5.2.1. The Principle of Galilean Relativity and the Galilean Group

We now consider the transformations of Newtonian relativity, the transformations which are appropriate for non-relativistic physics. (They can be shown to be the $c \to \infty$ limit of the Poincaré transformations which are the transformations of special relativistic physics.) We require the physics (transition probabilities) to be invariant under transformations of the coordinates corresponding to space and time translations, spatial rotations and boosts.

Consider two observers O and O' using inertial frames of reference in which they label the same space-time event by $(\vec{r}, t)$ and $(\vec{r}\,', t')$, respectively. Galilean invariance means that the laws of physics have the same form in the O as O' frames provided their coordinates are related by

The linear transformation (Galilean transformations)

$$x_i' = R_{ij} x_j + V_i t + a_i$$
$$t' = t + b$$

The parameters a_i, b, V_i, R_{ij} are real numbers such that R_{ij} are the elements of a 3×3 real orthogonal matrix, $R^T = R^{-1}$.

Any 3×3 real orthogonal matrix has only 3 independent matrix elements, and so can be parameterized by 3 real numbers, for instance the Euler angles. Hence we have that

a_i labels space translations = 3 real parameters

b labels time translations (origin or zero of time) = 1 real parameter

V_i labels velocity boosts = 3 real parameters

R_{ij} labels spatial rotations = 3 real parameters

Hence we have that the most general Galilean transformation of coordinates is specified by 10 real parameters. Thus specifying $\{R, \vec{V}, \vec{a}, b\}$ completely specifies which Galilean transformation is being considered. We will label the Galilean transformation by $\{\overleftrightarrow{R}, \vec{V}, \vec{a}, b\}$.

The set of Galilean transformations forms a group, naturally called the Galilean group. To be a group we must find the composition law for Galilean transformations. That is if we make 2 such transformations this should be another Galilean transformation with its 10 parameters specified in terms of the 10 parameters of the first 2 Galilean transformations. So consider two consecutive Galilean transformations from O to O′ then O′ to O″

$$O \to O' : \; x_i' = R_{ij} x_j + V_i t + a_i$$

$$t' = t + b$$

$O' \to O''$:

$$x_i'' = \bar{R}_{ij} x_j' + \bar{V}_i t' + \bar{a}_i$$

$$t'' = t' + \bar{b}$$

Substituting for (x_i', t') in terms of (x_i, t) yields

$$x_i'' = \bar{R}_{ij}(R_{jk} x_k + V_j t + a_j)$$

$$+ \bar{V}_i (t+b) + \bar{a}_i$$

$$= (\bar{R}_{ij} R_{jk}) x_k + (\bar{R}_{ij} V_j + \bar{V}_i) t$$

$$+ (\bar{R}_{ij} a_j + \bar{V}_i b + \bar{a}_i)$$

So

$$x_i'' = (\bar{R}R)_{ij} x_j + (\bar{R}_{ij} V_j + \bar{V}_i) t$$

$$+ (\bar{R}_{ij} a_j + \bar{V}_i b + \bar{a}_i)$$

and

$$t'' = t + (b + \bar{b}) .$$

But we recognize this as another Galilean transformation directly from $O \to O''$; that is

$O \to O''$:

$$x''_i = \bar{\bar{R}}_{ij} x_j + \bar{\bar{V}}_i t + \bar{\bar{a}}_i$$

$$t'' = t + \bar{\bar{b}}$$

where this transformation $\{\bar{\bar{R}}, \bar{\bar{V}}, \bar{\bar{a}}, \bar{\bar{b}}\}$ is specified by

$$\bar{\bar{R}} = \bar{R} R$$

$$\bar{\bar{V}}_i = \bar{R}_{ij} V_j + \bar{V}_i$$

$$\bar{\bar{a}}_i = \bar{R}_{ij} a_j + \bar{V}_i b + \bar{a}_i$$

$$\bar{\bar{b}} = b + \bar{b}$$

(note $(\bar{R}R)^T = (R^T \bar{R}^T) = (R^{-1} \bar{R}^{-1}) = (\bar{R}R)^{-1}$ is orthogonal, as required).

Hence, we have the composition law for the Galilean group

Denoting the Galilean transformation by $\{R, \vec{V}, \vec{a}, b\}$ we have the product law

1) $\{\bar{R}, \vec{\bar{V}}, \vec{\bar{a}}, \bar{b}\}\{R, \vec{V}, \vec{a}, b\}$

$$= \{\bar{R}R, \overrightarrow{\bar{R}V + \bar{V}}, \overrightarrow{\bar{R}a + \bar{V}b + \bar{a}}, \bar{b} + b\}.$$

To show that the set of Galilean transformations forms a group we further must show that

2) The identity element is $\{1, \vec{0}, \vec{0}, 0\}$

since $\{1, \vec{0}, \vec{0}, 0\}\{R, \vec{V}, \vec{a}, b\}$

$$= \{\underset{\bar{R}}{1}R, \overrightarrow{\underset{\bar{R}}{1}V + \underset{\bar{V}}{0}}, \overrightarrow{\underset{\bar{R}}{1}a + \underset{\bar{V}}{0}b + \underset{\bar{a}}{0}}, \underset{\bar{b}}{0} + b\}$$

$$= \{R, \vec{V}, \vec{a}, b\}.$$

The identity transformation on the coordinates is just that

$$x_i' = x_i$$

$$t' = t.$$

3) The inverse element to $\{R, \vec{V}, \vec{a}, b\}$ is

$$\{\vec{R},\vec{V},\vec{a},b\}^{-1} \equiv \{R^{-1}, -\overrightarrow{R^{-1}V}, -\overrightarrow{R^{-1}a + R^{-1}Vb}, -b\}$$

since

$$\{\vec{R},\vec{V},\vec{a},b\}^{-1}\{R,\vec{V},\vec{a},b\}$$

$$= \{\underbrace{R^{-1}R}_{\bar{R}}, \overbrace{\overrightarrow{R^{-1}V} - \overrightarrow{R^{-1}V}}^{\bar{\vec{V}}}, \overrightarrow{R^{-1}a} - \overrightarrow{R^{-1}Vb} \underbrace{- \overrightarrow{R^{-1}a} + \overrightarrow{R^{-1}Vb}}_{\bar{\vec{a}}}, \underbrace{-b+b}_{\bar{b}}\}$$

$$= \{1, \vec{0}, \vec{0}, 0\}$$ the identity element.

The inverse transformation on the coordinates yields (the transformation from $O' \to O$)

$$x_i = (R^{-1})_{ij}\, x'_j + (-R^{-1}V)_i\, t' + (-R^{-1}a + R^{-1}Vb)_i$$

$$= (R^{-1})_{ij}\,(x'_j - V_j(t'-b) - a_j)$$

$$t = t' - b$$

while the transformation from $O \to O'$ is

$$x_i' = R_{ij} x_j + V_i t + a_i$$

$$t' = t + b .$$

4) Since matrix multiplication is associative, we have that the Galilean transformations obey the associative law of multiplication. Hence they form a group, the Galilean group.

We will next show that the Galilean group is represented by unitary operators $U(R, \vec{V}, \vec{a}, b)$ on the Hilbert space of states. Let $\{R, \vec{V}, \vec{a}, b\}$ be an arbitrary element of the Galilean group which takes a particle of mass m at rest and gives the particle a velocity $\vec{v}'$, pictorially,

$$\{R, \vec{V}, \vec{a}, b\} \begin{pmatrix} \text{particle} \\ \text{at} \\ \text{rest} \end{pmatrix} = \begin{pmatrix} \text{particle with} \\ \text{velocity} \\ \vec{v}' \end{pmatrix} .$$

Let the pure velocity boost $\{1, \vec{v}', \vec{0}, 0\}$ be such that its inverse brings the particle back to rest

$$\underbrace{\{\mathbb{1}, \vec{v}', \vec{0}, 0\}^{-1}}_{= \{\mathbb{1}, -\vec{v}', \vec{0}, 0\}} \begin{pmatrix} \text{particle with} \\ \text{velocity} \\ \vec{v}' \end{pmatrix} = \begin{pmatrix} \text{particle} \\ \text{at} \\ \text{rest} \end{pmatrix}$$

Hence

$$\{\mathbb{1}, -\vec{v}', \vec{0}, 0\}\{R, \vec{V}, \vec{a}, b\} \begin{pmatrix} \text{particle} \\ \text{at} \\ \text{rest} \end{pmatrix} = \begin{pmatrix} \text{particle} \\ \text{at} \\ \text{rest} \end{pmatrix}.$$

