V. Symmetry In Quantum Mechanics

5.1 Transformations between equivalent descriptions and Wigner's Theorem

In general, there are two ways to view equivalent descriptions of a system along with its preparation and measuring apparati. The first viewpoint is called the active view. In this case one first prepares the system in state $|\psi\rangle$ say, and performs whatever measurements are desired, for instance an experiment can be performed to determine the probability that the state $|\psi\rangle$ is in some particular eigenstate $|\phi\rangle$ of an observable. This probability is

$$|\langle\phi|\psi\rangle|^2.$$

All preparation and measurement in this case is first performed at a particular time t and point in space $\vec{r}$ with respect to a fixed coordinate system. Next with the same coordinate system, the preparation and measurement apparatus is moved from $(\vec{r},t)$ to a new space coordinate $\vec{r}'$ and time t'.

Corresponding to the state $|\psi\rangle$ prepared at $\vec{r}$ and t, the same state is prepared at $\vec{r}'$ and t', denote it $|\psi'\rangle$. The same experiment can be performed to determine the probability the system is in an eigenstate of the same observable at $\vec{r}'$ and t' as was done at $\vec{r}$ and t, call this eigenstate $|\phi'\rangle$. The probability is

$$|\langle\phi'|\psi'\rangle|^2.$$

We demand that the two measurements must be the same,

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2,$$

Since these are equivalent descriptions of the same physical situation.

Alternatively we can view such an equivalent description from a <u>passive viewpoint</u>. In this case we have the one system and measurement apparatus viewed by two observers O and O' using different inertial coordinate systems $(\vec{r}, t)$ and $(\vec{r}\,', t')$ respectively. Observer O measures $|\langle\phi|\psi\rangle|^2$ while O' measures $|\langle\phi'|\psi'\rangle|^2$. Since they are describing the same situation we demand

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$$

as previously. ($|\psi'\rangle$ is a state in the Hilbert space of O that looks like $|\psi\rangle$ to O'.)

Clearly, in the active view, moving the system and apparatus one way is congruent, in the passive view, to the observers' being moved in the opposite (inverse) way. For instance if the apparatus is rotated about the z-axis through angle φ, that is the same as observer O' using an inertial frame rotated about the z-axis of O (with common origin) by angle $-\varphi$.

Note, this discussion can be generalized to not only include observers using different space-time inertial frames but also other relationships between the pairs of observers. For instance they could use two different conventions for the sign of electric charge. Or as in the case of the 3-dimensional isotropic SHO we could have two observers using different SU(3) conventions, that is the states $|\psi'\rangle$ will be related to $|\psi\rangle$ by SU(3) rotations.

Further we note that at this point it is not necessary for the observers to have the same form for their Hamiltonians. (as long as $t'=t$!) For instance, if the observers are related by a rotation it is not necessary for the Hamiltonian to have rotational invariance. In fact, we have not even spoken about dynamics in the above discussion.

The question we now ask is to the mathematical nature of the relationship between these different though equivalent descriptions. That is we desire the properties of the operator that maps the states $|\psi\rangle$ into the states $|\psi'\rangle$ so that

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2.$$

If in fact we measured the scalar products directly we would require $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$ and immediately note that the operator mapping $|\psi\rangle \to |\psi'\rangle$ must be unitary

$$|\psi'\rangle = U|\psi\rangle \text{ with } U^\dagger = U^{-1},$$

since it preserves inner products in $\mathcal{H}$.

However it is only $|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle|$ that is needed and more general operators can possibly occur.

Fortunately there is a simple result due to Wigner that states there is only one other type of transformation beyond unitary ones that need be considered, anti-unitary transformation operators.

Suppose we have two equivalent descriptions of the same physical system such that observer O uses kets $|\psi\rangle$, $|\phi\rangle$ etc. while observer O′ associates kets $|\psi'\rangle$, $|\phi'\rangle$ etc. with states which in his description have identical properties as $|\psi\rangle$, $|\phi\rangle$ etc. do in O's description, as discussed above. Since the descriptions are equivalent we have the equality of the transition probabilities in both cases

$$\frac{|\langle\phi'|\psi'\rangle|^2}{\langle\phi'|\phi'\rangle\langle\psi'|\psi'\rangle} = \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle}.$$

