent operators

$$J_1 \equiv J_{23}$$

$$J_2 \equiv J_{31}$$

$$J_3 \equiv J_{12} \quad ,$$

that is

$J_i \equiv \frac{1}{2}\epsilon_{ijk} J_{jk}$. Inverting

yields $J_{ij} = \epsilon_{ijk} J_k$.

So the commutation relation becomes

$$\frac{1}{4}\epsilon_{kmn}\epsilon_{lij}[J_{mn}, J_{ij}] = [J_k, J_l]$$

$$= -\frac{i\hbar}{4}\Big[\epsilon_{kmi}\epsilon_{lij} J_{mj} - \epsilon_{kin}\epsilon_{lij} J_{nj}$$

$$+ \epsilon_{kmj}\epsilon_{lij} J_{im} - \epsilon_{kjn}\epsilon_{lij} J_{in}\Big]$$

$$- \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

So relabelling dummy indices, we have

$$[J_k, J_l] \stackrel{?}{=} -\frac{i\hbar}{4}\Big[-\epsilon_{kmi}\epsilon_{lji} J_{mj} - \epsilon_{kmi}\epsilon_{lji} J_{mj} + \epsilon_{kmi}\epsilon_{lji} J_{jm} + \epsilon_{kmi}\epsilon_{lji} J_{jm}\Big]$$

$$- \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

$$= +i\hbar\, \epsilon_{kmi}\epsilon_{lji} J_{mj} - \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

$$= i\hbar(\delta_{kl}\delta_{mj} - \delta_{kj}\delta_{ml}) J_{mj} - \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

$$= i\hbar(\delta_{kl}\cancel{J_{mm}}^{\,0} - J_{lk}) - \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

$$= i\hbar\, \epsilon_{klm} J_m - \frac{i\hbar^2}{4}\epsilon_{kmn}\epsilon_{lij} f_{ijmn}$$

Now define

$$\epsilon_{kmn}\epsilon_{lij} f_{ijmn} \equiv \tilde{f}_{kl} = -\tilde{f}_{lk}$$

So
$$\epsilon_{kmi}\epsilon_{lji}\tilde{f}_{mj} = -\tilde{f}_{lk} = \tilde{f}_{kl}$$
Thus we finally define
$$F_m = \frac{1}{2}\epsilon_{mkl}\left(-\frac{\hbar}{4}\tilde{f}_{kl}\right).$$
inverting we have
$$-\frac{\hbar}{4}\tilde{f}_{kl} = \epsilon_{klm}F_m$$
So
$$\boxed{[J_k, J_l] = i\hbar\epsilon_{klm}(J_m + F_m)}$$
with
$$F_m = \frac{1}{2}\epsilon_{mkl}\left(-\frac{\hbar}{4}\tilde{f}_{kl}\right)$$
$$= -\frac{\hbar}{8}\epsilon_{mkl}\epsilon_{kpn}\epsilon_{lij}f_{ijpn}$$
$$= +\frac{\hbar}{8}(\delta_{mp}\delta_{ln} - \delta_{mn}\delta_{lp})\epsilon_{lij}f_{ijpn}$$
$$= \frac{\hbar}{8}(\epsilon_{lij}f_{ijml} - \epsilon_{lij}f_{ijlm})$$
$$\boxed{F_m = -\frac{\hbar}{4}\epsilon_{ijl}f_{ijlm}}$$

Thus we see that there is nothing preventing us from re-defining the generators of Rotations by absorbing the c-number phase factor. Thus

$$J_i' \equiv J_i + F_i$$

J_i' is still Hermitian $J_i' = J_i'^{\dagger}$

and since F_i is a c-number the commutator structure remains in tact

$$[J_i', J_j'] = i\hbar \epsilon_{ijk} J_k' .$$

Hence we can always choose the definition of $\vec{J}$ (the phase of U) so that the phase factor in the group multiplication law vanishes. Thus, dropping the primes, we have that

$$U(R(\vec{\theta}_3)) = U(R(\vec{\theta}_2))\, U(R(\vec{\theta}_1))$$

and for infinitesimal rotations $R = 1 + \omega$

$$U(1+\omega) = \mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\,\omega_{ij} J_{ij}$$

but $J_{ij} = \epsilon_{ijk} J_k$ and $\omega_k = \frac{-1}{2}\epsilon_{kij}\omega_{ij}$

