5.201. The Principle of Galilean Relativity and The Galilean Group

We now consider the transformation of Newtonian relativity, the transformation which is appropriate for non-relativistic physics. These can be shown to be the $c \to \infty$ limit of the Poincaré transformations which are the transformations of special relativistic physics. We require the physics (Hamilton probability) to be invariant under transformation of the coordinates corresponding to space and time translations and boosts.

Consider two observers $O$ and $O'$ using inertial frames of reference in which they label the same space-time event by $(F, t)$ and $(F', t')$, respectively. Galilean invariance means that the laws of physics have the same form in both $O$ and $O'$, provided their coordinates are related by
The linear transformations (Galilean transformations)

\[ X' = R_{ij} x_j + V_i t + a_i \]
\[ t' = t + b \]

The parameters \( a_i, b, V_i, R_{ij} \) are real numbers such that \( R_{ij} \) are the elements of a \( 3 \times 3 \) real orthogonal matrix, \( R^T = R^{-1} \).

Any \( 3 \times 3 \) real orthogonal matrix has only 3 independent matrix elements, and so can be parameterized by 3 real numbers. For instance, they form angles. Hence we have that

- \( a_i \): labels space translations = 3 real parameters
- \( b \): labels time translations (origin or zero of time) = 1 real parameter
- \( V_i \): labels velocity boosts = 3 real parameters
- \( R_{ij} \): labels spatial rotations = 3 real parameters
Hence we have that the most general Galilean transformation of coordinates is specified by 10 real parameters. Thus specifying $E, R, V, \alpha, \beta, \gamma$ completely specifies which Galilean transformation is being considered. We will label the Galilean transformation by $E, R, V, \alpha, \beta, \gamma$.

The set of Galilean transformations form a group, naturally called the Galilean group. To be a group we must find the composition laws for Galilean transformations. That is, if we make 2 such transformations this should be a single Galilean transformation with its 10 parameters specified in terms of the 10 parameters of the first 2 Galilean transformations. So consider two consecutive Galilean transformations from 0 to $O'$ then

$0 \to O' : x' = R_i^j x_j + V_i t + a_i$

$t' = t + b$
$0' \rightarrow 0''$:

\[ x_i'' = \overline{R}_{ij} x_j' + \overline{V}_i t' + \overline{a}_i. \]

\[ t'' = t' + b. \]

Substituting for \((x'_i, t')\) in terms of \((x_i, t)\) yields

\[ x_i'' = \overline{R}_{ij} (\overline{R}_{jk} x_k + V_j t + a_j) \]

\[ + \overline{V}_i (t + b) + \overline{a}_i. \]

So

\[ x_i'' = (\overline{R}R)_{ij} x_j + (\overline{R}_{ij} V_j + \overline{V}_i) t \]

\[ + (\overline{R}_{ij} a_j + V_i b + \overline{a}_i). \]

and

\[ t'' = t + (b + b). \]
But we recognize this as another Galilean transformation directly from \( O \rightarrow O'' \); that is
\[
x_i'' = \bar{R}_{ij} x_j + \bar{V}_i t + \bar{a}_i
\]
\[
t'' = t + \bar{b}
\]
where this transformation \( \bar{R}, \bar{V}, \bar{a}, \bar{b} \)
is specified by
\[
\bar{R} = R R
\]
\[
\bar{V}_i = \bar{R}_{ij} V_j + \bar{V}_i
\]
\[
\bar{a}_i = \bar{R}_{ij} a_j + \bar{V}_i b + \bar{a}_i
\]
\[
\bar{b} = b + \bar{b}
\]
(note \((\bar{R} R)^T = (R^T \bar{R}^T) = (\bar{R}^T R^T) = \bar{R}^T R^T\) is orthogonal, as required).

Hence we have the composition law for the Galilean group.
Denoting the Galilean transformation by 
\[ E \mathbf{R}, \mathbf{V}, \mathbf{a}, b^3 \] we have the product rule:

\[ E \mathbf{R}, \mathbf{V}, \mathbf{a}, b^3 \cdot E \mathbf{R}, \mathbf{V}, \mathbf{a}, b^3 \]

\[ = E \mathbf{R}, \mathbf{R} \mathbf{V} + \mathbf{V}, \mathbf{R} \mathbf{a} + \mathbf{V} b + \mathbf{a}, b + b^3. \]

To show that the set of Galilean transformations forms a group we further must show:

2) The identity element is \[ E 1, \mathbf{0}, \mathbf{0}, 0^3 \]

since \[ E 1, \mathbf{0}, \mathbf{0}, 0^3 \cdot E \mathbf{R}, \mathbf{V}, \mathbf{a}, b^3 \]

\[ = E \mathbf{R}, \mathbf{V} + \mathbf{0}, \mathbf{a} + \mathbf{0} b + \mathbf{0}, 0 + b^3 \]

\[ = E \mathbf{R}, \mathbf{V}, \mathbf{a}, b^3. \]

The identity transformation in the coordinates is just that

\[ x^i = x^i \]

\[ t' = t \].
The inverse element to $\mathcal{E}(\hat{R}, \hat{V}, \hat{a}, b)$ is
\[ \mathcal{E}(\hat{R}^{-1}, -\hat{R}^{-1} \hat{V}, -\hat{R}^{-1} \hat{a} + \hat{R}^{-1} \hat{V} b, -b^2) \]

Since
\[ \mathcal{E}(\hat{R}, \hat{V}, \hat{a}, b^2) = \mathcal{E}(\hat{R}^{-1}, R^{-1} V - R^{-1} \hat{V}, R^{-1} \hat{a} - R^{-1} \hat{V} b, -b + b^2) \]
\[ = \mathcal{E}(1, 0, 0, 0, 0) \text{ the identity element.} \]

The inverse transformation on the coordinates yields (the transformation from $0 \to 0$)
\[ x_i = (R^{-1})_{ij} x'_j + (-R^{-1} \hat{V})_{ij} e_j' \]
\[ + (-R^{-1} \hat{a} + R^{-1} \hat{V} b)_i \]
\[ = (R^{-1})_{ij} (x'_j - V_j (e'_j - b) - a_j) \]
\[ t = e'_j - b \]
while the transformation from $O \rightarrow O'$ is
\[ x'_i = R_{ij} x_j + V_i t + a_i \]
\[ t' = t + b. \]

4) Since matrix multiplication is associative, we have that the Galilean transformations obey the associative law of multiplication. Hence they form a group, the Galilean group.

We will next show that the Galilean group is represented by unitary operators $U(R, \vec{V}, \vec{a}, b)$ on the Hilbert space of states. Let $\mathbf{E} \mathbf{R} \mathbf{V} \mathbf{a}, b$ be an arbitrary element of the Galilean group which takes a particle of mass $m$ at rest and gives the particle a velocity $\vec{V}'$, pictorially,
\[ \mathbf{E} \mathbf{R} \mathbf{V} \mathbf{a}, b \] (particle at rest) = (particle with velocity $\vec{V}'$).

Let the pure velocity boost $[1, \vec{V}, 0, 0, 0, 0]$ be such that its inverse brings the
particle back to rest

\[ \xi, \bar{\nu}, \bar{0}, 0, 0, 3 \overset{-5 \leq \bar{\nu} \leq 0}{\rightarrow} (\text{particle with velocity}) = (\text{particle at rest}) \]

\[ = \xi, -\bar{\nu}, \bar{0}, 0, 0, 3 \]

Hence

\[ \xi, -\bar{\nu}, \bar{0}, 0, 0, 3 \in \mathbb{R}, \bar{V}, \bar{a}, \bar{b}, 3 \]

Since Galilean transformations form a group

\[ \xi, -\bar{\nu}, \bar{0}, 0, 0, 3 \in \mathbb{R}, \bar{V}, \bar{a}, \bar{b}, 3 \] is a Galilean transformation which leaves the particle at rest.