Since Galilean transformations form a group $\{\mathbb{1}, -\vec{v}', \vec{0}, 0\}\{R, \vec{V}, \vec{a}, b\}$ is a Galilean transformation which leaves the particle at rest. The most general such Galilean transformation is a product of a rotation and a translation $\Rightarrow$

$$\{\mathbb{1}, -\vec{v}', \vec{0}, 0\}\{R, \vec{V}, \vec{a}, b\}$$

$$= \underbrace{\{R', \vec{0}, \vec{0}, 0\}}_{\text{space Rotation}} \underbrace{\{\mathbb{1}, \vec{0}, \vec{a}', 0\}}_{\text{space translation}} \underbrace{\{\mathbb{1}, \vec{0}, \vec{0}, b'\}}_{\text{time translation}}$$

Thus we have an arbitrary Galilean transformation can be written as the product of a pure velocity boost, a pure space rotation, a pure spatial rotation and a pure time translation

$$\underbrace{\{R, \vec{V}, \vec{a}, b\}}_{\substack{\text{arbitrary} \\ \text{Galilean} \\ \text{transformation}}} = \underbrace{\{\mathbb{1}, \vec{v}', \vec{0}, 0\}}_{\substack{\text{pure velocity} \\ \text{boost}}} \underbrace{\{R, \vec{0}, \vec{0}, 0\}}_{\substack{\text{spatial} \\ \text{Rotation}}} \underbrace{\{\mathbb{1}, \vec{0}, \vec{a}', 0\}}_{\substack{\text{space} \\ \text{translation}}} \underbrace{\{\mathbb{1}, \vec{0}, \vec{0}, b'\}}_{\substack{\text{time} \\ \text{translation}}}$$

Hence the quantum mechanical operator $U(R, \vec{V}, \vec{a}, b)$ representing the Galilean transformation in the Hilbert space can be written as

$$\underbrace{U(R, \vec{V}, \vec{a}, b)}_{\substack{\text{arbitrary} \\ \text{Galilean} \\ \text{transformation}}} = \omega \, \underbrace{U(1, \vec{v}', \vec{0}, 0)}_{\substack{\text{pure velocity} \\ \text{boost}}} \underbrace{U(R, \vec{0}, \vec{0}, 0)}_{\substack{\text{spatial} \\ \text{Rotation}}} \underbrace{U(1, \vec{0}, \vec{a}', 0)}_{\substack{\text{space} \\ \text{translation}}} \times \underbrace{U(1, \vec{0}, \vec{0}, b')}_{\text{time translation}}$$

ω: arbitrary phase factor $|\omega| = 1$.

Since any pure boost, rotation, space or time translation can be written as the square of a pure boost, rotation space or time translation, we have that

$$U(1,\vec{v}',\vec{0},0) = U^2(1,\vec{\bar{v}},\vec{0},0)$$

$$U(R',\vec{0},\vec{0},0) = U^2(\bar{R},\vec{0},\vec{0},0)$$

$$U(1,\vec{0},\vec{a}',0) = U^2(1,\vec{0},\vec{\bar{a}},0)$$

$$U(1,\vec{0},\vec{0},b') = U^2(1,\vec{0},\vec{0},\bar{b}) .$$

Now Wigner's Theorem states that $U(1,\vec{\bar{v}},\vec{0},0)$, $U(\bar{R},\vec{0},\vec{0},0)$, $U(1,\vec{0},\vec{\bar{a}},0)$ and $U(1,\vec{0},\vec{0},\bar{b})$ are either unitary or anti-unitary. But the square of either a unitary or an anti-unitary operator is <u>unitary</u>. Hence the pure velocity boosts, rotations and space and time translation operators are <u>unitary operators</u>.

Since the product of unitary operators is again unitary, an arbitrary Galilean transformation is represented by an unitary operator in Hilbert space i.e.

$$\underline{U(R, \vec{V}, \vec{a}, b) \text{ is unitary}} .$$

Note that this same type of argument can be applied to the operator representing any continuous symmetry transformation to show that it is unitary.

Next we will determine the commutation relations that the generators of the Galilean group obey by studying the group product law. Here we will be careful about the arbitrary phase factors that result from the fact that $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ represent the same physical state. So if we have 2 consecutive Galilean transformations $O \xrightarrow[\{R,\vec{V},\vec{a},b\}]{} O' \xrightarrow[\{\bar{R},\vec{\bar{V}},\vec{\bar{a}},\bar{b}\}]{} O''$

its effect is equivalent up to a phase

to one transformation from $O \xrightarrow[\{\bar{\bar{R}},\vec{\bar{\bar{V}}},\vec{\bar{\bar{a}}},\bar{\bar{b}}\}]{} O''$

with

$$\{\bar{\bar{R}}, \vec{\bar{\bar{V}}}, \vec{\bar{\bar{a}}}, \bar{\bar{b}}\} = \{\bar{R}, \vec{\bar{V}}, \vec{\bar{a}}, \bar{b}\}\{R, \vec{V}, \vec{a}, b\}$$

$$= \{\bar{R}R, \overrightarrow{\bar{R}V+\bar{V}}, \overrightarrow{\bar{R}a+\bar{V}b+\bar{a}}, \bar{b}+b\}.$$

That is the state vectors are equal up to a phase

$$|\psi'\rangle = e^{i\omega}\, U(R,\vec{V},\vec{a},b)\,|\psi\rangle$$

$$|\psi''\rangle = e^{i\bar{\omega}}\, U(\bar{R},\vec{\bar{V}},\vec{\bar{a}},\bar{b})\,|\psi'\rangle$$

but

$$|\psi''\rangle = e^{i\bar{\bar{\omega}}}\, U(\bar{\bar{R}},\vec{\bar{\bar{V}}},\vec{\bar{\bar{a}}},\bar{\bar{b}})\,|\psi\rangle,$$

hence the U's form a representation of the group multiplication law up to a phase

$$U(\bar{R},\vec{\bar{V}},\vec{\bar{a}},\bar{b})\, U(R,\vec{V},\vec{a},b)$$

$$= e^{i\alpha}\, U(\bar{\bar{R}},\vec{\bar{\bar{V}}},\vec{\bar{\bar{a}}},\bar{\bar{b}})$$

with the phase factors

$$\alpha = \alpha(\bar{R}, \vec{\bar{V}}, \vec{\bar{a}}, \bar{b}; R, \vec{V}, \vec{a}, b).$$

In particular $U(1, \vec{0}, \vec{0}, 0) = e^{i\alpha}1$, by convention we choose $\alpha = 0$ so that all phases

$$1 \equiv e^{i\alpha(R, \vec{V}, \vec{a}, b; 1, \vec{0}, \vec{0}, 0)},$$

Further since U is unitary we have

$$U(R, \vec{V}, \vec{a}, b)\, U^{\dagger}(R, \vec{V}, \vec{a}, b) = \underline{1}$$

$$\Rightarrow \quad e^{i\alpha(R, \vec{V}, \vec{a}, b; R^{-1}, -\overrightarrow{R^{-1}V}, \overrightarrow{-R^{-1}a + R^{-1}Vb}, -b)} = 1.$$

Next we consider elements of the Galilean group which differ only infinitesimally from the identity. Hence

$$\begin{aligned} R_{ij} &\equiv \delta_{ij} + \omega_{ij} & \text{with} \quad & |\omega_{ij}| \ll 1 \\ V_i &= \nu_i & & |\nu_i| \ll 1 \\ a_i &= \epsilon_i & & |\epsilon_i| \ll 1 \\ b &= \delta & & |\delta| \ll 1 \end{aligned}$$

where ω_{ij}, ν_i, ϵ_i, δ are all real parameters. Since R is orthogonal, $R^T = R^{-1}$, we have to first order in ω_{ij}

$$(R^{-1})_{ij} = \delta_{ij} - \omega_{ij} = (R^T)_{ij} = \delta_{ij} + \omega_{ji}$$

$\Rightarrow$ $\boxed{\omega_{ij} = -\omega_{ji}}$ The ω_{ij} matrix is an <u>anti-symmetric</u> 3x3 matrix of real parameters. This implies ω_{ij} has only 3 - independent <u>real</u> <u>elements</u>

$$\omega_{ij} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix},$$

So again we have 10 independent infinitesimal real parameters specifying the Galilean transformation

transformation	Infinitesimal Parameter	independent number
Rotation	$\omega_{ij} = -\omega_{ji}$	3
Boost	v_i	3
space-translation	ϵ_i	3
time-translation	δ	1
		10.