(Here we allow $|\psi\rangle, |\phi\rangle, |\psi'\rangle, |\phi'\rangle$ to have general normalization.) The states in the two descriptions are related by an operator U, that is

$$|\psi'\rangle = U|\psi\rangle$$

$$|\phi'\rangle = U|\phi\rangle,$$

and so on for each state vector. Since it is the physical states that are equivalent, the above does not uniquely define U. We can re-scale U by a complex number Z_ψ for each vector $|\psi\rangle$ and obtain the same equivalence of physical states (rays in Hilbert space). That is $U|\psi\rangle$ and $Z_\psi U|\psi\rangle$ represent the same physical state. Using this freedom we find that

<u>Wigner's Theorem</u>: If U is an operator satisfying

$$\frac{|\langle \phi' | \psi' \rangle|^2}{\langle \phi' | \phi' \rangle \langle \psi' | \psi' \rangle} = \frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle}$$

with $|\psi'\rangle = U|\psi\rangle$ for each $|\psi\rangle$ in $\mathcal{H}$, then one may choose the arbitrary phases to define the operator U',

$$U'|\psi\rangle \equiv Z_\psi U|\psi\rangle = Z_\psi |\psi'\rangle,$$

so that U' is either <u>unitary</u> or <u>anti-unitary</u>.

A <u>unitary operator U</u> is such that

$$\langle \phi' | \psi' \rangle = \langle \phi | U^\dagger U | \psi \rangle = \langle \phi | \psi \rangle$$

that is $U^\dagger U = 1 = U U^\dagger \Rightarrow U^\dagger = U^{-1}$,

and is linear,

$$U(|\psi\rangle + \lambda|\phi\rangle) = U|\psi\rangle + \lambda U|\phi\rangle,$$

with $\lambda \in \mathbb{C}$. An <u>anti-unitary operator</u> U is such that for $|\psi'\rangle = U|\psi\rangle$,

$$|\phi'\rangle = U|\phi\rangle$$

one has $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$ and U is <u>anti-linear</u>,

$$U(|\psi\rangle + \lambda|\phi\rangle) = U|\psi\rangle + \lambda^* U|\phi\rangle$$

with $\lambda \in \mathbb{C}$.

<u>Proof</u>: We first choose the magnitude of z_ψ so that the states are normalized. Let

$$|\psi''\rangle \equiv U'|\psi\rangle \equiv \left(\frac{\langle\psi|\psi\rangle}{\langle\psi'|\psi'\rangle}\right)^{1/2} U|\psi\rangle,$$

$$= \left(\frac{\langle\psi|\psi\rangle}{\langle\psi'|\psi'\rangle}\right)^{1/2} |\psi'\rangle$$

Then

$$\langle\psi''|\psi''\rangle = \frac{\langle\psi|\psi\rangle}{\langle\psi'|\psi'\rangle}\langle\psi'|\psi'\rangle$$

$$= \langle\psi|\psi\rangle$$

and

$$\frac{|\langle\phi'|\psi'\rangle|^2}{\langle\phi'|\phi'\rangle\langle\psi'|\psi'\rangle} = \frac{\frac{\langle\phi'|\phi'\rangle}{\langle\phi|\phi\rangle}\frac{\langle\psi'|\psi'\rangle}{\langle\psi|\psi\rangle}|\langle\phi''|\psi''\rangle|^2}{\langle\phi'|\phi'\rangle\ \langle\psi'|\psi'\rangle}$$

$$= \frac{|\langle\phi''|\psi''\rangle|^2}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle}.$$

Hence the equivalence of transition probabilities,

$$\frac{|\langle\phi'|\psi'\rangle|^2}{\langle\phi'|\phi'\rangle\langle\psi'|\psi'\rangle} = \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle}\Big)$$

reduces to

$$|\langle\phi''|\psi''\rangle| = |\langle\phi|\psi\rangle|$$

and $\langle\psi''|\psi''\rangle = \langle\psi|\psi\rangle$. So

$|\psi''\rangle = U'|\psi\rangle$ and $|\psi\rangle$ have the same normalization, we only need to be concerned with phases.

Re-labelling $|\psi''\rangle \to |\psi'\rangle$ and $U' \to U$ then we ~~next~~ consider the ~~phases~~ z_ψ; $|z_\psi| = 1$. Again we define a U' operator so that

$$|\psi''\rangle = U'|\psi\rangle \equiv z_\psi U|\psi\rangle = z_\psi|\psi'\rangle$$

with $|z_\psi| = 1$ and $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$ now.