The most general second Galilean transformation is a product of a rotation and a translation

\[ \xi, -\bar{\nu}, \bar{0}, 0, 0, 3 \overset{\mathbb{R}, \bar{V}, \bar{a}, \bar{b}, 3}{\rightarrow} \]

\[ \overset{\text{space rotation}, \text{space translation}, \text{time translation}}{\rightarrow} \]

\[ \xi, \bar{0}, \bar{0}, 0, 0, 3 \overset{1, \bar{0}, \bar{a}, 0, 0, 0, 3 \xi}{\rightarrow} \bar{0}, \bar{a}, 0, 0, 0, 3 \]

\[ \overset{\text{space rotation}, \text{space translation}, \text{time translation}}{\rightarrow} \]
Thus we have an arbitrary Galilean transformation can be written as the product of a pure velocity boost, a pure spatial rotation, a pure spatial translation and a pure time translation.

\[ \{ R, V, a, b \} = \{ 1, \vec{V}, 0, 0 \} R \{ 0, \vec{0}, 0, 0 \} \{ 1, a, 0, 0 \} \{ 0, 1, 0, b \} \]

Arbitrary Galilean Transformation

pure velocity boost

spatial rotation

space translation

time translation

Hence the quantum mechanical operator \( U(R, \vec{V}, a, b) \) representing the Galilean transformation in the Hilbert space can be written as

\[ U(R, \vec{V}, a, b) = \omega U(1, \vec{V}, 0, 0) U(R, \vec{0}, 0, 0) U(1, a, 0, 0) \times U(0, 1, 0, b) \]

Arbitrary Galilean transformation

pure velocity boost

spatial rotation

space translation

time translation

arbitrary phase factor \( \omega \) = 1.
Since any pure boost, rotation, space or time translation can be written as the square of a pure boost, rotation, space or time translation, we have that

\[ U(1, \overline{v}, \overline{0}, 0) = U^2(1, \overline{v}, \overline{0}, 0) \]

\[ U(R', \overline{0}, \overline{0}, 0) = U^2(R', \overline{0}, \overline{0}, 0) \]

\[ U(1, \overline{0}, \overline{a}, 0) = U^2(1, \overline{0}, \overline{a}, 0) \]

\[ U(1, \overline{0}, \overline{0}, \overline{b}) = U^2(1, \overline{0}, \overline{0}, \overline{b}) \]

Now Wigner's Theorem states that

\[ U(1, \overline{0}, \overline{0}, 0), U(R', 0, 0, 0), U(1, 0, \overline{a}, 0) \]

and \( U(1, \overline{0}, \overline{0}, \overline{b}) \) are either unitary or anti-unitary. But the square of either a unitary or an anti-unitary operator is unitary. Hence the square velocity boosts, rotations and space and time translations
operators are unitary operators.

Since the product of unitary operators is again unitary, an arbitrary Galilean transformation is represented by an unitary operator in Hilbert space i.e.

\[ U(\mathbf{R}, \mathbf{v}, \mathbf{a}, b) \text{ is unitary.} \]

Note that this same type of argument can be applied to the operator representing any continuous symmetry transformation to show that it is unitary.

Next we will determine the commutation relations that the generators of the Galilean group obey by studying the group product law. Here we will be careful about the arbitrary phase factors that result from the fact that \( \exp(i \mathbf{a} \cdot \mathbf{r}) \) and \( \exp(i \mathbf{v} \cdot \mathbf{r}) \) represent the same physical state. So if we have 2 consecutive Galilean transformations \( 0 \xrightarrow{\mathbf{R}, \mathbf{v}, \mathbf{a}, b_1} 0' \xrightarrow{\mathbf{R}, \mathbf{v}, \mathbf{a}, b_2} 0'' \), its effect is equivalent up to a phase.
to one transformation from $O \xrightarrow{\mathbf{E}_{\overline{R}, \overline{V}, \overline{a}, t^3}} O^{\overline{11}}$

with

$\mathbf{E}_{\overline{R}, \overline{V}, \overline{a}, t^3} = \mathbf{E}_{\overline{R}, \overline{V}, \overline{a}, t^3} \mathbf{E}_{\overline{R}, \overline{V}, \overline{a}, b^3}$

$= \mathbf{E}_{\overline{R}, \overline{V} + \overline{V}, \overline{R} \overline{a} + \overline{V} \overline{b} + \overline{a}, t + b^3}$.

That is, the state vectors are equal up to a phase

$|14\rangle = e^{i\omega} U(R, \overline{V}, \overline{a}, b) |14\rangle$

$|14''\rangle = e^{i\omega} U(R, \overline{V}, \overline{a}, b) |14'\rangle$

but

$|14''\rangle = e^{i\omega} U(R, \overline{V}, \overline{a}, b) |14\rangle$

hence the $U$s form a representation of the group multiplication law up to a phase

$U(R, \overline{V}, \overline{a}, b) U(R, \overline{V}, \overline{a}, b)$

$= e^{i\omega} U(R, \overline{V}, \overline{a}, b)$
Next, we consider elements of the Galois group which differ only infinitesimally from the identity. Then, we have\\

\[ \text{Further, since this uniting, we have} \]

\[ 1 = e^{i\alpha(R,\alpha, b; \bar{R}, -R, -R, R, -R, -R, R, R, b, -b)} \]

In particular, we choose \( \alpha = 0 \) so that

\[ x = x(R, \alpha, b; R, \alpha, b) \]

with the phase factors.
\[(R^{-1})_{ij} = \delta_{ij} - W_{ij} = (R^T)_{ij} = \delta_{ij} + W_{ij}\]
\[\Rightarrow W_{ij} = -W_{ji}\]

The \(W_{ij}\) matrix is an anti-symmetric \(3 \times 3\) matrix of real parameters. This implies \(W_{ij}\) has only 3 independent real elements.

\[
W_{ij} = \begin{pmatrix}
0 & W_{12} & W_{13} \\
-W_{12} & 0 & W_{23} \\
-W_{13} & -W_{23} & 0
\end{pmatrix}
\]

So again we have 10 independent infinitesimal real parameters specifying the Galilean transformation.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Infinitesimal Parameter</th>
<th>Independent Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation</td>
<td>(W_{ij} = -W_{ji})</td>
<td>3</td>
</tr>
<tr>
<td>Boost</td>
<td>(V_i)</td>
<td>3</td>
</tr>
<tr>
<td>Space-Translation</td>
<td>(E_i)</td>
<td>3</td>
</tr>
<tr>
<td>Time-Translation</td>
<td>(S)</td>
<td>1</td>
</tr>
</tbody>
</table>

\[10\]

Assuming continuity of the physical transformations of our system we
we have
\[ U(1+w, \vec{u}, \vec{e}, \delta) \rightarrow 1 \]
as
\[ w, \vec{u}, \vec{e}, \delta \rightarrow 0 \]. But \( U \) is unitary so
\[ 1 = U^\dagger U = [1 + (U - I)]^\dagger [1 + (U - I)] \]
\[ = 1 + (U - I)^\dagger + (U - I) \], to first order.

\[ U - I = i (\text{Hermitian operator}) \].

Hence for Galilean transformations close to the identity we find
\[ U(1+w, \vec{u}, \vec{e}, \delta) = 1 + \frac{i}{2} \omega_{ij} \frac{J_{ij}}{\hbar} + i 2 \epsilon^{ijk} \frac{K_i}{\hbar} \frac{P_j}{\hbar} \]
\[ -i \epsilon^{ijk} \frac{P_k}{\hbar} + i \delta \frac{H}{\hbar} \]

where \( J_{ij} = -J_{ji} \), \( K_i, P_i \) are the 10 Hermitian operators which generate the Galilean transformations.
\[ J_\tilde{j} = \tilde{J}_j^+ \]
\[ K_i = K_i^+ \]
\[ P_\tilde{i} = P_i^+ \]
\[ H = H^+ \]

Now suppose we consider the product of three transfer matrices

\[ \mathcal{E}_R, \mathcal{V}, \mathcal{a}, b_3 \rightarrow \mathcal{E}_L, \mathcal{V}, \mathcal{a}, b_3 \]

\[ \mathcal{E}_R, \mathcal{V}, \mathcal{a}, b_3 \]

\[ \mathcal{E}_L, \mathcal{V}, \mathcal{a}, b_3 \]