Assuming continuity of the physical transformations of our system to we

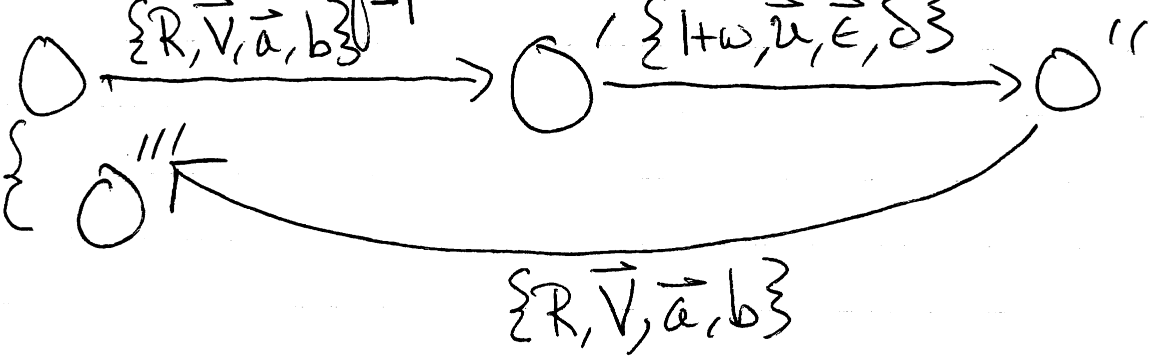
have
$$U(1+\omega,\vec{v},\vec{\epsilon},\delta) \to \mathbb{1} \quad \text{as}$$
$\omega,\vec{v},\vec{\epsilon},\delta \to 0$. But U is unitary so
$$\mathbb{1} = U^\dagger U = [1+(U-1)]^\dagger[1+(U-1)]$$
$$= \mathbb{1} + (U-1)^\dagger + (U-1) \quad \text{, to first order.}$$
$\Rightarrow$
$$U - \mathbb{1} = i\,(\text{Hermitian operator}).$$
Hence for Galilean transformations close to the identity we find

$$U(1+\omega,\vec{v},\vec{\epsilon},\delta) = \mathbb{1} + \frac{i}{2}\omega_{ij}\frac{J_{ij}}{\hbar} + iv_i\frac{K_i}{\hbar} - i\epsilon_i\frac{P_i}{\hbar} + i\delta\frac{H}{\hbar}$$

where $J_{ij} = -J_{ji}$, K_i, P_i, H are the 10 Hermitian operators which generate the Galilean transformations,

$$J_{ij} = J_{ij}^{+}$$
$$K_i = K_i^{+}$$
$$P_i = P_i^{+}$$
$$H = H^{+} \, .$$

Now suppose we consider the product of three transfer motions



O''' is infinitesimally close to O

That is

$$\{R, \vec{V}, \vec{a}, b\}\{1+\omega, \vec{u}, \vec{\epsilon}, \delta\}\{R, \vec{V}, \vec{a}, b\}^{-1}$$

using the group product law

$$= \{R + R\omega, \overrightarrow{Ru + V}, \overrightarrow{R\epsilon + V\delta + a}, b + \delta\} \times$$
$$\times \{R^{-1}, \overrightarrow{-R^{-1}V}, \overrightarrow{-R^{-1}a + R^{-1}Vb}, -b\}$$

$$= \{1 + R\omega R^{-1}, \overrightarrow{-R\omega R^{-1}V + Ru}, \overrightarrow{-R\omega R^{-1}a + R\omega R^{-1}Vb + R\epsilon - Rub + V\delta}, \delta\}.$$

Since this is again a Galilean transformation infinitesimally close to 0 it has infinitesimal parameters

$$\omega' = R\omega R^{-1}$$

$$u' = Ru - R\omega R^{-1}V$$

$$\epsilon' = R\epsilon - Rub - R\omega R^{-1}a + R\omega R^{-1}Vb + V\delta$$

$$\delta' = \delta \quad .$$

The unitary operators representing these transformations obey the same product law up to a phase

$$U(R, \vec{V}, \vec{a}, b)\, U(1+\omega, \vec{u}, \vec{\epsilon}, \delta)\, U^{-1}(R, \vec{V}, \vec{a}, b)$$

$$= e^{i\hat{\alpha}(\omega, \vec{u}, \vec{\epsilon}, \delta; R, \vec{V}, \vec{a}, b)} \times$$

$$\times U(1+\omega', \vec{u}', \vec{a}', \delta')$$

where the phase

$$\hat{\alpha}(\omega,\vec{u},\vec{\epsilon},\delta;R,\vec{V},\vec{a},b)$$
$$= \alpha(R,\vec{V},\vec{a},b;1+\omega,\vec{u},\vec{\epsilon},\delta)$$
$$+\alpha(R+R\omega,\overrightarrow{Ru+V},\overrightarrow{R\epsilon+a},\delta+b;$$
$$R^{-1},-\overrightarrow{R^{-1}V},-\overrightarrow{R^{-1}(a-Vb)},-b)$$

satisfies

$$\hat{\alpha}(0,\vec{0},\vec{0},0;R,\vec{V},\vec{a},b)=\alpha(R,\vec{V},\vec{a},b;\mathbb{1},\vec{0},\vec{0},0)$$
$$+\alpha(R,\vec{V},\vec{a},b;R^{-1},-\overrightarrow{RV},$$
$$-\overrightarrow{R^{-1}(a-Vb)},-b)$$
$$=0.$$

Hence $\hat{\alpha}(\omega,\vec{u},\vec{\epsilon},\delta;R,\vec{V},\vec{a},b)$ can be expanded about 0 to yield

$$e^{+i\hat{\alpha}(\omega,\vec{u},\vec{\epsilon},\delta;R,\vec{V},\vec{a},b)} = 1+\tfrac{i}{2}\omega_{ij}f_{ij}(R,\vec{V},\vec{a},b)$$
$$+iu_i g_i(R,\vec{V},\vec{a},b)-i\epsilon_i r_i(R,\vec{V},\vec{a},b)$$

$$+ i\delta h(R, \vec{V}, \vec{a}, b)$$

where $f_{ij} = -f_{ji}, g_i, r_i, h$ are real numbers. So we find, using

$$U(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = 1 + \frac{i}{2}\frac{\omega_{ij}J_{ij}}{\hbar} + i\frac{u_i K_i}{\hbar} - i\frac{\epsilon_i P_i}{\hbar} + i\frac{\delta H}{\hbar},$$

that

$$U(R, \vec{V}, \vec{a}, b)\left[1 + \frac{i}{2}\frac{1}{\hbar}\omega_{ij}J_{ij} + i\frac{u_i}{\hbar}K_i - i\frac{\epsilon_i}{\hbar}P_i + i\frac{\delta}{\hbar}H\right] \times U^{-1}(R, \vec{V}, \vec{a}, b)$$

$$= e^{i\hat{\alpha}}\, U(1+\omega', \vec{u}', \vec{a}', \delta')$$

$$= \left(1 + \frac{i}{2}\omega_{ij}f_{ij} + iu_i g_i - i\epsilon_i r_i + i\delta h\right) \times$$

$$\times \Big(\mathbb{1} + \frac{i}{2}\frac{1}{\hbar}(R\omega R^{-1})_{ij}J_{ij} + i\frac{1}{\hbar}(Ru - R\omega R^{-1}V)_i K_i$$
$$- i\frac{1}{\hbar}(R\epsilon - Rub - R\omega R^{-1}a + R\omega R^{-1}Vb + V\delta)_i P_i$$
$$+ i\frac{1}{\hbar}\delta H\Big)$$

That is to first order in $\omega, \vec{u}, \vec{\epsilon}, \delta$

$$U(R,\vec{V},\vec{a},b)\left[\tfrac{1}{2}\omega_{ij}J_{ij} + u_i K_i - \epsilon_i P_i + \delta H\right]U^{-1}(R,\vec{V},\vec{a},b)$$

$$= \tfrac{1}{2}\hbar\omega_{ij}f_{ij} + \hbar u_i g_i - \hbar\epsilon_i r_i + \hbar\delta h$$
$$+ \tfrac{1}{2}R_{li}R^{-1}_{jm}\omega_{ij}J_{lm} + (R_{li}u_i - R_{li}R^{-1}_{jm}V_m\omega_{ij})K_l$$

$$-(R_{li}\epsilon_i - R_{li}u_i b - R_{li}R^{-1}_{jm}a_m\omega_{ij}$$
$$+ R_{li}R^{-1}_{jm}\omega_{ij}V_m b + V_l\delta)P_l$$
$$+ \delta H \; .$$

Gathering terms we find

$$U(R,\vec{V},\vec{a},b)\left[\tfrac{1}{2}\omega_{ij}J_{ij}+u_iK_i-\epsilon_iP_i+\delta H\right]U^{-1}(R,\vec{V},\vec{a},b)$$

$$= \tfrac{1}{2}\omega_{ij}\Big[\hbar f_{ij} + \tfrac{1}{2}(R_{li}R^{-1}_{jm}-R_{lj}R^{-1}_{im})J_{lm}$$
$$-(R_{li}R^{-1}_{jm}-R_{lj}R^{-1}_{im})V_m K_l$$
$$+(R_{li}R^{-1}_{jm}-R_{lj}R^{-1}_{im})a_m P_l$$
$$+(R_{li}R^{-1}_{jm}-R_{lj}R^{-1}_{im})V_m b P_l\Big]$$

$$+u_i\left[\hbar g_i + R_{li}K_l + R_{li}bP_l\right]$$

$$-\epsilon_i\left[\hbar r_i + R_{li}P_l\right]$$
$$+\delta\left[\hbar h - V_lP_l + H\right] \quad .$$