Suppose O prepares a complete set of orthonormal states $\{|\phi_k\rangle\}$:

$$\langle\phi_k|\phi_l\rangle = \delta_{kl}$$

and $\sum_k |\phi_k\rangle\langle\phi_k| = \mathbb{1}$, then

by the equivalence of the

descriptions, O′ prepares the corresponding equivalent basis set $\{|\phi'_k\rangle\}$:

$$\langle\phi'_k|\phi'_l\rangle = \delta_{kl}$$

and $$\sum_k |\phi'_k\rangle\langle\phi'_k| = 1 .$$

(Note by our choice of magnitudes of Z_{ϕ_k} we have from the equivalence relations

$$|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle|$$ that

$$|\langle\phi'_k|\phi'_l\rangle| = |\langle\phi_k|\phi_l\rangle| = 0$$

for $k \neq l$ and $\langle\phi'_k|\phi'_k\rangle = \langle\phi_k|\phi_k\rangle$

$$= 1 ,$$

that is the set $\{|\phi'_k\rangle\}$ is orthonormal

$$\langle\phi'_k|\phi'_l\rangle = \delta_{kl} .)$$

The point being the <u>phases</u> of

these sets can be chosen as we wish. In addition the phases of $|\psi\rangle$ and $|\psi'\rangle$ are free to be chosen as we wish. The equivalence of the O and O' descriptions only requires the unit ray containing $|\psi\rangle$ to be mapped into the unit ray containing $|\psi'\rangle$.

So O can expand $|\psi\rangle$ as

$$|\psi\rangle = \sum_k \langle\phi_k|\psi\rangle|\phi_k\rangle$$

while O' expands $|\psi'\rangle = U|\psi\rangle$ as

$$U|\psi\rangle = |\psi'\rangle = \sum_k \langle\phi'_k|\psi'\rangle|\phi'_k\rangle .$$

The equivalence relation implies

$$|\langle\phi'_k|\psi'\rangle| = |\langle\phi_k|\psi\rangle| ,$$

the magnitudes of the expansion coefficients are equal.

Let $|\phi_1\rangle$ be such that $\langle\phi_1|\psi\rangle \neq 0$ (since $|\psi\rangle \neq 0$; there exists such a $|\phi_k\rangle$). Consider the states

$$|\xi_k\rangle \equiv |\phi_1\rangle + |\phi_k\rangle \quad , \; k = 2, 3, \ldots ,$$

The corresponding O′ states are

$$\mathcal{U}(|\phi_1\rangle + |\phi_k\rangle) = |\xi_k'\rangle ,$$

which have the expansion

$$|\xi_k'\rangle = \sum_l{}' \langle\phi_l'|\xi_k'\rangle |\phi_l'\rangle , \; k = 2, 3, \ldots$$

The equivalence relation implies

$$\begin{aligned}
|\langle\phi_l'|\xi_k'\rangle| &= |\langle\phi_l|\xi_k\rangle| \\
&= |\langle\phi_l|(|\phi_1\rangle + |\phi_k\rangle)| \\
&= |\langle\phi_l|\phi_1\rangle + \langle\phi_l|\phi_k\rangle| \\
&= |\delta_{l1} + \delta_{lk}| , \; k = 2, 3, \ldots
\end{aligned}$$

Thus only $l=1$ or $l=k$ expansion coefficients for $|\xi_k'\rangle$ can be non-zero, this yields

$$|\xi_k'\rangle = U(|\phi_1\rangle + |\phi_k\rangle)$$

$$= \omega_k|\phi_1'\rangle + \tau_k|\phi_k'\rangle$$

where

$$\omega_k = \langle\phi_1'|\xi_k\rangle$$

$$\tau_k = \langle\phi_k'|\xi_k\rangle$$

with $|\omega_k| = 1 = |\tau_k|$; the ω_k & τ_k

are simply phase factors. We can absorb these phase factors into the Z_{ϕ_k}, define U' so that

$$U'|\phi_1\rangle = Z_{\phi_1}U|\phi_1\rangle \equiv U|\phi_1\rangle$$

$$U'|\phi_k\rangle = Z_{\phi_k}U|\phi_k\rangle \equiv \frac{\tau_k}{\omega_k}U|\phi_k\rangle$$

$$U'|\xi_k\rangle = Z_{\xi_k}U|\xi_k\rangle \equiv \frac{1}{\omega_k}U|\xi_k\rangle$$

i.e. the phase factors $Z_{\phi_1} \equiv 1$, $Z_{\phi_k} \equiv \frac{\tau_k}{\omega_k}$,