This is

\[ \mathcal{E}_R, \mathcal{V}, \mathcal{a}, b_3 \]

\[ \mathcal{E}_L, \mathcal{V}, \mathcal{a}, b_3 \]

using the group product law

\[ \mathcal{E}_R, R_w, R_{2u} + V, R_{4e} + V_{5a} + a, b + \delta_3 \]

\[ \mathcal{E}_R^{-1}, -R^{-1}V, -R^{-1}a + R^{-1}V_b, -b_3 \]
\[-568\]

\[
\begin{align*}
\frac{\xi_1 + R\omega R^{-1}}{\xi_1 + R\omega R^{-1}V + R\nu,} & \quad -R\omega R^{-1}a + R\omega R^{-1}Vb + R\epsilon - R\nu b + \nu \delta, \\
= & \quad \xi_1 + R\omega R^{-1}V + R\nu, \\
& \quad -R\omega R^{-1}a + R\omega R^{-1}Vb + R\epsilon - R\nu b + \nu \delta, \\
= & \quad \xi_1 + R\omega R^{-1}V + R\nu, \\
& \quad -R\omega R^{-1}a + R\omega R^{-1}Vb + R\epsilon - R\nu b + \nu \delta.
\end{align*}
\]

Since this is again a Galilean transformation infinitesimally close to 0 it has infinitesimal parameters

\[
\omega' = R\omega R^{-1}
\]

\[u' = Ru - R\omega R^{-1}V
\]

\[e' = Re - R\nu b - R\omega R^{-1}a + R\omega R^{-1}Vb + \nu \delta
\]

\[\delta' = \delta.
\]

The unitary operators representing these transformations after the same product laws up to a phase

\[U(R, \tilde{V}, \tilde{a}, b) U(1 + \omega, \tilde{u}, \tilde{e}, \delta) U^{-1}(R, \tilde{V}, \tilde{a}, b) \]

\[= e^{i \mathcal{A}(\omega, \tilde{u}, \tilde{e}, \delta; R, \tilde{V}, \tilde{a}, b)} \times U(1 + \omega', \tilde{u}', \tilde{a}', \delta')
\]
where the phase
\[ 2(\omega, \hat{\omega}, \hat{\delta}, \delta; R, \hat{V}, \hat{a}, b) \]
\[ = \chi(R, \hat{V}, \hat{a}, b; 1 + i\omega, \hat{\omega}, \hat{\delta}, \delta) \]
\[ + \chi(R + R\omega, R\hat{\omega} + V, R\hat{\delta} + \delta; R^{-1}, -R^{-1}V, -R^{-1}(a - Vb), -b) \]

satisfies
\[ 2(0, \hat{\omega}, \hat{\delta}, 0; R, \hat{V}, \hat{a}, b) = \chi(R, \hat{V}, \hat{a}, b; 1, \hat{\delta}, 0, 0) \]
\[ + \chi(R, \hat{V}, \hat{a}, b; R^{-1}, -R\hat{V}, -R^{-1}(a - Vb), -b) \]
\[ = 0. \]

Hence \( 2(\omega, \hat{\omega}, \hat{\delta}, \delta; R, \hat{V}, \hat{a}, b) \) can be expanded about 0 to yield
\[ e^{i2(\omega, \hat{\omega}, \hat{\delta}, \delta; R, \hat{V}, \hat{a}, b)} = 1 + \frac{i}{2} \omega \hat{\omega} \text{det}(R, \hat{V}, \hat{a}, b) \]
\[ + \text{adj}(R, \hat{V}, \hat{a}, b) - \text{det}(R, \hat{V}, \hat{a}, b) \]
\[-S>0-
+i \delta h(R, \tilde{V}, \tilde{a}, b)\]

where \( f_{ij} = -f_{ji}, g_i, r_i, h \) are real numbers. So we find, using

\[U(1+w, \tilde{V}, \tilde{a}, S) = 1 + \frac{i}{\hbar} \omega_{ij} \tilde{T}_{ij} + i \frac{2\tilde{e}_i}{\hbar} K_i - i \frac{\tilde{e}_i P_i}{\hbar} + i \frac{\delta h}{\hbar}\]

that

\[U(R, \tilde{V}, \tilde{a}, b) \left[ 1 + \frac{i}{\hbar} \omega_{ij} \tilde{T}_{ij} + i \frac{2\tilde{e}_i}{\hbar} K_i - i \frac{\tilde{e}_i P_i}{\hbar} + i \frac{\delta h}{\hbar} \right] U^\dagger(R, \tilde{V}, \tilde{a}, b)\]

\[= e^{i\frac{2}{\hbar} U(1+w, \tilde{V}, \tilde{a}, S')}\]

\[= \left( 1 + \frac{i}{\hbar} \omega_{ij} f_{ij} + i \xi_i g_i - i \xi_i r_i + i \delta h \right)\]

\[\times \left( 1 + \frac{i}{\hbar} (R w R^{-1})_{ij} \tilde{T}_{ij} + i \frac{1}{\hbar} (R u - R w R^{-1} V)_i K_i - i \frac{1}{\hbar} (R e - R u b - R w R^{-1} a + R w R^{-1} V b + V b) P_i + i \frac{1}{\hbar} \delta h \right)\]
That is to first order in \( \omega, \omega, \varepsilon, \delta \)

\[
UR, \tilde{V}, \tilde{a}, \tilde{b}) [\frac{1}{2} \omega_j J_j + \omega_i k_i - \varepsilon_i P_i + 8H J] (\vec{P}_{

= \frac{1}{2} \omega_j f_j + \omega_i g_i - \varepsilon_i r_i + \delta h

+ \frac{1}{2} R_{i k} R^{-1}_{j m} \omega_j T_{k m} + (R_{i k} \omega_i - R_{i k} R^{-1}_{j m} V_{m j}) \Lambda

- (R_{i k} \varepsilon_i - R_{i k} \omega_i b - R_{i k} R^{-1}_{j m} \omega j

+ R_{i k} R^{-1}_{j m} \omega_j V_{m b} + V b) \Lambda

+ \Delta H .

Gathering terms we find
\[ \text{U}(\vec{R}, \vec{v}, \vec{a}, b) \left[ \frac{1}{2} w_{ij} J_{ij} + u_i k_i - e_i P_i + S h \right] \text{U}^{-1}(\vec{R}, \vec{v}, \vec{a}, b) \]

\[ = \frac{1}{2} w_{ij} \left[ t h_{ij} + \frac{1}{2} (R_i R_j - R_j R_i) J_{mn} \right. \]

\[ - (R_i R_j - R_j R_i) V_m K_e \]

\[ + (R_i R_j - R_j R_i) A m P_e \]

\[ + (R_i R_j - R_j R_i) V_m b P \]

\[ + u_i \left[ t h_{gi} + R_i K e + R_i b P \right] \]

\[ - e_i \left[ t h_{ri} + R_i P \right] \]

\[ + S \left[ t h_{h} - V e P + H \right] \]

Since \( w_{ij}, u_i, e_i, S \) are independent parameters, we identify their coefficients of the left and right hand sides.

(Recall \( R_i R_j = R_j R_i = R_{ij} \))
1) \( \frac{1}{2} w_{ij} : \)
\[
U_{(R, \tilde{V}, \tilde{a}, \tilde{b})} J_{ij} U^{(R, \tilde{V}, \tilde{a}, \tilde{b})}
\]
\[
= \text{t} f_{ij}(R, \tilde{V}, \tilde{a}, \tilde{b}) + R_{ij} R_{mj} (J_{lm} - V_{ml} K_{le} + V_{lm} K_{me} + c m P_{e} - a_2 P_{m} + b V_{m} P_{e} - b V_{e} P_{m})
\]

2) \( U_{i} : \)
\[
U_{(R, \tilde{V}, \tilde{a}, \tilde{b})} K_{i} U^{(R, \tilde{V}, \tilde{a}, \tilde{b})}
\]
\[
= \text{t} g_{ij}(R, \tilde{V}, \tilde{a}, \tilde{b}) + K_{le} R_{le} + b P_{e} R_{le}
\]

3) \( E_{i} : \)
\[
U_{(R, \tilde{V}, \tilde{a}, \tilde{b})} P_{i} U^{(R, \tilde{V}, \tilde{a}, \tilde{b})}
\]
\[
= \text{t} r_{i}(R, \tilde{V}, \tilde{a}, \tilde{b}) + P_{e} R_{le}
\]