(i.e. p.-620- $\omega_{ij} = -\epsilon_{ijk}\omega_k$) so

$$\tfrac{1}{2}\omega_{ij} J_{ij} = \tfrac{1}{2}\omega_{ij}\epsilon_{ijk} J_k = -\omega_k J_k$$
$$= -\vec{\omega}\cdot\vec{J}$$

So

$$\boxed{U(1+\omega) = \mathbb{1} - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}}$$

with the generators of rotations $\vec{J}$ obeying the rotation, O(3) algebra

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k,$$

corresponding to the (true) representation of the group O(3) multiplication law.

Since $[\vec{\theta}\cdot\vec{J}, \vec{\theta}\cdot\vec{J}] = 0$, we can build up any finite rotation transformation corresponding to the rotation $R(\vec{\theta})$ by successive infinitesimal rotations about the same axis $\hat{\theta}$ thru angle θ/n that is

For finite rotations $R(\vec{\theta})$ we have

$$U(R(\vec{\theta})) = \lim_{n\to\infty}\left[U\left(R\left(\frac{\vec{\theta}}{n}\right)\right)\right]^n$$

$$= \lim_{n\to\infty}\left[\mathbb{1} - \frac{i}{\hbar}\frac{1}{n}\vec{\theta}\cdot\vec{J}\right]^n$$

$$\boxed{U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}}\ .$$

Of course since $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$

$$e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}} \neq e^{-\frac{i}{\hbar}\theta_x J_x}\, e^{-\frac{i}{\hbar}\theta_y J_y}\, e^{-\frac{i}{\hbar}\theta_z J_z}\ .$$

In passing we note that since an arbitrary rotation may also be specified by its Euler angles, and the final rotation can be built up successively by rotations about the z, ξ, z' axes we have

$$U(R(\theta,\varphi,\psi)) = U(R(\psi\hat{e}_{z'}))\,U(R(\theta\hat{e}_{\xi}))\,U(R(\varphi\hat{e}_z))$$

$$= e^{-\frac{i}{\hbar}\psi J_{z'}}\, e^{-\frac{i}{\hbar}\theta J_{\xi}}\, e^{-\frac{i}{\hbar}\varphi J_z}\ .$$

Since $\vec{J}$ is an operator it transforms according to

$$\vec{J}' = U(R(\vec{\theta}))\,\vec{J}\,U^{-1}(R(\vec{\theta}))$$

hence we can relate J_ξ to J_x since the ξ-axis was obtained from the x-axis by rotating about the z-axis by angle φ

$$J_\xi = e^{-\frac{i}{\hbar}\varphi J_z}\,J_x\,e^{+\frac{i}{\hbar}\varphi J_z}.$$

Consequently

$$e^{-\frac{i}{\hbar}\theta J_\xi} = e^{-\frac{i}{\hbar}\varphi J_z}\,e^{-\frac{i}{\hbar}\theta J_x}\,e^{+\frac{i}{\hbar}\varphi J_z}.$$

Similarly $J_{z'}$ is related to J_z by a Rotation about the ξ-axis through angle θ

$$J_{z'} = e^{-\frac{i}{\hbar}\theta J_\xi}\,J_z\,e^{+\frac{i}{\hbar}\theta J_\xi}$$

using the above Relation for J_ξ this becomes

$$J_{z'} = e^{-\frac{i}{\hbar}\varphi J_z} e^{-\frac{i}{\hbar}\theta J_x} e^{+\frac{i}{\hbar}\varphi J_z} \times J_z$$
$$\times e^{-\frac{i}{\hbar}\varphi J_z} e^{+\frac{i}{\hbar}\theta J_x} e^{+\frac{i}{\hbar}\varphi J_z}$$

