As we shall see, anti-unitary operators will only be needed when considering transformations that reverse the sign of the time. Otherwise we will only need unitary operators to relate the equivalent descriptions of our system.

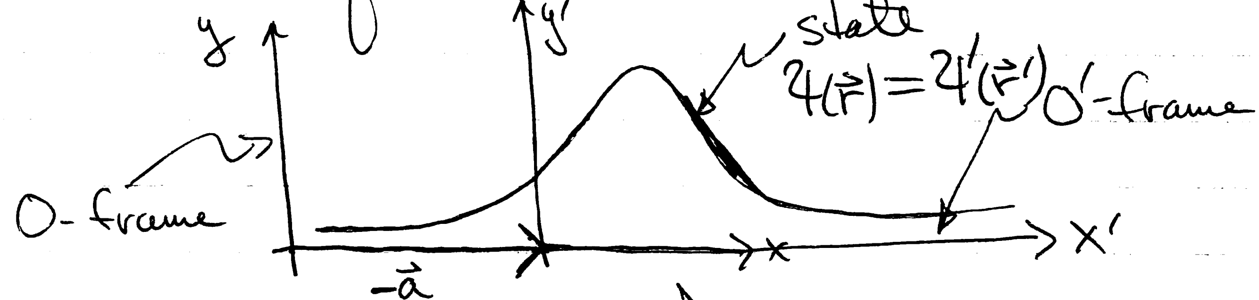
5.2 Space-time Symmetries

Of particular importance are the equivalent descriptions that are related to the geometrical symmetries of our physical system. In the most general case we will consider two observers to be related by space & time translations, space rotations and uniform constant velocity boosts. The geometrical transformation equations between such inertial frames of reference will imply the existence of the quantum mechanical operators generating the corresponding symmetry transformations on the states of the system. As we anticipate the momentum operator will be related to space-translations the Hamiltonian to time-translations, the angular momentum operator to space Rotations for example. As we have seen in the case of central potentials, when the Hamiltonian of the system (the force law if you will) reflects one of these geometrical symmetries (for example if it is invariant under spatial Rotations of the coordinates as in the central potential case) the symmetry transformation operator (the orbital angular momentum in our example) commutes with the Hamiltonian and can be used in part to form a CSCO with H whose eigenvalues will label the possible states of the system. Since the operators commute with H they are time independent according to the Heisenberg equations of motion. They, as well as their expectation values (by Ehrenfest's theorem), are constants of the motion; they are conserved quantities. (The system cannot make a transition from one eigenstate of the operator to another, ex. angular momentum is conserved.)

Before undertaking a systematic study of such geometrical symmetry transformations let's make the above discussion more concrete by examining the relation between two observers translated in space from each other by a constant vector $\vec{a}$. Alternatively from the active viewpoint, we can consider preparing the system in state $|\psi\rangle$ and measuring the observables given by operator A at point $\vec{r}$. Then we can translate the preparation and measurement apparatus across the room to coordinate $\vec{r}+\vec{a}$. The state that was prepared and measured at $\vec{r}$ is now prepared and measured at $\vec{r}' = \vec{r}+\vec{a}$; call it state $|\psi'\rangle$. In a wavemechanics formulation the first case is described by the wavefunction $\psi(\vec{r})$ while the second identical but translated system is described by $\psi'(\vec{r})$. Since the states prepared are identical, we have

$$\psi'(\vec{r}+a) = \psi(\vec{r}),$$

we only translated all the equipment etc. from $\vec{r}$ to $\vec{r}+a$. Thus, if $\psi(\vec{r})$ had a maximum at $\vec{r}_0$, then the same state of the system prepared at the translated site has a maximum at $\vec{r}_0 + \vec{a}$. (Clearly from the passive view the system is left stationary, the second observer has a coordinate frame displaced by $-\vec{a}$ w.r.t. the first observer



but $\vec{r}\,' = \vec{r} + \vec{a}$.)