4) \( \delta : U_{(R, \tilde{V}, \tilde{a}, \tilde{b})} H U^{(R, \tilde{V}, \tilde{a}, \tilde{b})} \)
\[
= \text{t} h - V_{e} P_{e} + H.
\]
The Galilean transformation $E_{RV, \tilde{V}, \tilde{a}, b^3}$ is completely arbitrary; suppose we choose it to be infinitesimal. Let

$$E_{RV, \tilde{V}, \tilde{a}, b^3} = \frac{1}{\hbar} [1 + \omega, \tilde{u}, \tilde{e}, \delta^3],$$

then

$$U(1 + \omega, \tilde{u}, \tilde{e}, \delta) = 1 + \frac{i}{2} \omega_{mn} \frac{J_{mn}}{\hbar} + i \nu n \frac{K_n}{\hbar}$$

$$- i \epsilon_n \frac{P_n}{\hbar} + i \delta \frac{H}{\hbar}$$

while

$$U^{-1}(1 + \omega, \tilde{u}, \tilde{e}, \delta) = U(1 - \omega, -\tilde{u}, -\tilde{e}, -\delta)$$

$$= 1 - \frac{i}{2} \omega_{mn} \frac{J_{mn}}{\hbar} - i \nu n \frac{K_n}{\hbar}$$

$$+ i \epsilon_n \frac{P_n}{\hbar} - i \delta \frac{H}{\hbar}$$

Thus, for any operator $\mathcal{O}$ we have

$$U(1 + \omega, \tilde{u}, \tilde{e}, \delta) \mathcal{O} U^{-1}(1 + \omega, \tilde{u}, \tilde{e}, \delta)$$

$$= \mathcal{O} + \frac{i}{2} \omega_{mn} \frac{1}{\hbar} [J_{mn}, \mathcal{O}]$$

$$+ i \nu n \frac{1}{\hbar} [K_n, \mathcal{O}] - i \epsilon_n \frac{1}{\hbar} [P_n, \mathcal{O}]$$

$$+ i \delta \frac{1}{\hbar} [H, \mathcal{O}].$$
Further the phase angles $f_{ij}, g_i, r_i, h$ can be Taylor expanded about $0$:

$$f_{ij}(1 + \omega, \hat{u}, \hat{e}, \delta) = \frac{1}{2} \omega_{mj} f_{ijmn} + u_{n} f_{ijn} - \epsilon_{n} f_{ij} + 8 f_{ij}$$

$$g_{i}(1 + \omega, \hat{u}, \hat{e}, \delta) = \frac{1}{2} \omega_{jn} g_{imn} + u_{n} g_{ij} - \epsilon_{n} g_{i} + 8 g_{i}$$

$$r_{i}(1 + \omega, \hat{u}, \hat{e}, \delta) = \frac{1}{2} \omega_{mn} r_{imn} + u_{n} r_{in} - \epsilon_{n} r_{i} + 8 r_{i}$$

$$h(1 + \omega, \hat{u}, \hat{e}, \delta) = \frac{1}{2} \omega_{mn} h_{mnm} + u_{n} h_{in} - \epsilon_{n} h_{i} + 8 h$$

With all quantities real numbers and with obvious symmetry properties:

$$f_{ijmn} = -f_{jimn} = f_{ijnm}$$

$$f_{ij} = -f_{ji}$$

$$f_{ijn} = -f_{jien}$$

$$f''_{ij} = -f''_{ji}$$

$$g_{imn} = -g_{jimn}$$

$$r_{imn} = -r_{jin}$$

$$h_{mn} = -h_{nm}$$
Substituting all of these expansions into the 4 equations at page -573- yields

\[ \frac{1}{2} \sum_{\alpha ij} U(iw, \bar{\nu}, \bar{e}, \bar{\delta}) J_{ij} U^{-1}(iw, \bar{\nu}, \bar{e}, \bar{\delta}) \]

\[ = J_{ij} + \frac{i}{\hbar} \omega_{mn} \left[ J_{mn}, J_{ij} \right] \]
\[ + i \hbar n \frac{1}{\hbar} \left[ k_{n}, J_{ij} \right] - i \hbar n \frac{1}{\hbar} \left[ p_{n}, J_{ij} \right] \]
\[ + i \hbar \frac{1}{\hbar} \left[ h, J_{ij} \right] \]
\[ = J_{ij} + \omega_{mn} \left[ \delta_{m} J_{nj} + \delta_{n} J_{mj} + \frac{1}{2} \tilde{f}_{mn} j_{ij} \right] \]
\[ - i \hbar \left[ k_{i} \delta_{nj} - k_{j} \delta_{ni} - \tilde{h} f_{ijn} \right] \]
\[ + i \hbar \left[ p_{i} \delta_{nj} - p_{j} \delta_{ni} - \tilde{h} f_{ijn} \right] \]
\[ + \delta_{i j} \tilde{h} f_{ij} \]

Since \( \omega_{mn} = - \omega_{nm} \), we anti-symmetrize the first term. Equating the coefficients of the independent parameters \( w, \nu, e, \delta \), we obtain the 4 commutation relations.
Proceeding similarly with the $K_i$ equation:

\[ k = \frac{1}{2} w_{mm} + [K_{m}, K_{j}] + \frac{1}{2} w_{mm} \frac{1}{2} [J_{m}, K_j] + i \left( \frac{1}{2} w_{mm} + [J_{m}, K_j] \right) \]

\[ = k_i + \frac{1}{2} w_{mm} + [J_{m}, K_j] + i \left( \frac{1}{2} w_{mm} + [J_{m}, K_j] \right) \]

\[ \Rightarrow \left[ P_n, J_s \right] = i t \left( k_i S_{m} - k_j S_{m} \right) - i t f_{m} \]

\[ \Rightarrow \left[ P_n, J_s \right] = i t \left( k_i S_{m} - k_j S_{m} \right) - i t f_{m} \]
Which yields the 4 commutators:

\[ [J_{mn}, K_i] = -i \hbar (\delta m n - \delta i m n) \]
\[ -i \hbar^2 g_{i m n} \]
\[ [K_n, K_i] = -i \hbar^2 g_{i n} \]
\[ [P_n, K_i] = -i \hbar^2 g'_{i n} \]
\[ [H, K_i] = -i \hbar \cdot P_i - i \hbar^2 g_i'' \]

Next we have

\( U(t + \omega, \tilde{w}, \tilde{e}, \delta) P_i U^{-1}(t + \omega, \tilde{w}, \tilde{e}, \delta) \)
\[ = P_i + \frac{i}{2} \omega_{mn} \frac{1}{\hbar} [J_{mn}, P_i] \]
\[ + i \tilde{w}_n \frac{1}{\hbar} [K_n, P_i] - i \tilde{e}_n \frac{1}{\hbar} [P_n, P_i] \]
\[ + i \delta \frac{1}{\hbar} [H, P_i] \]
\[ = P_i + \frac{i}{2} \omega_{mn} (\delta m P_n - \delta i P_m + \hbar \text{rimn}) \]
\[ + \text{an thrin} - \text{enthrin} + \text{circ}'' \]
Which yields
\[ [J_{mn}, P_i] = -i \hbar (\sin P_m - \sin P_n) - i \hbar^2 R_{in} \]
\[ [K_n, P_i] = -i \hbar^2 R_{in} \]
\[ [P_n, P_i] = -i \hbar^2 R'_{in} \]
\[ [H, P_i] = -i \hbar^2 R''_{in} \]

And finally
\[ U(\omega, \vec{u}, \vec{e}, \delta) H U^{-1}(\omega, \vec{u}, \vec{e}, \delta) \]
\[ = H + \frac{i}{\hbar} \omega_{mn} \frac{1}{\hbar} [J_{mn}, H] \]
\[ + i \omega_{n} \frac{1}{\hbar} [K_n, H] - i e_n \frac{1}{\hbar} [P_n, H] \]
\[ + i \delta \frac{1}{\hbar} [H, H] \]
\[ = H + \frac{i}{\hbar} \omega_{mn} (\hbar h_{mn}) \]
\[ + 2 \omega_n (P_n + \hbar h_n) - e_n \hbar' h_n + \delta \hbar h'' \]
which yields
\[
[J_{mn}, H] = -i \hbar^2 h_{mn}
\]
\[
[K_n, H] = +i \hbar p_n - i \hbar^2 h_n
\]
\[
[p_n, H] = -i \hbar^2 h'_n
\]
\[
[H, H] = -i \hbar^2 h''
\]