Since ω_{ij}, u_i, ϵ_i, δ are independent parameters, we identify their coefficients of the left and right hand sides (Recall $R^{-1}_{jm} = R^T_{jm} = R_{mj}$)

1) $\frac{1}{2}\omega_{ij}$:

$$U(R,\vec{V},\vec{a},b)\, J_{ij}\, U^{-1}(R,\vec{V},\vec{a},b)$$
$$= \hbar f_{ij}(R,\vec{V},\vec{a},b) + R_{li}R_{mj}\big(J_{lm}$$
$$- V_m K_l + V_l K_m + a_m P_l - a_l P_m$$
$$+ b V_m P_l - b V_l P_m\big)$$

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2) u_i :

$$U(R,\vec{V},\vec{a},b)\, K_i\, U^{-1}(R,\vec{V},\vec{a},b)$$
$$= \hbar g_i(R,\vec{V},\vec{a},b) + K_l R_{li} + b P_l R_{li}$$

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3) ϵ_i :

$$U(R,\vec{V},\vec{a},b)\, P_i\, U^{-1}(R,\vec{V},\vec{a},b)$$
$$= \hbar r_i(R\vec{V},\vec{a},b) + P_l R_{li}$$

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4) δ : $U(R,\vec{V},\vec{a},b)\, H\, U^{-1}(R\vec{V},\vec{a},b)$

$$= \hbar h - V_l P_l + H .$$

The Galilean transformation $\{R, \vec{V}, \vec{a}, b\}$ is completely arbitrary; suppose we choose it to be infinitesimal. Let

$$\{R, \vec{V}, \vec{a}, b\} = \{1+\omega, \vec{u}, \vec{\epsilon}, \delta\}$$, then

$$U(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = 1 + \frac{i}{2}\omega_{mn}\frac{J_{mn}}{\hbar} + i u_n \frac{K_n}{\hbar} - i\epsilon_n \frac{P_n}{\hbar} + i\delta\frac{H}{\hbar}$$

while

$$U^{-1}(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = U(1-\omega, -\vec{u}, -\vec{\epsilon}, -\delta)$$

$$= 1 - \frac{i}{2}\omega_{mn}\frac{J_{mn}}{\hbar} - i u_n \frac{K_n}{\hbar} + i\epsilon_n \frac{P_n}{\hbar} - i\delta\frac{H}{\hbar}$$

Thus for any operator $\mathcal{O}$ we have

$$U(1+\omega, \vec{u}, \vec{\epsilon}, \delta)\,\mathcal{O}\,U^{-1}(1+\omega, \vec{u}, \vec{\epsilon}, \delta)$$

$$= \mathcal{O} + \frac{i}{2}\omega_{mn}\frac{1}{\hbar}[J_{mn}, \mathcal{O}] + i u_n \frac{1}{\hbar}[K_n, \mathcal{O}] - i\epsilon_n \frac{1}{\hbar}[P_n, \mathcal{O}] + i\delta\frac{1}{\hbar}[H, \mathcal{O}] .$$

Further the phase angles f_{ij}, g_i, r_i, h can be Taylor expanded about 0

$$f_{ij}(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = \tfrac{1}{2}\omega_{mn} f_{ijmn} + u_n f_{ijn} - \epsilon_n f'_{ijn} + \delta f''_{ij}$$

$$g_i(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = \tfrac{1}{2}\omega_{mn} g_{imn} + u_n g_{in} - \epsilon_n g'_{in} + \delta g''_i$$

$$r_i(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = \tfrac{1}{2}\omega_{mn} r_{imn} + u_n r_{in} - \epsilon_n r'_{in} + \delta r''_i$$

$$h(1+\omega, \vec{u}, \vec{\epsilon}, \delta) = \tfrac{1}{2}\omega_{mn} h_{mn} + u_n h_n - \epsilon_n h'_n + \delta h''$$

with all quantities real numbers and with obvious symmetry properties

$$f_{ijmn} = -f_{jimn} = -f_{ijnm}$$

$$f_{ijn} = -f_{jin}$$

$$f'_{ijn} = -f'_{jin}$$

$$f''_{ij} = -f''_{ji}$$

$$g_{imn} = -g_{inm}$$

$$r_{imn} = -r_{inm}$$

$$h_{mn} = -h_{nm} .$$

Substituting all of these expansions into the 4 equations on page -573- yields

$\frac{1}{2}\omega_{ij}$ 1) $U(1+\omega,\vec{u},\vec{\epsilon},\delta)\,J_{ij}\,U^{-1}(1+\omega,\vec{u},\vec{\epsilon},\delta)$

$$= J_{ij} + \frac{i}{2}\omega_{mn}\frac{1}{\hbar}[J_{mn},J_{ij}] + iu_n\frac{1}{\hbar}[K_n,J_{ij}] - i\epsilon_n\frac{1}{\hbar}[P_n,J_{ij}] + i\delta\frac{1}{\hbar}[H,J_{ij}]$$

$$= J_{ij} + \omega_{mn}\left[\delta_{in}J_{mj} + \delta_{jn}J_{im} + \frac{\hbar}{2}f_{ij\,mn}\right] - u_n\left[K_i\delta_{nj} - K_j\delta_{ni} - \hbar f_{ijn}\right] + \epsilon_n\left[P_i\delta_{nj} - P_j\delta_{ni} - \hbar f'_{ijn}\right] + \delta\left[\hbar f''_{ij}\right]$$

Since $\omega_{mn} = -\omega_{nm}$ we anti-symmetrize the first term. Equating ~~the~~ the coefficients of ~~the~~ independent parameters ω_{ij}, u_i, ϵ_i, δ; we obtain the 4 commutation relations

$$[J_{mn}, J_{ij}] = -i\hbar(\delta_{in} J_{mj} + \delta_{jn} J_{im} - \delta_{im} J_{nj} - \delta_{jm} J_{in}) - i\hbar^2 f_{ijmn}$$

$$[K_n, J_{ij}] = i\hbar(K_i \delta_{nj} - K_j \delta_{ni}) - i\hbar^2 f_{ijn}$$

$$[P_n, J_{ij}] = i\hbar(P_i \delta_{nj} - P_j \delta_{ni}) - i\hbar^2 f'_{ijn}$$

$$[H, J_{ij}] = -i\hbar^2 f''_{ij}$$

Proceding similiarly with the K_i equation

x_i 2) $$U(1+\omega, \vec{u}, \vec{\epsilon}, \delta) K_i U^{-1}(1+\omega, \vec{u}, \vec{\epsilon}, \delta)$$

$$= K_i + \frac{i}{2}\omega_{mn}\frac{1}{\hbar}[J_{mn}, K_i] + iu_n\frac{1}{\hbar}[K_n, K_i] - i\epsilon_n\frac{1}{\hbar}[P_n, K_i] + i\delta\frac{1}{\hbar}[H, K_i]$$

$$= K_i + \frac{1}{2}\omega_{mn}(\delta_{in} K_m - \delta_{im} K_n + \hbar g_{imn}) + u_n(\hbar g_{in}) - \epsilon_n(\hbar g'_{in}) + \delta(P_i + \hbar g''_i)$$

Which yields the 4 commutators

$$[J_{mn}, K_i] = -i\hbar(\delta_{in}K_m - \delta_{im}K_n) - i\hbar^2 g_{imn}$$

$$[K_n, K_i] = -i\hbar^2 g_{in}$$

$$[P_n, K_i] = -i\hbar^2 g'_{in}$$

$$[H, K_i] = -i\hbar P_i - i\hbar^2 g''_i$$

Next we have

ϵ_i 3) $U(1+\omega, \vec{u}, \vec{\epsilon}, \delta) P_i U^{-1}(1+\omega, \vec{u}, \vec{\epsilon}, \delta)$

$$= P_i + \frac{i}{2}\omega_{mn}\frac{1}{\hbar}[J_{mn}, P_i] + iu_n\frac{1}{\hbar}[K_n, P_i] - i\epsilon_n\frac{1}{\hbar}[P_n, P_i] + i\delta\frac{1}{\hbar}[H, P_i]$$

$$= P_i + \frac{1}{2}\omega_{mn}(\delta_{in}P_m - \delta_{im}P_n + \hbar r_{imn}) + u_n\hbar r_{in} - \epsilon_n\hbar r'_{in} + \delta\hbar r''_i$$