From the transformation law of the wavefunctions we can determine the unitary operator relating the states

$$|\psi'\rangle = U(\vec{a})|\psi\rangle .$$

In the coordinate representation we have

$$\underbrace{\langle \vec{r}|\psi'\rangle}_{=\psi'(\vec{r})} = \int d^3r' \langle \vec{r}|U(\vec{a})|\vec{r}\,'\rangle \underbrace{\langle \vec{r}\,'|\psi\rangle}_{=\psi(\vec{r}\,')}$$

but

$$\psi'(\vec{r}) = \psi(\vec{r}-\vec{a}) \quad \text{from above.}$$

Taylor expanding $\psi(\vec{r}-\vec{a})$ about $\vec{r}$, we have

$$\psi'(\vec{r}) = \psi(\vec{r}-a) = \psi(\vec{r}) - \vec{a}\cdot\vec{\nabla}_{\vec{r}}\psi(\vec{r}) + \frac{1}{2!}\left(-\vec{a}\cdot\vec{\nabla}_{\vec{r}}\right)^2\psi(\vec{r}) + \cdots$$

$$= e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\psi(\vec{r})$$

which we write as

$$= \int d^3r'\, e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\delta^3(\vec{r}-\vec{r}\,')\,\psi(\vec{r}\,').$$

Thus we secure in the coordinate representation

$$\langle\vec{r}|U(\vec{a})|\vec{r}\,'\rangle = e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\delta^3(\vec{r}-\vec{r}\,')$$

$$= e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\langle\vec{r}|\vec{r}\,'\rangle.$$

In the coordinate basis $\langle\vec{r}|\vec{P} = -i\hbar\vec{\nabla}_{\vec{r}}\langle\vec{r}|$

So

$$\langle\vec{r}|U(\vec{a})|\vec{r}\,'\rangle = \langle\vec{r}|e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}|\vec{r}\,'\rangle$$

And we have the operator relation

$$\boxed{U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}}$$

The momentum operator is said to be the generator of space translations for states in the Hilbert space. Since $\vec{P} = \vec{P}^{\dagger}$ is Hermitian, $U(\vec{a})$ is unitary $U^{\dagger}(\vec{a}) = U^{-1}(\vec{a})$ that is

$$\langle\phi'|\psi'\rangle = \langle\phi|U^{\dagger}(\vec{a})U(\vec{a})|\psi\rangle$$

$$= \langle\phi|\psi\rangle, \quad \text{and}$$

hence the equivalence relation between transition probabilities follows

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2.$$

Note if the two observers' frames differ by an infinitesimal spatial displacement $\vec{\epsilon}$ then to first order

$$U(\vec{\epsilon}) = 1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}$$

while $U^{-1}(\vec{\epsilon}) = U^{\dagger}(\vec{\epsilon}) = U(-\vec{\epsilon}) = 1 + \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}$.

The translation of the system takes place "instantaneously". Making explicit the afore-unmentioned time variable we have $|\psi(t)\rangle$ describing the state of the system at time t in the frame of observer O who uses coordinates $\vec{r}$. On the other hand we have that $|\psi'(t)\rangle = U(\vec{a})|\psi(t)\rangle$ describes the same state of the system at time t but in the frame of observer O' who uses coordinates $\vec{r}\,' = \vec{r} - \vec{a}$.

We can then ask if $|\psi(t)\rangle$ evolves according to Hamiltonian H, does the spatially translated state $|\psi'(t)\rangle$ correspond to possible motions of the system. That is does the system as described by O' evolve in time into the transformation of the time evolution of the state $|\psi(t)\rangle$? Pictorially we have

$$\begin{tikzcd}
|\psi(t)\rangle \arrow[r, "H"] \arrow[d, "U(\vec{a})"'] & |\psi(t+\epsilon)\rangle \arrow[d, "U(\vec{a})"] \\
|\psi'(t)\rangle \arrow[r, "H"] & |\psi'(t+\epsilon)\rangle
\end{tikzcd}$$

And clearly in general $|\psi'(t)\rangle$ does not evolve according to H into $|\psi'(t+\epsilon)\rangle$.

When does $|\psi'(t)\rangle \xrightarrow{H} |\psi'(t+\epsilon)\rangle$?