We first note some consistency requirements that are straightforward:
\[
[K_n, J_{ij}] = -[J_{ij}, K_n] \Rightarrow f_{ijn} = -g_{nij} = g_{nji}
\]
\[
[p_n, J_{ij}] = -[J_{ij}, p_n] \Rightarrow f'_{ijn} = -r_{nij} = r_{nji}
\]
\[
[H, J_{ij}] = -[J_{ij}, H] \Rightarrow f''_{ij} = -h_{ij} = h_{ji}
\]
\[
[p_n, K_i] = -[K_i, p_n] \Rightarrow g_{in} = -r_{nic}
\]
\[
[H, K_i] = -[K_i, H] \Rightarrow g_{i''} = -h_i
\]
\[
[H, p_i] = -[p_i, H] \Rightarrow r_{i''} = -h_i
\]
\[
[H, H] = 0 \quad \Rightarrow h'' = 0
\]
\[ [K_i, K_j] = -[K_j, K_i] \Rightarrow g_{in} = -g_{ni} \]
\[ [P_i, P_j] = -[P_j, P_i] \Rightarrow r_{in} = -r_{ni} \]

Further since \( J_{ij} = -J_{ji} \), there are only 3 independent \( J_{ij} \) operators. Define

\[ J_1 \equiv J_{23} \]
\[ J_2 \equiv J_{31} \]
\[ J_3 \equiv J_{12} \]

That is

\[ J_i = \frac{1}{2} \epsilon_{ijk} J_{jk} \]

where \( \epsilon_{ijk} \) is the anti-symmetric permutation tensor (Levi-Civita symbol)

\[ \epsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is a cyclic even permutation of } (1,2,3) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \]
Using
\[ E_{ij} E_{ij'} = 8 j j' S_{kk'} - 8 j k' S_{ij} \]
we have
\[ J_{ij} = E_{ij} J_k. \]

The commutation relations become
\[
\begin{align*}
[J_i, J_j] &= i \hbar E_{ijk} (J_k + F_k) \\
[J_i, K_j] &= i \hbar E_{ijk} (K_k + G_k) \\
[J_i, P_j] &= i \hbar E_{ijk} (P_k + R_k) \\
[J_i, H] &= i \hbar H_i \\
[K_i, K_j] &= i \hbar E_{ijk} G_k \\
[K_i, P_j] &= i \hbar M_{ij} \\
[K_i, H] &= i \hbar (P_i + \tilde{H}_i) \\
[P_i, P_j] &= i \hbar E_{ijk} \tilde{R}_k \\
[P_i, H] &= i \hbar H_i. \\
\end{align*}
\]
\[ [H, H] = 0, \]
Where $F_k, G_k, R_k, H_k, G_k, M_j, A_i, R_k, H_i$ are all real numbers related to the arbitrary phases by

\[
F_k = -\frac{\hbar}{8} 
\]

\[
G_k = -\frac{\hbar}{4} 
\]

\[
R_k = -\frac{\hbar}{4} 
\]

\[
H_k = -\frac{\hbar}{2} 
\]

Finally, we apply a more subtle consistency requirement, which is the Jacobi identity.
The Jacobi identity is cyclic identity
\[ [A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0 \]
for any \( A, B, C \).

1) So for \( A = J_i, B = H, C = P_j \) we get
\[ 0 = [J_i, [H, P_j]] + [P_j, [J_i, H]] + [H, [P_j, J_i]] \]
\[ = [J_i, -i\hbar H'_j] + [P_j, H H_i] + [H, -i\hbar \epsilon_{ijk} P_k + R_k] \]
\[ = [J_i, -i\hbar H'_j] + [P_j, H H_i] + [H, -i\hbar \epsilon_{ijk} P_k + R_k] \]
\[ = [J_i, -i\hbar \epsilon_{ijk} P_k] + [H, -i\hbar \epsilon_{ijk} H_k] \]
\[ \Rightarrow 0 = \frac{H_k}{\hbar} \]
\[ \Rightarrow H_k = 0 \]

2) For \( A = J_i, B = H, C = J_j \)
\[ 0 = [J_i, [H, J_j]] + [J_j, [J_i, H]] + [H, [J_j, J_i]] \]
\[ \Rightarrow 0 = [J_i, -i\hbar J'_j] + [J_j, H H_i] + [H, -i\hbar \epsilon_{ijk} J_k + P_k] \]
\[ \Rightarrow -i\hbar \epsilon_{ijk} [H, J_k] = 0 = -\hbar^2 \epsilon_{ijk} H_k \]
\[ \Rightarrow H_k = 0 \]
3) For $A = P_i$, $B = H$, $C = K_j$

$$O = [P_i, [H, K_j]] + [K_j, [P_i, H]] + [H, [K_j, P_i]]$$
$$= [P_i, -i\hbar(P_j + H_j)] + [K_j, i\hbar H_i] + [H, +i\hbar M_j]$$

$$\Rightarrow [P_i, P_j] = O = i\hbar \epsilon_{ijk} \hat{R}_k$$

$$\Rightarrow \hat{R}_k = 0$$

4) For $A = J_i$, $B = K_j$, $C = K_k$

$$O = [J_i, [K_j, K_k]] + [K_k, [J_i, K_j]] + [K_j, [K_k, J_i]]$$
$$= [J_i, i\hbar \epsilon_{ijk} \hat{G}_e + [K_k, i\hbar \epsilon_{ijk} \hat{K} + \hat{G}_e]]$$

$$\Rightarrow O = i\hbar \epsilon_{ijk} [K_k, K_i] - i\hbar \epsilon_{ijk} [K_j, K_k]$$

$$= -\hbar^2 \epsilon_{ijk} \epsilon_{km} \hat{G}_m + \hbar^2 \epsilon_{ike} \epsilon_{jem} \hat{G}_m$$

$$= \hbar^2 (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk} - \delta_{ij} \delta_{km} + \delta_{im} \delta_{kj}) \hat{G}_m$$

$$= \hbar^2 (\delta_{ik} \delta_{G_j} - \delta_{ij} \delta_{G_k}) \Rightarrow \hat{G}_{ik} = 0.$$
Further, we can re-define $J_i$, $K_i$, $P_i$ by adding to the initial constants $F_i$, $G_i$, $R_i$ respectively. They will still leave the algebra Hermitian and will not affect the consistency checks so far. So letting

$$J_i \rightarrow J_i' = J_i + F_i$$

$$K_i \rightarrow K_i' = K_i + G_i$$

$$P_i \rightarrow P_i' = P_i + R_i$$

The algebra becomes

$$[J_i', J_j'] = i\hbar e_{ijk} J_k'$$

$$[J_i', K_j'] = i\hbar e_{ijk} K_k'$$

$$[J_i', P_j'] = i\hbar e_{ijk} P_k'$$

$$[J_i', H] = 0$$

$$[K_i', K_j'] = 0$$

$$[K_i', P_j'] = i\hbar M_{ij}$$

$$[K_i', H] = +i\hbar (P_i' + (\tilde{A}_i - R_i))$$

$$[P_i', P_j'] = 0$$

$$[P_i', H] = 0$$

$$[H, H] = 0$$
Finally applying the Jacobi identity once more for $A = H$, $B = K^j$, $C = J^i$ we have

$$O = [H, [K^i, J^j]] + [J^j, [H, K^i]] + [K^i, [J^j, H]]$$

$$= [H, -i\hbar e_{ijk}k^k] + [J^j, -i\hbar (P_i + (I_i - R_i))] + [K^i, \delta^j_i]$$

$$\Rightarrow$$

$$O = +i\hbar e_{ijk}i\hbar (P_k + (I_k - R_k))$$

$$\Rightarrow$$

$$\left( \tilde{H}_k - \tilde{R}_k \right) = 0$$

6) And for $A = P^i$, $B = K^j$, $C = J^k$

$$O = [P^i, [K^j, J^k]] + [J^k, [P^i, K^j]] + [K^j, [J^k, P^i]]$$

$$= [P^i, -i\hbar e_{kjl}k^l] + [J^k, -i\hbar M^i_j] + [K^j, i\hbar e_{kjl}P^l]$$

$$= -i\hbar e_{kjl}(-i\hbar)M^i_l + i\hbar e_{kjl}i\hbar M^i_j$$

$$\Rightarrow$$

$$O = -\hbar^2 (e_{kjl}M^i_l + e_{kjl}M^i_j)$$
\[ \begin{align*}
\text{Now for} & \quad i = 1, j = 2, k = 3 \quad \Rightarrow -M_{11} + M_{22} = 0 \\
& \quad i = 2, j = 3, k = 1 \quad \Rightarrow -M_{22} + M_{33} = 0 \\
& \quad i = 1, j = 2, k = 2 \quad \Rightarrow -M_{11} - M_{12} = 0 \\
& \quad i = 2, j = 1, k = 2 \quad \Rightarrow -M_{21} = 0 \\
& \quad i = 1, j = 3, k = 3 \quad \Rightarrow M_{13} = 0 \\
& \quad i = 1, j = 3, k = 2 \quad \Rightarrow M_{12} = 0 \\
& \quad i = 3, j = 2, k = 2 \quad \Rightarrow M_{21} = 0 \\
\Rightarrow \\
M_{ij} = m S_{ij}, \quad m = \text{constant} \\
\end{align*} \]