Which yields

$[J_{mn}, P_i] = -i\hbar(\delta_{in}P_m - \delta_{im}P_n) - i\hbar^2 r_{imn}$

$[K_n, P_i] = -i\hbar^2 r_{in}$

$[P_n, P_i] = -i\hbar^2 r'_{in}$

$[H, P_i] = -i\hbar^2 r''_i$

And finally

δ 4) $U(1+\omega, \vec{u}, \vec{\epsilon}, \delta) H U^{-1}(1+\omega, \vec{u}, \vec{\epsilon}, \delta)$

$= H + \frac{i}{2}\omega_{mn}\frac{1}{\hbar}[J_{mn}, H]$

$+ iu_n \frac{1}{\hbar}[K_n, H] - i\epsilon_n \frac{1}{\hbar}[P_n, H]$

$+ i\delta\frac{1}{\hbar}[H, H]$

$= H + \frac{1}{2}\omega_{mn}(\hbar h_{mn})$

$+ u_n(-P_n + \hbar h_n) - \epsilon_n \hbar h'_n + \delta\hbar h''$

which yields

$$[J_{mn}, H] = -i\hbar^2 h_{mn}$$

$$[K_n, H] = +i\hbar P_n \;\; -i\hbar^2 h_n$$

$$[P_n, H] = -i\hbar^2 h'_n$$

$$[H, H] = -i\hbar^2 h''$$

We first note some consistency requirements that are straightforward

$$[K_n, J_{ij}] = -[J_{ij}, K_n] \Rightarrow f_{ijn} = -g_{nij} = g_{nji}$$

$$[P_n, J_{ij}] = -[J_{ij}, P_n] \Rightarrow f'_{ijn} = -r_{nij} = r_{nji}$$

$$[H, J_{ij}] = -[J_{ij}, H] \Rightarrow f''_{ij} = -h_{ij} = h_{ji}$$

$$[P_n, K_i] = -[K_i, P_n] \Rightarrow g'_{in} = -r_{ni}$$

$$[H, K_i] = -[K_i, H] \Rightarrow g''_i = -h_i$$

$$[H, P_i] = -[P_i, H] \Rightarrow r''_i = -h'_i$$

$$[H, H] = 0 \quad \Rightarrow h'' = 0$$

$$[K_i, K_j] = -[K_j, K_i] \Rightarrow g_{in} = -g_{ni}$$

$$[P_i, P_j] = -[P_j, P_i] \Rightarrow r'_{in} = -r'_{ni} .$$

Further since $J_{ij} = -J_{ji}$ there are only 3 independent J_{ij} operators; define

$$J_1 \equiv J_{23}$$

$$J_2 \equiv J_{31}$$

$$J_3 \equiv J_{12}$$

that is

$$J_i \equiv \tfrac{1}{2} \epsilon_{ijk} J_{jk}$$

where ϵ_{ijk} is the anti-symmetric permutation tensor (Levi-Civita symbol)

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an (cyclic) even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Using

$$\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$$

we have

$$J_{ij} = \epsilon_{ijk} J_k .$$

The commutation relations become

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk}(J_k + F_k)$$

$$[J_i, K_j] = i\hbar\,\epsilon_{ijk}(K_k + G_k)$$

$$[J_i, P_j] = i\hbar\,\epsilon_{ijk}(P_k + R_k)$$

$$[J_i, H] = i\hbar H_i$$

$$[K_i, K_j] = i\hbar\,\epsilon_{ijk}\tilde{G}_k$$

$$[K_i, P_j] = i\hbar M_{ij}$$

$$[K_i, H] = +i\hbar(P_i + \tilde{H}_i)$$

$$[P_i, P_j] = i\hbar\,\epsilon_{ijk}\tilde{R}_k$$

$$[P_i, H] = i\hbar H'_i \quad , \quad [H, H] = 0,$$

where $F_k, G_k, R_k, H_i, \tilde{G}_k, M_{ij}, \tilde{H}_i, \tilde{R}_k, H'_i$ are all real numbers related to the arbitrary phases by

$$F_k \equiv -\frac{\hbar}{8} \epsilon_{ijk} \epsilon_{ipl} \epsilon_{jmn} f_{mnpl} = -\frac{\hbar}{4} \epsilon_{jmn} f_{mnjk}$$

$$G_k \equiv -\frac{\hbar}{4} \epsilon_{ijk} \epsilon_{ilm} g_{jlm} = -\frac{\hbar}{2} g_{iik}$$

$$R_k \equiv -\frac{\hbar}{4} \epsilon_{ijk} \epsilon_{ilm} r_{jlm} = -\frac{\hbar}{2} r_{iik}$$

$$H_i \equiv -\frac{\hbar}{2} \epsilon_{ijk} h_{jk}$$

$$\tilde{G}_k \equiv -\frac{\hbar}{2} \epsilon_{ijk} g_{ji}$$

$$M_{ij} \equiv -\hbar r_{ji}$$

$$\tilde{H}_i \equiv \hbar h_i$$

$$\tilde{R}_k \equiv -\frac{\hbar}{2} \epsilon_{ijk} r'_{ji}$$

$$H'_i \equiv -\hbar h'_i \quad .$$

Finally we apply a more subtle consistency requirement, that of the Jacobi identity.

The Jacobi identity is cyclic identity

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

for any A, B, C.

1) So for $A = J_i$, $B = H$, $C = P_j$ we get

$$0 = [J_i,[H,P_j]] + [P_j,[J_i,H]] + [H,[P_j,J_i]]$$

$$= [J_i, -i\hbar H_j'] + [P_j, i\hbar H_i'] + [H, -i\hbar \epsilon_{ijk}(P_k + R_k)]$$

(first two terms $\to 0$)

$$\Rightarrow \quad 0 = [H, -i\hbar \epsilon_{ijk} P_k] = -\hbar^2 \epsilon_{ijk} H_k'$$

$$\Rightarrow \boxed{H_k' = 0}$$

2) For $A = J_i$, $B = H$, $C = J_j$

$$0 = [J_i,[H,J_j]] + [J_j,[J_i,H]] + [H,[J_j,J_i]]$$

$$\Rightarrow 0 = [J_i, -i\hbar H_j] + [J_j, i\hbar H_i] + [H, -i\hbar \epsilon_{ijk}(J_k + Y_k)]$$

(struck-out terms $\to 0$)

$$\Rightarrow \quad -i\hbar \epsilon_{ijk}[H, J_k] = 0 = -\hbar^2 \epsilon_{ijk} H_k$$

$$\Rightarrow \boxed{H_k = 0}$$

3) For $A = P_i,\ B = H,\ C = K_j$

$$0 = [P_i,[H,K_j]] + [K_j,[P_i,H]] + [H,[K_j,P_i]]$$

$$= [P_i, -i\hbar(P_j + \cancelto{0}{\tilde{H}_j})] + [K_j, \cancelto{0}{i\hbar H'_i}] + [H, \cancelto{0}{+i\hbar M_{ji}}]$$

$$\Rightarrow \quad [P_i,P_j] = 0 = i\hbar\epsilon_{ijk}\tilde{R}_k$$

$$\Rightarrow \boxed{\tilde{R}_k = 0}$$

4) For $A = J_i,\ B = K_j,\ C = K_k$

$$0 = [J_i,[K_j,K_k]] + [K_k,[J_i,K_j]] + [K_j,[K_k,J_i]]$$

$$= [J_i, \cancelto{0}{i\epsilon_{jkl}\tilde{G}_l}] + [K_k, i\hbar\epsilon_{ijl}(K_l + \cancel{G_l})]$$

$$+ [K_j, -i\hbar\epsilon_{ikl}(K_l + \cancelto{0}{G_l})]$$

$\Rightarrow$

$$0 = i\hbar\epsilon_{ijl}[K_k,K_l] - i\hbar\epsilon_{ikl}[K_j,K_l]$$

$$= -\hbar^2\epsilon_{ijl}\epsilon_{klm}\tilde{G}_m + \hbar^2\epsilon_{ikl}\epsilon_{jlm}\tilde{G}_m$$

$$= +\hbar^2(\delta_{ik}\delta_{jm} - \cancel{\delta_{im}\delta_{jk}} - \delta_{ij}\delta_{km} + \cancel{\delta_{im}\delta_{kj}})\tilde{G}_m$$

$$= \hbar^2(\delta_{ik}\tilde{G}_j - \delta_{ij}\tilde{G}_k) \Rightarrow \boxed{\tilde{G}_k = 0}.$$