To answer this, consider the time derivative of $|\psi'(t)\rangle$

$$
\begin{aligned}
i\hbar \frac{d}{dt}|\psi'(t)\rangle &= i\hbar \frac{d}{dt} U(\vec{a})|\psi(t)\rangle \\
&= U(\vec{a})\, \underbrace{i\hbar \frac{d}{dt}|\psi(t)\rangle}_{=H|\psi(t)\rangle} \\
&= U(\vec{a}) H \underbrace{U^\dagger(\vec{a})\, U(\vec{a})}_{=1} |\psi(t)\rangle \\
&= U(\vec{a}) H U^\dagger(\vec{a}) |\psi'(t)\rangle
\end{aligned}
$$

$$
i\hbar \frac{d}{dt}|\psi'(t)\rangle \equiv H'|\psi'(t)\rangle .
$$

Thus $|\psi'(t)\rangle$ evolves according to

$H' = U(\vec{a}) H U^\dagger(\vec{a})$; only if $U(\vec{a}) H U^\dagger(\vec{a}) = H$ does $|\psi'(t)\rangle$ obey the same Schrödinger equation as $|\Psi(t)\rangle$, then $|\Psi'(t)\rangle$ is a physically possible state of the system.

Multiplying $H' = H$ by $U(\vec{a})$ on the right yields

$$U(\vec{a})H = HU(\vec{a})$$

$$\Rightarrow \quad [U(\vec{a}), H] = 0.$$ So only if $U(\vec{a})$ commutes with H is $|\Psi'(t)\rangle$ still a physically allowed state. For infinitesimal displacements $U(\vec{\epsilon}) = 1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}$ and

$$U(\vec{\epsilon})HU^{\dagger}(\vec{\epsilon}) = H$$

$$= \left(1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right)H\left(1 + \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right)$$

to first order $\Rightarrow$

$$= H - \frac{i}{\hbar}[\vec{\epsilon}\cdot\vec{P}, H]$$

$\Rightarrow$

$$[\vec{P}, H] = 0.$$

On the one hand this tells us that the momentum is a constant of the motion. Thus a physical system that can be displaced in space and still be a possible physical system is characterized by a constant momentum as well as energy. Since R and $\vec{P}$ dos not commute, $[\vec{P}, H] = 0$ implies H is a function of $\vec{P}$ only. The simplest case is

$$H = \frac{1}{2m}\vec{P}^2 \quad ; \text{ a free particle.}$$

Clearly, $V(\vec{R}) \neq 0$ corresponds to external forces acting on the system and translation of the system through the force field does not leave the system unaffected. So on the other hand, the momentum operator is the generator of translations in space and $[\vec{P}, H] = 0$ implies that space-translations are a symmetry of the system. That is the Hamiltonian exhibits translation invariance. As in classical mechanics symmetries of the system imply conservation laws, $[H, \vec{P}] = 0$, and vice versa.

Not only is the state preparation apparatus translated in space but also the measuring apparatus that is the observables. We denote the observables at $\vec{r}$ by operators A and after translation to $\vec{r}+\vec{a}$ by operators A'. By definition the states were prepared to have the same properties at the two locations. Thus the matrix elements of the observables are the same (the eigenvalue spectrum is unchanged),

$$\langle\phi'|A'|\psi'\rangle = \langle\phi|A|\psi\rangle .$$

But this gives

$$\langle\phi'|A'|\psi'\rangle = \langle\phi|U^{+}(\vec{a})A'U(\vec{a})|\psi\rangle$$
$$= \langle\phi|A|\psi\rangle$$

hence, as we found with the Hamiltonian,

$$\boxed{A' = U(\vec{a})AU^{+}(\vec{a})} .$$

So an observable, A, at location $\vec{r}$ becomes the observable $A' = U(\vec{a})\, A\, U^\dagger(\vec{a})$ when translated to location $\vec{r}+\vec{a}$.
A is said to be <u>translationally invariant</u> if