Thus, dropping the primes on \( T_i, K_i, P_i \) we obtain the fiducial form of the algebra of the generators of the Galilean group.
\[ [J_i, J_j] = i \hbar \epsilon_{ijk} J_k \]

\[ [J_i, K_j] = i \hbar \epsilon_{ijk} K_k \]

\[ [J_i, P_j] = i \hbar \epsilon_{ijk} P_k \]

\[ [J_i, H] = 0 \]

\[ [K_i, K_j] = 0 \]

\[ [K_i, P_j] = +i \hbar m \delta_{ij} \]

\[ [K_i, H] = +i \hbar P_i \]

\[ [P_i, P_j] = 0 \]

\[ [P_i, H] = 0 \]

\[ [H, H] = 0 \]

*Note: if we assumed the representation to be true (not a representation up to a phase), the commutation relations would be the same except \([K_i, P_j] = 0\) instead of \(= i \hbar m \delta_{ij}\). This is for the*
Geliean group the phases are non-trivial.

Consequences of the Geliean group multiplication law:

1) Consider the transformation law for the position operator. This can be found from considering a particle in \( O \) at position \( \vec{r} \) at time \( t \) which in \( O' \) at \( \vec{r}' = \vec{r} + \vec{V}t + \vec{a} \) at time \( t' = t + b \).

In particular the wavefunction in \( O' \) at \( \vec{r}', t' \) is the same as the wavefunction in \( O, \vec{r}, t \) at \( \vec{r} + \vec{a}, t + b \).

\[
\langle \vec{r}', t' | \Psi \rangle = 2'(\vec{r}', t') \\
= \langle \vec{r}, t | \Psi \rangle = 2'(\vec{r}, t)
\]

(i.e. if \( 2'(\vec{r}, t) \) has a peak at \( \vec{r}, t \) as observed by \( O \), then \( O' \) observes the peak in \( 2'(\vec{r}', t') \) at \( \vec{r}' + \vec{a}, t' + b \).

Hence \( 2'(\vec{r}, t) = 2'(R^{-1}(\vec{r} - \vec{V}(t - b) + \vec{a}), t + b) \).
This is just

\[ q'(\hat{F}, t) = \langle \hat{F}, t | q' \rangle = \langle \hat{F}, t | \mathcal{U}(R, \hat{V}, \hat{a}, b) | q' \rangle \]

\[ = \langle \mathcal{U}(R, \hat{V}, \hat{a}, b)(\hat{F} - \hat{V}(t-b) - \hat{a}), t-b \rangle \]

\[ = \langle \mathcal{U}(R, \hat{V}, \hat{a}, b)(\hat{F} - \hat{V}(t-b) - \hat{a}), t-b \rangle \]

\[ \Rightarrow \]

\[ e^{i \alpha} \mathcal{U}^{-1}(R, \hat{V}, \hat{a}, b) | \hat{F}, t \rangle \]

\[ = | \mathcal{U}(R, \hat{V}, \hat{a}, b)(\hat{F} - \hat{V}(t-b) - \hat{a}), t-b \rangle \]

That is simply

\[ e^{i \alpha} \]

\[ | \hat{F}', t' \rangle = \mathcal{U}(R, \hat{V}, \hat{a}, b) | \hat{F}, t \rangle \]

The position operator on this state yields

\[ \hat{R}(t-b) \mathcal{U}^{-1}(R, \hat{V}, \hat{a}, b) | \hat{F}, t \rangle \]

\[ = \mathcal{U}^{-1}(R, \hat{V}, \hat{a}, b) \mathcal{U}(R, \hat{V}, \hat{a}, b) \mathcal{U}^{-1}(R, \hat{V}, \hat{a}, b) | \hat{F}, t \rangle \]

\[ \Rightarrow \]
\[ U(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \mathbf{\hat{R}}(t-b) U^\dagger(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \mathbf{\hat{f}}(t) \]
\[ = \mathbf{R}\mathbb{^t}(\mathbf{\hat{f}} - \mathbf{\hat{V}}(t-b) - \mathbf{\hat{a}}) \mathbf{\hat{f}}(t) \]

but \[ \mathbf{\hat{f}} \mathbf{\hat{f}}(t) = \mathbf{R}(t) \mathbf{\hat{f}}(t) \] ; then

\[ \Rightarrow \]

\[ U(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \mathbf{\hat{R}}(t-b) U^\dagger(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \]
\[ = \mathbf{R}\mathbb{^t}(\mathbf{\hat{R}(t)} - \mathbf{\hat{V}(t-b)} - \mathbf{\hat{a}}) \]

Relabelling \( t-b \rightarrow t \) we have

\[ U(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \mathbf{\hat{X}}_i(t) U^\dagger(\mathbf{R}, \mathbf{V}, \mathbf{a}, b) \]
\[ = \mathbf{R}\mathbb{_{ij}}(\mathbf{\hat{X}}_j(t+b) - \mathbf{\hat{V}}_j(t) - \mathbf{\hat{a}}_j) \]

For infinitesimal Galilean transformations \( \mathbf{R} = 1 + \mathbf{\hat{R}}, \mathbf{\hat{V}} = \mathbf{\hat{u}}, \mathbf{\hat{a}} = \mathbf{\hat{e}}, b = \delta \) we have

commutation relations on the LHS while we have Taylor expansions on the RHS.
\[ X_i(t) + \frac{i}{\hbar} \sum_{m} \left[ J_{mn}, X_i(t) \right] + i \sum_{n} \left[ K_n, X_i(t) \right] \]

\[ - i \epsilon_i \frac{1}{\hbar} \left[ P_n, X_i(t) \right] + i \delta_{ij} \left[ H, X_i(t) \right] \]

\[ = X_i(t) + \sum \frac{d}{dt} X_i(t) - \omega_{ij} X_j(t) \]

\[ - 2 \epsilon_i + \epsilon_i \]

Equating the coefficients of the independent \( u_j \), \( \epsilon_i \), \( \delta \) we find \( \omega_{mn} \)

1. \[ \left[ J_{mn}, X_i(t) \right] = i \hbar \left( \delta_{mi} X_n(t) - \delta_{ni} X_m(t) \right) \]

\[ \Rightarrow \]

\[ \left[ J_i, X_j(t) \right] = \hbar \epsilon_{ijk} X_k(t) \]

Thus \( \hat{R}(t) \) is a vector operator under spatial rotations.

2. \[ \left[ K_n, X_i(t) \right] = + i \hbar \delta_{in} + \]
3) 
\[ \left[ P_n, X_i(t) \right] = -i \hbar \delta_{ni} \]

The canonical commutation relation; hence we identify \( P_i \) with the momentum operator.

4) 
\[ \left[ H, X_i(t) \right] = -i \hbar \frac{\partial}{\partial t} X_i(t) \]

The Heisenberg equations of motion, hence we identify \( H \) with the Hamiltonian and confirm that in the passive view we are in the Heisenberg picture.