Further, we can re-define J_i, K_i, P_i by adding to them the constants F_i, G_i, R_i respectively. This will still leave them as Hermitian and will not affect the consistency checks so far. So letting

$$J_i \rightarrow J_i' = J_i + F_i$$

$$K_i \rightarrow K_i' = K_i + G_i$$

$$P_i \rightarrow P_i' = P_i + R_i$$

The algebra becomes

$$[J_i', J_j'] = i\hbar \epsilon_{ijk} J_k'$$

$$[J_i', K_j'] = i\hbar \epsilon_{ijk} K_k'$$

$$[J_i', P_j'] = i\hbar \epsilon_{ijk} P_k'$$

$$[J_i', H] = 0$$

$$[K_i', K_j'] = 0 \quad , \quad [K_i', P_j'] = i\hbar M_{ij}$$

$$[K_i', H] = +i\hbar (P_i' + (\tilde{H}_i - R_i))$$

$$[P_i', P_j'] = 0 \; ; \; [P_i', H] = 0 \; , \; [H, H] = 0$$

Finally applying the Jacobi identity once ~~more~~ ~~for~~ $A=H'$, $B=K_i'$, $C=J_j'$ we have

5) $$0=[H,[K_i',J_j']]+[J_j',[H,K_i']]+[K_i',[J_j',H]]$$

$$=[H,-i\hbar\epsilon_{jik}K_k']+[J_j',-i\hbar(P_i'+(\tilde{H}_i-\tilde{R}_i))]+[K_i',0]$$

(with $(\tilde{H}_i-\tilde{R}_i)$ and $[K_i',0]$ struck through and marked $\to 0$)

$\Rightarrow$

$$0=+i\hbar\epsilon_{jik}\,i\hbar\left(P_k'+(\tilde{H}_k-\tilde{R}_k)\right)-i\hbar\,i\hbar\epsilon_{jik}P_k'$$

(the P_k' terms struck through)

$$\Rightarrow\quad \boxed{(\tilde{H}_k-\tilde{R}_k)=0}$$

6) And for $A=P_i'$, $B=K_j'$, $C=J_k'$

$$0=[P_i',[K_j',J_k']]+[J_k',[P_i',K_j']]+[K_j',[J_k',P_i']]$$

$$=[P_i',-i\hbar\epsilon_{kjl}K_l']+[J_k',-i\hbar M_{ji}]+[K_j',i\hbar\epsilon_{kil}P_l']$$

(with $[J_k',-i\hbar M_{ji}]$ struck through and marked $\to 0$)

$$=-i\hbar\epsilon_{kjl}(-i\hbar)M_{li}+i\hbar\epsilon_{kil}\,i\hbar M_{jl}$$

$$\Rightarrow\quad 0=-\hbar^2(\epsilon_{kjl}M_{li}+\epsilon_{kil}M_{jl})$$

Now for $i=1, j=2, k=3 \Rightarrow -M_{11}+M_{22}=0$

$i=2, j=3, k=1 \Rightarrow -M_{22}+M_{33}=0$

$i=1, j=1, k=2 \Rightarrow -M_{31}-M_{13}=0$

$i=3, j=1, k=1 \Rightarrow -M_{12}=0$

$i=2, j=1, k=1 \Rightarrow M_{13}=0$

$i=1, j=3, k=3 \Rightarrow M_{32}=0$

$i=1, j=2, k=2 \Rightarrow -M_{23}=0$

$i=3, j=2, k=2 \Rightarrow M_{21}=0$

$$\Rightarrow \boxed{M_{ij}=m\,\delta_{ij}\ ,\ m=\text{constant}}$$

Thus dropping the primes on J_i, K_i, P_i, we obtain the final form of the algebra of the generators of the Galilean group

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\hbar \epsilon_{ijk} K_k$$

$$[J_i, P_j] = i\hbar \epsilon_{ijk} P_k$$

$$[J_i, H] = 0$$

$$[K_i, K_j] = 0$$

$$[K_i, P_j] = +i\hbar m \delta_{ij}$$

$$[K_i, H] = +i\hbar P_i$$

$$[P_i, P_j] = 0$$

$$[P_i, H] = 0$$

$$[H, H] = 0$$

Note, if we assumed the representation to be true (not a representation up to a phase), the commutation relations would be the same except $[K_i, P_j] = 0$ instead of $= i\hbar m \delta_{ij}$. Thus for the

Galilean group the phases are non-trivial.

Consequences of the Galilean group multiplication law:

1) Consider the transformation law for the position operator. This can be found from considering a particle in O at position $\vec{r}$ at time t which in O' is at $\vec{r}' = R\vec{r} + \vec{V}t + \vec{a}$ at time $t' = t + b$. In particular the wavefunction ψ' in O' at $(\vec{r}', t')$ is the same as the wavefunction in O, ψ, at $\vec{r}$ and t.

$$\langle \vec{r}', t' | \psi' \rangle = \psi'(\vec{r}', t')$$

$$= \langle \vec{r}, t | \psi \rangle = \psi(\vec{r}, t)$$

(i.e. if $\psi(\vec{r}, t)$ has a peak at $\vec{r}_0$ at t_0 as observed by O, then O' observes the peak in $\psi'(\vec{r}', t')$ at $\vec{r}_0'$ and t_0'.)

Hence

$$\psi'(\vec{r}, t) = \psi(R^{-1}(\vec{r} - \vec{V}(t-b) - \vec{a}), t - b)$$

This is just

$$\psi'(\vec{r},t) = \langle \vec{r},t|\psi'\rangle = \langle \vec{r},t|e^{i\alpha}\mathcal{U}(R,\vec{v},\vec{a},b)|\psi\rangle$$

$$= \psi(R^{-1}(\vec{r}-\vec{V}(t-b)-\vec{a}),t-b)$$

$$= \langle R^{-1}(\vec{r}-\vec{V}(t-b)-\vec{a}),t-b|\psi\rangle$$

$\Rightarrow$

$$\boxed{e^{-i\alpha}\mathcal{U}^{-1}(R,\vec{V},\vec{a},b)|\vec{r},t\rangle = |R^{-1}(\vec{r}-\vec{V}(t-b)-\vec{a}),t-b\rangle}$$

That is simply

$$|\vec{r}',t'\rangle = e^{+i\alpha}\mathcal{U}(R,\vec{V},\vec{a},b)|\vec{r},t\rangle .$$

The position operator on this state yields

$$\vec{R}(t-b)\,\mathcal{U}^{-1}(R,\vec{V},\vec{a},b)|\vec{r},t\rangle$$

$$= R^{-1}(\vec{r}-\vec{V}(t-b)-\vec{a})\,\mathcal{U}^{-1}(R,\vec{V},\vec{a},b)|\vec{r},t\rangle$$

$\Rightarrow$

$$U(R,\vec{V},\vec{a},b)\vec{R}(t-b)\,U^{-1}(R,\vec{V},\vec{a},b)\;|\vec{r},t\rangle$$
$$= R^{-1}(\vec{r}-\vec{V}(t-b)-\vec{a})\,|\vec{r},t\rangle$$

but $\vec{r}\,|\vec{r},t\rangle = \vec{R}(t)\,|\vec{r},t\rangle$; then

$\Rightarrow$

$$U(R,\vec{V},\vec{a},b)\vec{R}(t-b)\,U^{-1}(R,\vec{V},\vec{a},b) = R^{-1}(\vec{R}(t)-\vec{V}(t-b)-\vec{a})$$

relabelling $t-b \rightarrow t$ we have

$$U(R,\vec{V},\vec{a},b)\,X_i(t)\,U^{-1}(R,\vec{V},\vec{a},b) = R^{-1}_{ij}\left(X_j(t+b)-V_j t - a_j\right).$$

For infinitesimal Galilean transformation $R = 1+\omega$, $\vec{V}=\vec{u}$, $\vec{a}=\vec{\epsilon}$, $b=\delta$ we have commutation relations on the LHS while we have Taylor expansions on the RHS

$$X_i(t) + \frac{i}{2}\frac{\omega_{mn}}{\hbar}[J_{mn}, X_i(t)] + i\frac{u_n}{\hbar}[K_n, X_i(t)]$$

$$- i\epsilon_n \frac{1}{\hbar}[P_n, X_i(t)] + i\delta\frac{1}{\hbar}[H, X_i(t)]$$

$$= X_i(t) + \delta\frac{d}{dt}X_i(t) - \omega_{ij}X_j(t)$$

$$- u_i t - \epsilon_i$$

Equating the coefficients of the independent ω_{ij}, u_i, ϵ_i, δ we find

ω_{mn} 1) $[J_{mn}, X_i(t)] = i\hbar(\delta_{mi}X_n(t) - \delta_{ni}X_m(t))$

$\Longrightarrow$

$$[J_i, X_j(t)] = \hbar i\epsilon_{ijk}X_k(t).$$

Thus $\vec{R}(t)$ is a vector operator under spatial rotations.

u_n 2) $[K_n, X_i(t)] = +i\hbar\delta_{in}t$

ϵ_n 3) $[P_n, X_i(t)] = -i\hbar\delta_{in}$, the canonical commutation relation; hence we identify P_i with the momentum operator.