$$A' = A = U(\vec{a})\, A\, U^\dagger(\vec{a}),$$

that is $[U(\vec{a}), A] = 0$ or equivalently if

$$[\vec{P}, A] = 0 .$$

Such observables are simultaneously measurable (diagonalizable) with the momentum $\vec{P}$.
For example the momentum operator is invariant

$$\vec{P}' = U(\vec{a})\, \vec{P}\, U^\dagger(\vec{a}) = \vec{P}$$

but the position operator is not

$$\vec{R}' = U(\vec{a})\, \vec{R}\, U^\dagger(\vec{a}) = \vec{R} - \frac{i}{\hbar}\underbrace{[\vec{a}\cdot\vec{P}, \vec{R}]}_{\substack{= a_i [P_i, \vec{R}] \\ = -i\hbar \vec{a}}}$$

$$\boxed{\vec{R}' = \vec{R} - \vec{a}} .$$

If we make two translations

$$\vec{r} \longrightarrow \vec{r}+\vec{a} \longrightarrow (\vec{r}+\vec{a})+\vec{b}$$

(that is introduce a third observer O'' translated wrt O' by $\vec{b}$), this should be equivalent to making one overall translation

$$\vec{r} \longrightarrow \vec{r}+\vec{c}$$

where $\vec{c} = \vec{a}+\vec{b}$. Hence the unitary operators $U(\vec{a})$ relating the states in the translated systems obey a similar product rule. For the system at the position $\vec{r}$ the state is $|\psi\rangle$, at $\vec{r}+\vec{a}$ the state is denoted $|\psi'\rangle$ and at $\vec{r}+\vec{a}+\vec{b}$ it is $|\psi''\rangle$ where

$$|\psi''\rangle = U(\vec{b})|\psi'\rangle = U(\vec{b})U(\vec{a})|\psi\rangle$$

but on the other hand

$$|\psi''\rangle = U(\vec{c})|\psi\rangle = U(\vec{a}+\vec{b})|\psi\rangle.$$

Thus $U(\vec{b})U(\vec{a}) = U(\vec{a}+\vec{b})$. Since

$\vec{a}+\vec{b}=\vec{b}+\vec{a} \Rightarrow U(\vec{a}+\vec{b})=U(\vec{b}+a)=U(\vec{a})U(\vec{b})$, that is the U operators commute

$$U(\vec{b})U(\vec{a})=U(\vec{a})U(\vec{b}).$$

So we have a product laws for the translation operators $U(\vec{a})$, they form a representation of the group of translations, that is they form a group under this multiplication law that is isomorphic to the translation group. The translation group T is the set of all displacement vectors $\vec{a}$ such that ($T=\{\vec{a} \mid \vec{a}=\text{displacement vector}\}$)

1) $\vec{a}+\vec{b}=\vec{c} \in T$ vector addition

2) $\vec{a}+\vec{0}=\vec{a}$ $\vec{0}=$ identity element $\in T$

3) $\vec{a}+(-\vec{a})=\vec{0}$ $-\vec{a}=$ inverse $\in T$

4) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ associative law

The quantum mechanical translation group $\mathcal{T}$ is the set of all $U(\vec{a})$ for $\vec{a}\in T$ with the composition laws given above such that

1) $\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b}) = \mathcal{U}(\vec{a}+\vec{b}) \in \mathcal{T}$

2) $\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{0}) = \mathcal{U}(\vec{a})$; $\mathcal{U}(\vec{0}) = \mathbb{1}$ the identity operator $\in \mathcal{T}$

3) $\mathcal{U}(\vec{a})\,\mathcal{U}(-\vec{a}) = \mathcal{U}(\vec{0}) = \mathbb{1}$; $\mathcal{U}(-\vec{a}) = \mathcal{U}^{-1}(\vec{a}) \in \mathcal{T}$

4) $\left(\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b})\right)\mathcal{U}(\vec{c}) = \mathcal{U}(\vec{a})\left(\mathcal{U}(\vec{b})\,\mathcal{U}(\vec{c})\right)$ associative law.