Since $[J_z, J_z] = 0$ this is

$$J_{z'} = e^{-\frac{i}{\hbar}\varphi J_z} e^{-\frac{i}{\hbar}\theta J_x} J_z \, e^{+\frac{i}{\hbar}\theta J_x} e^{+\frac{i}{\hbar}\varphi J_z}$$

Hence

$$e^{-\frac{i}{\hbar}\psi J_{z'}} = e^{-\frac{i}{\hbar}\varphi J_z} e^{-\frac{i}{\hbar}\theta J_x} e^{-\frac{i}{\hbar}\psi J_z} e^{+\frac{i}{\hbar}\theta J_x} e^{+\frac{i}{\hbar}\varphi J_z}$$

Putting this together, we find

$$U(R(\theta,\varphi,\psi)) = \Big(e^{-\frac{i}{\hbar}\varphi J_z} e^{-\frac{i}{\hbar}\theta J_x} e^{-\frac{i}{\hbar}\psi J_z} \times$$
$$\times e^{+\frac{i}{\hbar}\theta J_x} \underbrace{e^{+\frac{i}{\hbar}\varphi J_z}\Big)\Big(e^{-\frac{i}{\hbar}\varphi J_z}}_{=\mathbb{1}} e^{-\frac{i}{\hbar}\theta J_x} e^{+\frac{i}{\hbar}\varphi J_z}\Big) \times$$
$$\times \Big(e^{-\frac{i}{\hbar}\varphi J_z}\Big)$$

Note: If H is Rotationally invariant:

$$H' = U(R(\vec{\theta}))\, H\, U^{\dagger}(R(\vec{\theta})) = H$$

$$\Leftrightarrow \quad [\vec{J}, H] = 0$$

Then Ehrenfest's Thm. $\Rightarrow$

$$\boxed{\frac{d}{dt}\langle \vec{J} \rangle = 0} \qquad \frac{d}{dt}\langle \vec{J}\rangle = \frac{i}{\hbar}\langle \underbrace{[H, \vec{J}]}_{0} \rangle + \langle \underbrace{\cancel{\frac{\partial \vec{J}}{\partial t}}}_{0} \rangle$$

$\langle \vec{J} \rangle$ is constant in time hence

$\langle$ total $\measuredangle$ momentum $\rangle$ is conserved.

We have

Homogeneity of Time + Time translation Invariance (H time indep.)	$\Leftrightarrow$	$\langle \underbrace{\text{Energy}}_{H} \rangle$ is conserved
Homogeneity of Space + Space Translation Invariance ($[\vec{P}, H] = 0$)	$\Leftrightarrow$	$\langle$ Total $\underbrace{\text{Linear Momentum}}_{\vec{P}}$ $\rangle$ is conserved
Isotropy of Space + Space Rotation Invariance ($[\vec{J}, H] = 0$)	$\Leftrightarrow$	$\langle$ Total $\underbrace{\measuredangle \text{ momentum}}_{\vec{J}}$ $\rangle$ is conserved

except for the first 3 exponentials, all the others collapse to the identity; so we find

$$\boxed{U(R(\theta,\varphi,\psi)) = e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_x}\, e^{-\frac{i}{\hbar}\psi J_z}}$$

5.3.3. Physical description of Spin

Under rotation the position vector $\vec{r}$ becomes $R\vec{r} = \vec{r}'$ that is if the components of the position vector $\vec{r}$ in a Cartesian coordinate system are x_i before the rotation the position vector, $\vec{r}'$, after rotation has components in this coordinate system given by $x_i' = R_{ij}\,x_j$ and we write $\vec{r}' = R\vec{r}$ as short hand for this. Hence the quantum mechanical position eigenstates $|\vec{r}\rangle$ and $|\vec{r}' = R\vec{r}\rangle$ are related by the unitary operator $U(R(\vec{\theta}))$ representing the rotation $R(\vec{\theta})$

$$|\vec{r}'\rangle = U(R(\vec{\theta}))\,|\vec{r}\rangle = |R(\vec{\theta})\vec{r}\rangle$$