In the active view we move the system and apparatus. Then state \( |2(t)\rangle \) in that view corresponds to the state \( |2(t)\rangle \) translated in time. The opposite way \( t = -b \)

\[ |2(t)\rangle \equiv |2(t)\rangle = U(1, \theta, \bar{\theta}, -t) |2(0)\rangle \]

(Since \( [H,H] = 0 \))

\[ e^{-i H t / \hbar} |2(0)\rangle \]

Again since \( i \hbar \frac{d}{dt} |2(t)\rangle = H |2(t)\rangle \), we can identify \( H \) as the Hamiltonian, the generator of time translations.
2) If a physical system is Galilean invariant then the Galilean algebra for it is true. For a single particle the commutators
\[ [p_i, p_j] = 0, \quad [H, \hat{\mathbf{p}}] = 0 \]
implies that
\[ H = H(\hat{\mathbf{p}}) \] only since \[ [p_i, \hat{\mathbf{p}}_j] = -i \hbar \delta_{ij} \].
Also we have \[ [k_i, p_j] = i \hbar m \delta_{ij} \] so
\[ [k_i, H] = \frac{\partial H}{\partial p_j} [k_i, p_j] = i \hbar m \frac{\partial H}{\partial p_i} \]
\[ + i \hbar p_i \]
\[ \Rightarrow \quad \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \quad \Rightarrow \quad H = \frac{\hat{\mathbf{p}}^2}{2m} \]
as we argued earlier. Further we now identify \( m \) with the mass of the particle.

Assuming that \( \mathbf{R} = \mathbf{R}(\hat{\mathbf{p}}, \hat{\mathbf{r}}) \) we have
\[ [k_k, \hat{x}_i] = \frac{\partial \hat{x}_i}{\partial p_j} [k_k, p_j] = i \hbar m \frac{\partial \hat{x}_i}{\partial p_k} \]
\[ i \hbar \delta_{ik} \] (from 1)
\[ \frac{\partial x_i}{\partial p_k} = \pm \frac{1}{m} \delta_{ik} \]

Also \[ [p_k, x_i] = \frac{\partial x_i}{\partial k_j} [p_k, k_j] = -i\hbar \frac{\partial x_i}{\partial k_k} \]
\[ \Rightarrow -i\hbar \delta_{ki} \text{ from 0} \]
\[ \Rightarrow \frac{\partial x_i}{\partial k_k} = \frac{1}{m} \delta_{ik} \]

These imply that \[ x_i(t) = \frac{k_i + p_i t}{m} \].

As noted earlier, a single particle moving in a central potential with the Hamiltonian \[ H = \frac{1}{2m} p^2 + V(r) \] is not Galilean invariant. In particular

\[ [p_i, H] = [p_i, V(r)] \neq 0, \]

which is a violation of the Galilean algebra. The reason this system is not Galilean invariant is that the Hamiltonian does...
not describe a closed system. The central potential must have a source which is neglected in the Hamiltonian. When the appropriate dynamics of the source of the central potential is accounted for the Galilean invariance will be restored. For example, the 2-body Hamiltonian

\[ H = \frac{1}{2m_1} \vec{p}_1^2 + \frac{1}{2m_2} \vec{p}_2^2 + V(\vec{r}_1 - \vec{r}_2) \]

is Galilean invariant. For instance in the coordinate basis

\[ [\vec{P}, H] = [\vec{p}_1 + \vec{p}_2, H] \]

\[ = [\vec{p}_1 + \vec{p}_2, V(\vec{r}_1 - \vec{r}_2)] \]

\[ = -i\hbar (\vec{\nabla}_{\vec{r}_1} + \vec{\nabla}_{\vec{r}_2}) V(\vec{r}_1 - \vec{r}_2) \]

\[ = -i\hbar (\vec{\nabla}_{\vec{r}_1} - \vec{\nabla}_{\vec{r}_2}) V(\vec{r}_1 - \vec{r}_2) \]

\[ = 0. \]

For a system to be Galilean invariant, the potential must be a function of coordinate differences and not depend
on an arbitrary origin or center of force. Indeed for several particles we can proceed analogously by introducing corresponding momenta \( \mathbf{P}_1, \ldots, \mathbf{P}_n \) and coordinates \( \mathbf{R}_1, \ldots, \mathbf{R}_n \) for \( n \) particles with masses \( m_1, \ldots, m_n \). As usual the canonical commutation relations are

\[
\left[ X_{ni}, P_{mj} \right] = i \hbar \delta_{mn} \delta_{ij} \quad m, n = 1, 2, 3, \quad i, j = 1, \ldots, N,
\]

all other commutators vanishing.

Then the Galilean algebra is satisfied by

\[
\mathbf{P} = \sum_{i=1}^{N} \mathbf{P}_i
\]

\[
\mathbf{K} = \sum_{i=1}^{N} m_i \mathbf{R}_i - \mathbf{P} t
\]

and

\[
H = \sum_{i=1}^{N} \frac{\mathbf{P}_i^2}{2m_i} + V(\mathbf{R}_i - \mathbf{R}_j, \mathbf{P}_i - \mathbf{P}_j)
\]

with \( V \) a scalar function of its arguments.
3) Finally let's construct the unitary operators representing the Galilean transformations from the algebraic properties of the generators. Consider first the subgroups of spatial translations.

i) Spatial translations are given by the operator $U(1, \delta, \vec{a}, 0)$.

For infinitesimal translation vectors $\vec{a}$ we had that

$$U(1, \delta, \vec{a}, 0) = 1 - i \cdot \vec{a} \cdot \frac{1}{\hbar} \hat{P} = 1 - i \frac{\vec{a}}{\hbar} \cdot \hat{P}$$

hence we find

$$[U(1, \delta, \vec{a}, 0)]^2 = 1 - \frac{2i}{\hbar} \frac{\vec{a}}{\hbar} \cdot \hat{P} = U(1, \delta, 2\vec{a}, 0).$$

For $N$ very large and eventually letting $N \to \infty$ we have $\delta = \frac{1}{N \vec{a}}$

$$[U(1, \delta, \frac{\vec{a}}{N}, 0)]^2 = U(1, \delta, \frac{2\vec{a}}{N}, 0).$$

Continuing to make successive infinitesimal transformations we find

$$\lim_{N \to \infty} [U(1, \delta, \frac{\vec{a}}{N}, 0)]^N = \lim_{N \to \infty} U(1, \delta, \frac{N\vec{a}}{N}, 0) = U(1, \delta, \vec{a}, 0).$$
On the other hand
\[ \lim_{\hbar \to 0} [U(1, \hat{\sigma}, \hat{\alpha}, 0)]^n = \lim_{\hbar \to 0} [1 - \frac{i}{\hbar} \hat{\alpha} \cdot \hat{P}]^n \]

since \( [\hat{P}, \hat{\alpha}] = 0 \) this is just the exponential

\[ = e^{\frac{i}{\hbar} \hat{\alpha} \cdot \hat{P}} \]

Hence the spatial translation operator is

\[ U(1, \hat{\sigma}, \hat{\alpha}, 0) = e^{\frac{i}{\hbar} \hat{\alpha} \cdot \hat{P}} \]

Further

\[ U(1, \hat{\sigma}, \hat{\alpha}, 0) U(1, \hat{\sigma}, \hat{\alpha}, 0) \]
\[ = e^{\frac{i}{\hbar} \hat{\alpha} \cdot \hat{P}} e^{\frac{i}{\hbar} \hat{\alpha} \cdot \hat{P}} \]
\[ = e^{\frac{i}{\hbar} (\hat{\alpha} + \hat{\alpha}) \cdot \hat{P}} \]

since \( [\hat{P}_i, \hat{P}_j] = 0 \).
The spatial translation operators form an abelian subgroup of transformations.

Recall that in general,
\[ \mathcal{U}(1, \vec{0}, \vec{a}, 0) \mathcal{U}(1, \vec{0}, \vec{a}, 0) \mathcal{U}^{-1}(1, \vec{0}, \vec{a}, 0) = e \]
\[ \mathcal{U}(1, \vec{0}, \vec{a}, 0) \mathcal{U}(1, \vec{0}, \vec{a}, 0) = e \]

hence we find that \[ \mathcal{U}(1, \vec{0}, \vec{a}, 0; 1, \vec{0}, \vec{a}, 0) = 0 \]

since \[ [P_i, P_j] = 0. \]

2) Time translations are given by the operator \[ \mathcal{U}(1, \vec{0}, \vec{0}, \vec{b}) \]

Since \[ [H, H] = 0 \] we proceed analogously to the spatial translation case.
\[ \mathcal{U}(1, \vec{0}, \vec{0}, b) = \lim_{N \to \infty} \left[ \mathcal{U}(1, \vec{0}, \vec{0}, \frac{b}{N}) \right]^N \]
\[ = \lim_{N \to \infty} \left[ 1 + \frac{i}{\hbar} \frac{b}{N} H \right]^N \]
\[ \mathcal{U}(1, \vec{0}, \vec{0}, b) = e^{\frac{i}{\hbar} H b}, \quad \text{since} \quad [H, H] = 0. \]

Further

\[ \mathcal{U}(1, \vec{0}, \vec{0}, b) \mathcal{U}(1, \vec{0}, \vec{0}, b) = \mathcal{U}(1, \vec{0}, \vec{0}, b + b), \]

the time translation operators form an abelian subgroup of the Galilean transformations. Again the phase in the multiplication law is zero since \([H, H] = 0\). 