δ 4) $[H, X_i(t)] = -i\hbar \frac{d}{dt} X_i(t)$,

the Heisenberg equations of motion, hence we identify H with the Hamiltonian and confirm that in the passive view we are in the Heisenberg picture.

In the active view we move the system and apparatus. Then state $|\psi'\rangle$ in that view corresponds to the state $|\psi\rangle$ translated in time the opposite way, $t = -b$

$$|\psi'\rangle \equiv |\psi(t)\rangle = U(\mathbb{1}, \vec{0}, \vec{0}, -t)|\psi(0)\rangle$$

(since $[H,H]=0$) $= e^{-iHt/\hbar}|\psi(0)\rangle$, again since $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$ we can identify H as the Hamiltonian, the generator of time translations.

2) If a physical system is Galilean invariant then the Galilean algebra for it is true. For a single particle the commutators

$[P_i, P_j] = 0$, $[H, \vec{P}] = 0$ implies that

$H = H(\vec{P})$ only since $[P_i, X_j(t)] = -i\hbar\delta_{ij}$.

Also we have $[K_i, P_j] = i\hbar m \delta_{ij}$ so

$$[K_i, H] = \frac{\partial H}{\partial P_j} \underbrace{[K_i, P_j]}_{= i\hbar m\delta_{ij}} = \hbar i m \frac{\partial H}{\partial P_i}$$

$\| \quad +i\hbar P_i$

$$\Rightarrow \quad \frac{\partial H}{\partial P_i} = +\frac{P_i}{m} \Rightarrow \boxed{H = \frac{\vec{P}^2}{2m}},$$

as we argued earlier. Further we now identify <u>m</u> with the <u>mass</u> of the particle.

Assuming that $\vec{R} = \vec{R}(\vec{P}, \vec{K})$ we have

$$[K_k, X_i] = \frac{\partial X_i}{\partial P_j} \underbrace{[K_k, P_j]}_{= i\hbar m\delta_{kj}} = i\hbar m \frac{\partial X_i}{\partial P_k}$$

$\| \quad i\hbar\delta_{ik} t$ from 1)

$$\Rightarrow \quad \boxed{\frac{\partial X_i}{\partial P_k} = \frac{t}{m}\delta_{ik}}$$

Also $[P_k, X_i] = \frac{\partial X_i}{\partial K_j}\underbrace{[P_k, K_j]}_{-i\hbar m\delta_{kj}} = -i\hbar m\frac{\partial X_i}{\partial K_k}$

$\quad\quad\quad \| $

$\quad -i\hbar\delta_{ki}$ from ①

$$\Rightarrow \quad \boxed{\frac{\partial X_i}{\partial K_k} = \frac{1}{m}\delta_{ik}}$$

These imply that $\boxed{X_i(t) = \frac{K_i + P_i t}{m}}$.

As noted earlier, a single particle moving in a central potential with the Hamiltonian $H = \frac{1}{2m}\vec{P}^2 + V(|\vec{R}|)$ is not Galilean invariant. In particular

$$[P_i, H] = [P_i, V(|\vec{R}|)] \neq 0,$$

which is a violation of the Galilean algebra. The reason this system is not Galilean invariant is that the Hamiltonian does not describe a closed system. The central potential must have a source which is neglected in the Hamiltonian. When the appropriate dynamics of the source of the central potential is accounted for the Galilean invariance will be restored. For example, the 2-body Hamiltonian

$$H = \frac{1}{2m_1}\vec{P}_1^{\,2} + \frac{1}{2m_2}\vec{P}_2^{\,2} + V(|\vec{R}_1 - \vec{R}_2|)$$

is Galilean invariant. For instance in the coordinate basis

$$[\vec{P}, H] = [\vec{P}_1 + \vec{P}_2, H]$$

$$= [\vec{P}_1 + \vec{P}_2, V(|\vec{r}_1 - \vec{r}_2|)]$$

$$= -i\hbar\,(\vec{\nabla}_{\vec{r}_1} + \vec{\nabla}_{\vec{r}_2})\,V(|\vec{r}_1 - \vec{r}_2|)$$

$$= -i\hbar\,(\vec{\nabla}_{\vec{r}_1} - \vec{\nabla}_{\vec{r}_1})\,V(|\vec{r}_1 - \vec{r}_2|)$$

$$= 0.$$

For a system to be Galilean invariant, the potential must be a function of coordinate differences and not depend on an arbitrary origin or center of force.

Indeed for several particles we can proceed analogously by introducing corresponding momenta $\vec{P}_1, \ldots, \vec{P}_N$ and coordinates $\vec{R}_1, \ldots, \vec{R}_N$ for N particles with masses $m_1, \ldots, m_N$. As usual the canonical commutation relations are

$$[X_{ni}, P_{mj}] = i\hbar \delta_{mn} \delta_{ij} \qquad \begin{matrix} m,n = 1,2,3. \\ i,j = 1,\ldots,N, \end{matrix}$$

all other commutators vanishing.

Then the Galilean algebra is satisfied by

$$\vec{P} = \sum_{i=1}^{N} \vec{P}_i$$

$$\vec{K} = \sum_{i=1}^{N} m_i \vec{R}_i - \vec{P} t$$

and

$$H = \sum_{i=1}^{N} \frac{1}{2m_i} \vec{P}_i^{\,2} + V(\vec{R}_i - \vec{R}_j, \vec{P}_i - \vec{P}_j)$$

with V a scalar function of its arguments.

3) Finally let's construct the unitary operators representing the Galilean transformations from the algebraic properties of the generators. Consider first the subgroup of spatial translations.

1) Spatial translations are given by the operator $U(1,\vec{0},\vec{a},0)$.

For infinitesimal translation vectors $\vec{\epsilon}$ we had that

$$U(1,\vec{0},\vec{\epsilon},0) = 1 - i\epsilon_i \frac{1}{\hbar} P_i = 1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}$$

hence we find

$$[U(1,\vec{0},\vec{\epsilon},0)]^2 = 1 - \frac{2i}{\hbar}\vec{\epsilon}\cdot\vec{P} = U(1,\vec{0},2\vec{\epsilon},0).$$

For N very large and eventually letting $N \to \infty$ we have for $\vec{\epsilon} = \frac{1}{N}\vec{a}$

$$[U(1,\vec{0},\frac{\vec{a}}{N},0)]^2 = U(1,\vec{0},\frac{2\vec{a}}{N},0).$$

Continuing to make successive infinitesimal transformations we find

$$\lim_{N\to\infty}[U(1,\vec{0},\frac{\vec{a}}{N},0)]^N = \lim_{N\to\infty} U(1,\vec{0},\frac{N\vec{a}}{N},0)$$
$$= U(1,\vec{0},\vec{a},0).$$

On the other hand

$$\lim_{N\to\infty}\left[\mathcal{U}(1,\vec{0},\tfrac{\vec{a}}{N},0)\right]^N = \lim_{N\to\infty}\left[1-\frac{i}{\hbar}\frac{1}{N}\vec{a}\cdot\vec{P}\right]^N$$

Since $[P_i,P_j]=0$ this is just the exponential function

$$= e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}.$$

Hence the spatial translation operator is

$$\boxed{\mathcal{U}(1,\vec{0},\vec{a},0)=e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}}.$$

Further

$$\mathcal{U}(1,\vec{0},\vec{\bar{a}},0)\,\mathcal{U}(1,\vec{0},\vec{a},0)$$

$$= e^{-\frac{i}{\hbar}\vec{\bar{a}}\cdot\vec{P}}\cdot e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$$

$$= e^{-\frac{i}{\hbar}(\vec{\bar{a}}+\vec{a})\cdot\vec{P}}$$

since $[P_i,P_j]=0$.

So

$$\boxed{U(1,\vec{0},\vec{\bar{a}},0)\,U(1,\vec{0},\vec{a},0) = U(1,\vec{0},\vec{\bar{a}}+\vec{a},0)}$$

The spatial translation operators form an abelian subgroup of transformations.

Recall that in general

$$U(1,\vec{0},\vec{\bar{a}},0)U(1,\vec{0},\vec{a},0) = e^{i\alpha(1,\vec{0},\vec{\bar{a}},0;1,\vec{0},\vec{a},0)}\,U(1,\vec{0},\vec{\bar{a}}+\vec{a},0),$$

hence we find that $\alpha(1,\vec{0},\vec{\bar{a}},0;1,\vec{0},\vec{a},0)=0$ since $[P_i,P_j]=0$.

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2) Time translations are given by the operator

$$U(1,\vec{0},\vec{0},b).$$

Since $[H,H]=0$ we proceed analogously to the spatial translation case

$$U(1,\vec{0},\vec{0},b) = \lim_{N\to\infty}\left[U(1,\vec{0},\vec{0},\tfrac{b}{N})\right]^N$$

$$= \lim_{N\to\infty}\left[1+\frac{i}{\hbar}\frac{b}{N}H\right]^N$$

$$\boxed{U(1,\vec{0},\vec{0},b) = e^{\frac{i}{\hbar}Hb},}$$ since $[H,H]=0$.