$Z_{\xi_k} \equiv \frac{1}{\omega_k}$ are chosen. Using

$$|\xi_k'\rangle = \omega_k |\phi_1'\rangle + \tau_k |\phi_k'\rangle$$

we have

$$\omega_k U' |\xi_k\rangle = U |\xi_k\rangle = |\xi_k'\rangle$$

$$= \omega_k |\phi_1'\rangle + \tau_k |\phi_k'\rangle$$

$$= \omega_k U |\phi_1\rangle + \tau_k U |\phi_k\rangle$$

$$= \omega_k U' |\phi_1\rangle + \omega_k U' |\phi_k\rangle$$

$$\Rightarrow \quad U' |\xi_k\rangle = U' |\phi_1\rangle + U' |\phi_k\rangle$$

that is $\boxed{U'(|\phi_1\rangle + |\phi_k\rangle) = U' |\phi_1\rangle + U' |\phi_k\rangle}$.

Once again relabel the $U' \to U$ etc.

Thus we now, having fixed the phases for the $\{|\phi_k'\rangle\}$, secured

$$U(|\phi_1\rangle + |\phi_k\rangle) = U|\phi_1\rangle + U|\phi_k\rangle.$$

We next return to the expansions for $|\psi\rangle$ and $|\psi'\rangle$

$$|\psi\rangle = \sum_{l}' \langle\phi_l|\psi\rangle|\phi_l\rangle$$

$$|\psi'\rangle = U|\psi\rangle$$

$$= \sum_{l}' \langle\phi_l'|\psi'\rangle|\phi_l'\rangle$$

$$= \sum_{l}' \langle\phi_l'|\psi'\rangle U|\phi_l\rangle .$$

The equivalence relation, again, implies

$$|\langle\phi_1'|\psi'\rangle| = |\langle\phi_1|\psi\rangle|$$

$$|\langle\phi_k'|\psi'\rangle| = |\langle\phi_k|\psi\rangle| \;, k=2,3,\ldots$$

Now we find for $|\xi_k'\rangle = U(|\phi_1\rangle+|\phi_k\rangle)$

$$|\langle\xi_k'|\psi'\rangle| = \left|\sum_{l}' \langle\phi_l'|\psi'\rangle\langle\xi_k'|\phi_l'\rangle\right|$$

but $|\xi_k'\rangle = U|\phi_1\rangle + U|\phi_k\rangle = |\phi_1'\rangle + |\phi_k'\rangle$ from the choice of phases $z_{\phi_1}, z_{\phi_k}, z_{\xi_k}$.

So $\langle\xi_k'|\phi_l'\rangle = \langle\phi_1'|\phi_l'\rangle + \langle\phi_k'|\phi_l'\rangle$

$$= \delta_{1l} + \delta_{kl}$$

yielding

$$|\langle \xi_k' | \psi' \rangle| = \left| \sum_l \langle \phi_l' | \psi' \rangle (\delta_{1l} + \delta_{kl}) \right|$$
$$= |\langle \phi_1' | \psi' \rangle + \langle \phi_k' | \psi' \rangle|$$

for $k = 2, 3, \ldots$.

In addition

$$|\langle \xi_k | \psi \rangle| = \left| \sum_l \langle \phi_l | \psi \rangle \langle \xi_k | \phi_l \rangle \right|$$
$$= \left| \sum_l \langle \phi_l | \psi \rangle \left(\langle \phi_1 | \phi_l \rangle + \langle \phi_k | \phi_l \rangle \right) \right|$$
$$= \left| \sum_l \langle \phi_l | \psi \rangle (\delta_{1l} + \delta_{kl}) \right|$$
$$= |\langle \phi_1 | \psi \rangle + \langle \phi_k | \psi \rangle| ,$$

for $k = 2, 3, \ldots$.

The equivalence relation requires

$$|\langle \xi_k' | \psi' \rangle| = |\langle \xi_k | \psi \rangle|$$

$\Rightarrow$

$$|\langle \phi_1' | \psi' \rangle + \langle \phi_k' | \psi' \rangle| = |\langle \phi_1 | \psi \rangle + \langle \phi_k | \psi \rangle|$$

for $k = 2, 3, \ldots$.