3) Velocity boosts are given by the operator

\[ \mathcal{U}(1, \vec{V}, \vec{0}, 0) \]. As with spatial translations we have the commutation relation \([K_i, K_j] = 0\).
Hence
\[ U(1, \vec{V}, \vec{0}, 0) = \lim_{N \to \infty} [U(1, \vec{V}, \vec{0}, 0)]^N \]
\[ = \lim_{N \to \infty} (1 + \frac{i}{\hbar N} \vec{V} \cdot \vec{K})^N \]
\[ U(1, \vec{V}, \vec{0}, 0) = e^{+\frac{i}{\hbar} \vec{V} \cdot \vec{K}} \] since \([K_i, K_j] = 0\).

Further
\[ U(1, \vec{V}, \vec{0}, 0) U(1, \vec{V}, \vec{0}, 0) = U(1, \vec{V} + \vec{V}, \vec{0}, 0) \]
since \([K_i, K_j] = 0\) the velocity boost operators form an abelian subgroup of the Galilean transformations. Again the multiplication law phase is zero due to \([K_i, K_j] = 0\).
4) Spatial rotations are given by

$$U(R, \vec{0}, \vec{0}, 0) \equiv U(R)$$.

Although we absorbed unwanted phases into the definition of $J_i$ so that

$$[J_i, J_j] = i \hbar \epsilon_{ijk} J_k$$, the situation is slightly more complicated since the $J_i$ do not commute.

Let's first consider rotations about the $z$-axis, denoting the orthogonal $3 \times 3$ rotation matrix $R(\theta)$ with $\theta = \theta \mathbb{Z}$

we have

$$x_i' = R_{ij}(\theta) x_j \quad \text{with}$$

$$R_{ij}(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
x \cos \theta - y \sin \theta \\
x \sin \theta + y \cos \theta \\
z
\end{pmatrix}$$
For infinitesimal $\Theta$ we have

$$R_{ij}(\Theta) = \delta_{ij} + \omega_{ij}(\Theta)$$

$$= \begin{pmatrix} 1 & -\Theta & 0 \\ \Theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Theta$$

Then

$$\omega_{ij}(\Theta) = \Theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $\omega_{12} = -\Theta$, $\omega_{21} = +\Theta$, all others $= 0$.

So a rotation in the $X_i - X_j$ plane is described by $\Theta \omega_{ij} = -\omega_{ji}$.

Now for an infinitesimal rotation about the $z$-axis we have the transformation operator

$$U(1 + \omega(\Theta)) = 1 + \frac{i}{\hbar} \omega_{ij}(\Theta) J_{ij}$$

$$= 1 + \frac{i}{\hbar} (\omega_{12} J_{12} + \omega_{21} J_{21})$$

$$= 1 + \frac{i}{\hbar} \omega_{12} J_{12}$$
So
\[ U(1 + \omega(\hat{e})) = 1 - \frac{i}{\hbar} \theta J_{12} \]
but recall \( J_i = \frac{1}{2} \epsilon_{ijk} J_{jk} \), that is
\[ J_1 = J_{23}, \quad J_2 = J_{31}, \quad J_3 = J_{12}, \quad \text{so} \]
\[ U(1 + \omega(\hat{e})) = 1 - \frac{i}{\hbar} \theta J_2 \]
\[ = 1 - \frac{i}{\hbar} \theta \hat{e} \cdot \hat{J}_2 \]

Now since \([J_3, J_3] = 0\) we can proceed as in the spatial translation case and compound successive infinitesimal rotations about the 2-axis to obtain a rotation about the z-axis through a finite angle \( \theta \).

\[ U(R(\hat{e})) = \lim_{N \to \infty} [U(R(\hat{e}))]^N \]
\[ = \lim_{N \to \infty} \left[ 1 - \frac{i}{\hbar} \theta J_2 \right]^N \]

\[ U(R(\hat{e})) = e^{-\frac{i}{\hbar} \theta J_2} \]
Similarly, any finite arbitrary rotation can be obtained by a single rotation through a finite angle $\Theta$ about a fixed direction (axis) $\hat{e}$. Thus the vector $\Theta \equiv \hat{e} \Theta$ completely specifies the arbitrary rotation under consideration. Since rotations about the same fixed axis commute, we can build up the finite rotation by composing successive infinitesimal rotations about this same axis. That is, although

$$[J_i, J_j] = i h \epsilon_{ijk} J_k,$$

we have

$$[\hat{e} \cdot J, \hat{e} \cdot J] = \hat{e} \cdot J_i [J_i, J_j]$$

$$= i h \hat{e} \cdot \epsilon_{ijk} J_k$$

Symmetric: $\epsilon_{ijk} = -\epsilon_{kji}$

$$= 0$$

Hence

$$U(R(\hat{e})) = \lim_{N \to \infty} [U(R(\hat{e}))]^N$$
\[ U(R(\theta)) = e^{-\frac{i}{\hbar} \cdot \vec{\theta} \cdot \vec{J}} \]

The unitary operator representing rotations.

Note: 1) \( \vec{\theta} \cdot \vec{J} = \theta_i \vec{J}_i = \frac{1}{2} \theta_i \varepsilon_{ijk} J_{jk} \)

\[ \equiv \frac{1}{2} \theta_{jk} J_{jk} \]

where \( \theta_{jk} = \varepsilon_{jki} \theta_i = -\theta_{kj} \)

and \( \theta_i = \frac{1}{2} \varepsilon_{ijk} \theta_{jk} \). So for example

\( \vec{\theta} = \theta_3 \hat{z} \) describes rotations about the z-axis that is in the \( X_1 - X_2 \) plane by angle \( \theta_3 = \theta_{12} = -\theta_{21} \).

2) Since \( [\vec{J}_i, \vec{J}_j] = i \hbar \varepsilon_{ijk} \vec{J}_k \)
\[
\frac{1}{\hbar} (\Theta_0 J) = e^{-\frac{i}{\hbar} (B_x J_x + B_y J_y + B_z J_z)} = e^{-\frac{i}{\hbar} B_x J_x} e^{-\frac{i}{\hbar} B_y J_y} e^{-\frac{i}{\hbar} B_z J_z},
\]

as we know from the Baker–Campbell–Hausdorff formula in general. Hence the group properties are complicated for the composition of 2 rotations, it is non-abelian.

So to summarize, we have considered systems invariant under Galilean transformations. They led us to identify the generator of spatial translations as the momentum operator and the generator of time translations as the Hamiltonian. In addition, the total angular momentum operator $\hat{J}$ generated spatial rotations. The passive view was used to describe the symmetry transformations and was shown to simply working in the Heisenberg picture. The time evolution of an operator
is given by \(-i\hbar \frac{dA(t)}{dt} = [H, A(t)]\). Hence, constants of the motion correspond to those operators which commute with the Hamiltonian. For a Galilean invariant system, the momentum \(\mathbf{p}\) and total angular momentum \(\mathbf{J}\) (as well as the energy \(H\)) are constants of the motion. From the dynamical point of view, by Ehrenfest's Theorem, this implies that the expectation value of these quantities is conserved, that is constant in time. Hence, we have for quantum mechanics, the relation between invariances or symmetries of the system and constants of the motion — or conservation laws.

\[
\begin{align*}
\text{Time Translation Invariance} & \quad \iff \quad H \text{ is constant} \iff \langle \text{Energy} \rangle \text{ is conserved} \\
\text{Space Translation Invariance} & \quad \iff \quad \mathbf{p} \text{ is constant} \iff \langle \text{Momentum} \rangle \text{ is conserved} \\
\text{Space Rotation Invariance} & \quad \iff \quad \mathbf{J} \text{ is constant} \iff \langle \text{Total angular momentum} \rangle \text{ is conserved}.
\end{align*}
\]