Further

$$U(1,\vec{0},\vec{0},\bar{b})\,U(1,\vec{0},\vec{0},b) = U(1,\vec{0},\vec{0},\bar{b}+b),$$

the time translation operators form an abelian subgroup of the Galilean transformations. Again the phase in the multiplication law is zero since $[H,H]=0$.

3) Velocity boosts are given by the operator $U(1,\vec{V},\vec{0},0)$. As with spatial translations we have the commutation relation $[K_i,K_j]=0$.

Hence

$$U(1,\vec{V},\vec{0},0) = \lim_{N\to\infty}\left[U(1,\tfrac{\vec{V}}{N},\vec{0},0)\right]^N$$

$$= \lim_{N\to\infty}\left(1+\frac{i}{\hbar}\frac{1}{N}\vec{V}\cdot\vec{K}\right)^N$$

$$\boxed{U(1,\vec{V},\vec{0},0) = e^{+\frac{i}{\hbar}\vec{V}\cdot\vec{K}}}$$ since $[K_i,K_j]=0$.

Further

$$U(1,\vec{V}',\vec{0},0)\,U(1,\vec{V},\vec{0},0) = U(1,\vec{V}'+\vec{V},\vec{0},0)$$

since $[K_i,K_j]=0$ the velocity boost operators form an abelian subgroup of the Galilean transformations. Again the multiplication law phase is zero due to $[K_i,K_j]=0$.

4) Spatial rotations are given by

$$\mathcal{U}(R,\vec{0},\vec{0},0) \equiv \mathcal{U}(R) .$$

Although we absorbed unwanted phases into the definition of J_i so that

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$, the situation

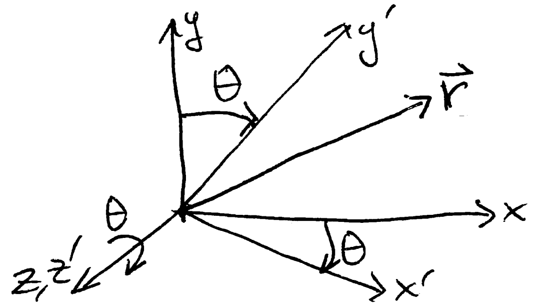
is slightly more complicated since the J_i does not commute.

Let's first consider rotations about the z-axis, denoting the orthogonal 3x3 rotation matrix $R(\vec{\theta})$ with $\vec{\theta} \equiv \theta \hat{z}$ we have

$$x'_i = R_{ij}(\vec{\theta})\, x_j \quad \text{with}$$

$$R_{ij}(\vec{\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix}$



For infinitesimal θ we have

$$R_{ij}(\vec{\theta}) = \delta_{ij} + \omega_{ij}(\vec{\theta})$$

$$= \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\theta$$

Thus

$$\omega_{ij}(\vec{\theta}) = \theta \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $\omega_{12} = -\theta$, $\omega_{21} = +\theta$ all others $= 0$.

So a rotation in the $x_i - x_j$ plane is described by $\omega_{ij} = -\omega_{ji}$.

Now for an infinitesimal rotation about the z-axis we have the transformation operator

$$U(1+\omega(\vec{\theta})) = 1 + \frac{i}{2}\frac{1}{\hbar}\omega_{ij}(\vec{\theta})J_{ij}$$

$$= 1 + \frac{i}{2}\frac{1}{\hbar}(\omega_{12}J_{12} + \omega_{21}J_{21})$$

$$= 1 + \frac{i}{\hbar}\omega_{12}J_{12}$$

So

$$\mathcal{U}(1+\omega(\vec{\theta})) = 1 - \frac{i}{\hbar}\theta J_{12} \; ,$$

but recall $J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$, that is

$J_1 = J_{23}$, $J_2 = J_{31}$, $J_3 = J_{12}$, so

$$\mathcal{U}(1+\omega(\vec{\theta})) = 1 - \frac{i}{\hbar}\theta J_z$$

$$= 1 - \frac{i}{\hbar}\vec{\theta}\cdot\vec{J} \; .$$

Now since $[J_3, J_3] = 0$ we can proceed as in the spatial translation case and compound successive infinitesimal rotations about the z-axis to obtain a rotation about the z-axis through a finite angle θ.

$$\mathcal{U}(R(\vec{\theta})) = \lim_{N\to\infty}\left[\mathcal{U}(R(\tfrac{\vec{\theta}}{N}))\right]^N$$

$$= \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\frac{\theta}{N}J_z\right]^N$$

$$\boxed{\mathcal{U}(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\theta J_z}}$$

Similarly any finite arbitrary rotation can be obtained by a single rotation through a finite angle Θ about a fixed direction (axis) $\hat{\Theta}$. Thus the vector $\vec{\Theta} \equiv \Theta\hat{\Theta}$ completely specifies the arbitrary rotation under consideration. Since rotations about the same fixed axis commute we can build up the finite rotation by compounding successive infinitesimal rotations about this same axis. That is although

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k \quad , \text{ we have}$$

$$[\vec{\Theta}\cdot\vec{J}, \vec{\Theta}\cdot\vec{J}] = \Theta_i \Theta_j [J_i, J_j]$$

$$= i\hbar \underbrace{\Theta_i\Theta_j}_{\text{symmetric}} \underbrace{\epsilon_{ijk}}_{\text{anti-symmetric}} J_k$$

$$= 0$$

Hence

$$U(R(\vec{\Theta})) = \lim_{N\to\infty} \left[U\left(R\left(\frac{\vec{\Theta}}{N}\right)\right)\right]^N$$

$$= \lim_{N\to\infty} \left[1 - \frac{i}{\hbar} \frac{1}{N} \vec{\Theta} \cdot \vec{J} \right]^N$$

$$\boxed{\mathcal{U}(R(\vec{\Theta})) = e^{-\frac{i}{\hbar} \vec{\Theta} \cdot \vec{J}},}$$

The unitary operator representing rotations.

Note: 1) $\vec{\Theta} \cdot \vec{J} = \Theta_i J_i = \frac{1}{2} \Theta_i \epsilon_{ijk} J_{jk}$

$$\equiv \frac{1}{2} \Theta_{jk} J_{jk}$$

where $\Theta_{jk} \equiv \epsilon_{jki} \Theta_i = -\Theta_{kj}$

and $\Theta_i = \frac{1}{2} \epsilon_{ijk} \Theta_{jk}$. So for example $\vec{\Theta} = \Theta_3 \hat{z}$ describes rotations about the z-axis, that is in the $x_1 - x_2$ plane by angle $\Theta_3 = \Theta_{12} = -\Theta_{21}$.

2) Since $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

$$e^{-\frac{i}{\hbar}(\vec{\theta}\cdot\vec{J})} = e^{-\frac{i}{\hbar}(\theta_x J_x + \theta_y J_y + \theta_z J_z)}$$

$$\neq e^{-\frac{i}{\hbar}\theta_x J_x}\, e^{-\frac{i}{\hbar}\theta_y J_y}\, e^{-\frac{i}{\hbar}\theta_z J_z},$$

as we know from the Baker-Campbell-Hausdorff formula in general. Hence the group properties are complicated for the composition of 2 rotations, it is non-abelian.

So to summarize, we have considered systems invariant under Galilean transformations. This led us to identify the generator of spatial translations as the momentum operator and the generator of time translations as the Hamiltonian. In addition the total angular momentum operator $\vec{J}$ generated spatial rotations. The passive view was used to describe the symmetry transformations and was shown to imply working in the Heisenberg picture. There the time evolution of an operator is given by $-i\hbar \frac{dA(t)}{dt} = [H, A(t)]$. Hence constants of the motion correspond to those operators which commute with the Hamiltonian. For a Galilean invariant system, the momentum $\vec{P}$ and total angular momentum $\vec{J}$ (as well as the energy H) are constants of the motion. From the active point of view by Ehrenfest's theorem this implies that the expectation value of these quantities is conserved, that is constant in time. Hence we have for quantum mechanics the relation between invariances or symmetries of the system and constants of the motion or conservation laws.

Time Translation Invariance $\Leftrightarrow$ H is constant of motion $\Leftrightarrow$ ⟨Energy⟩ is conserved

Space Translation Invariance $\Leftrightarrow$ $\vec{P}$ is constant of motion $\Leftrightarrow$ ⟨Momentum⟩ is conserved

Space Rotation Invariance $\Leftrightarrow$ $\vec{J}$ is constant of motion $\Leftrightarrow$ ⟨total angular momentum⟩ is conserved.