Since the order of vectors under addition is irrelevant $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ T is called an abelian group. Likewise since the operators $\mathcal{U}(\vec{a})$ and $\mathcal{U}(\vec{b})$ commute

$$\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b}) = \mathcal{U}(\vec{b})\,\mathcal{U}(\vec{a}),$$

$\mathcal{T}$ is also an abelian group; it is a representation of the group T on vectors in $\mathcal{H}$.

The group composition law implies algebraic properties for the generators of the transformations. In the

abelian translation group case we have
$$U(\vec{a})\,U(\vec{b}) = U(\vec{a}+\vec{b}) \text{ where}$$
$U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$. For infinitesimal transformations $U(\vec{a})$ and $U(\vec{b})$, $\vec{a}, \vec{b}$ infinitesimal vectors, we can expand the operators to first order in $\vec{a}$ or $\vec{b}$
$$U(\vec{a})\,U(\vec{b}) = \left(1-\frac{i}{\hbar}\vec{a}\cdot\vec{P}\right)\left(1-\frac{i}{\hbar}\vec{b}\cdot\vec{P}\right)$$
$$= \mathbb{1} - \frac{i}{\hbar}\left(\vec{a}\cdot\vec{P} + \vec{b}\cdot\vec{P}\right)$$
$$+ \frac{i^2}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P})$$
$$= 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P} - \frac{1}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P})$$
On the other hand this equals
$$U(\vec{a}+\vec{b}) = 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P} + \frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\left((\vec{a}+\vec{b})\cdot\vec{P}\right)^2$$
$$= 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P}$$
$$- \frac{1}{2\hbar^2}\left((\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) + (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})\right)$$
This must $= U(\vec{a})\,U(\vec{b})$

The only way this can be true is if

$$-\frac{1}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P})$$
$$= -\frac{1}{2\hbar^2}\left[(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) + (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})\right]$$

$$\Rightarrow (\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) = (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})$$

$$\Rightarrow a_i b_j P_i P_j = a_i b_j P_j P_i$$

Since $\vec{a}, \vec{b}$ are arbitrary $\Rightarrow$

$$P_i P_j = P_j P_i \quad \text{or}$$

$$\boxed{[P_i, P_j] = 0}$$

Of course in this simple case we know the result already for the P-T commutation relations. In general though the symmetry group multiplication law implies the commutation relations for the generators.

Alternatively, given the commutation relations for a set of symmetry generators, we can reconstruct the group multiplication law. For example knowing that

$$[P_i, P_j] = 0 \quad \text{implies that}$$

$$e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}\, e^{-\frac{i}{\hbar}\vec{b}\cdot\vec{P}} = e^{-\frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P}}$$

by the Baker-Campbell-Hausdorff formula. Hence

$$U(\vec{a})\,U(\vec{b}) = U(\vec{a}+\vec{b}).$$

If we do not introduce time translations into our group of transformations we have not asked for them to be symmetries of the Hamiltonian. That is if the multiplication law between the time translation operator and the other transformation operators has not been specified then neither has the commutator of the Hamiltonian H with the generators of the other transformations.

However in general for either relativistic or non-relativistic physics we find experimentally that time translations and space translations can be made independently of each other, thus we find the commutation relation $[H, \vec{P}] = 0$.

The forces we observe in nature are between bodies, there is no background force field in space preventing $[\vec{P}, H] = 0$. Hence, from a general point of view, we will imbed time translations in our group of transformations, in a way, of course, that is consistent with the precepts of Newtonian, that is, non-relativistic physics. When studying certain models we may go to a particular frame of reference, thus replacing the two-body potential for example, with a background one body potential. We can then ask for an appropriate subset of the transformation generators to commute with the reduced Hamiltonian. Of course we can just study the "instantaneous" transformations by decoupling the time translations from set of operators.

Hence we will study transformations from one inertial frame to another. They are known as the Galilean group of transformations. Since they involve time translations we will be imposing commutation relations of the "instantaneous" transformation generators with the Hamiltonian. Hence we will be imposing a certain set of symmetry properties or restrictions on the Hamiltonian.

We can always choose a subset of transformations, a subgroup of the Galilean group, relevant to the system under study, thus easing the restrictions on H.