Calling $c_k \equiv \langle \phi_k | \psi \rangle$ and $c_k' \equiv \langle \phi_k' | \psi' \rangle$ for $k = 1, 2, 3, \ldots$, this is

$$|c_1' + c_k'| = |c_1 + c_k| \quad , \text{for } k = 2, 3, \ldots$$

along with the relations $|c_k'| = |c_k|$ for $k = 1, 2, 3, \ldots$. This is just, for $k = 2, 3, \ldots$

$$\cancel{|c_1'|^2} + \cancel{|c_k'|^2} + (c_1'^* c_k' + c_k'^* c_1')$$
$$= \cancel{|c_1|^2} + \cancel{|c_k|^2} + (c_1^* c_k + c_k^* c_1)$$

$$\Rightarrow \quad \boxed{c_1'^* c_k' + c_k'^* c_1' = c_1^* c_k + c_k^* c_1}$$

Multiplying by $c_1'^* c_k'$ $\Rightarrow$

$$0 = (c_1'^* c_k')^2 - (c_1'^* c_k')(c_1^* c_k + c_k^* c_1) + |c_1'^* c_k'|^2.$$

The roots to this equation are

$$c_1'^* c_k' = \begin{cases} c_1^* c_k \\ c_1 c_k^* \end{cases} .$$

That is

$$\boxed{\frac{C_k'}{C_1'} = \begin{cases} \dfrac{C_k}{C_1} \\ \left(\dfrac{C_k}{C_1}\right)^* \end{cases}} \qquad k=2,3,\ldots$$

or

$$\boxed{\frac{\langle \phi_k' | \psi' \rangle}{\langle \phi_1' | \psi' \rangle} = \begin{cases} \dfrac{\langle \phi_k | \psi \rangle}{\langle \phi_1 | \psi \rangle} \\ \left(\dfrac{\langle \phi_k | \psi \rangle}{\langle \phi_1 | \psi \rangle}\right)^* \end{cases}} \qquad k=2,3,\ldots$$

Finally, we can choose the overall phases of $|\psi\rangle$ and $|\psi'\rangle$ so that

$C_1 = C_1'$; that is $\langle \phi_1' | \psi' \rangle = \langle \phi_1 | \psi \rangle$

and it is Real $\langle \phi_1 | \psi \rangle = \langle \phi_1 | \psi \rangle^* = \langle \psi | \phi_1 \rangle$.

Hence we find that equivalence relation becomes

$$C_k' = \begin{cases} C_k \\ C_k^* \end{cases}$$

That is

$$\langle \phi_k' | \psi' \rangle = \begin{cases} \langle \phi_k | \psi \rangle \\ \langle \phi_k | \psi \rangle^* \end{cases},$$

So we have ~~two~~ cases

Case 1) $\langle \phi_k' | \psi' \rangle = \langle \phi_k | \psi \rangle$; the expansion of the transformation equation is

$$|\psi'\rangle = \mathcal{U}|\psi\rangle = \sum_k' \langle \phi_k' | \psi' \rangle |\phi_k'\rangle$$

$$= \sum_k' \underbrace{\langle \phi_k' | \psi' \rangle}_{= \langle \phi_k | \psi \rangle} \mathcal{U} |\phi_k\rangle$$

$$\mathcal{U}|\psi\rangle = \sum_k' \langle \phi_k | \psi \rangle \mathcal{U} |\phi_k\rangle$$

but $|\psi\rangle = \sum_k' \langle \phi_k | \psi \rangle |\phi_k\rangle$ on the LHS

$$\Rightarrow \boxed{\mathcal{U}\left(\sum_k' \langle \phi_k | \psi \rangle |\phi_k\rangle\right) = \sum_k' \langle \phi_k | \psi \rangle \mathcal{U} |\phi_k\rangle}$$

Case 2) $\langle \phi_k' | \psi' \rangle = \langle \phi_k | \psi \rangle^* \Rightarrow$

$$|\psi'\rangle = U|\psi\rangle = \sum_k {}' \langle \phi_k' | \psi' \rangle |\phi_k'\rangle$$
$$= \sum_k {}' \langle \phi_k | \psi \rangle^* U|\phi_k\rangle$$

So

$$\boxed{U\left(\sum_k {}' \langle \phi_k | \psi \rangle |\phi_k\rangle\right) = \sum_k {}' \langle \phi_k | \psi \rangle^* U|\phi_k\rangle}$$

Case 1) corresponds to the unitary case and case 2) the anti-unitary case.

Moreover, consider another vector $|\phi\rangle = \sum_k {}' \langle \phi_k | \phi \rangle |\phi_k\rangle$. Then the transformed vector $|\phi'\rangle$ is either

(unitary case) $$|\phi'\rangle = U|\phi\rangle = \sum_k {}' \langle \phi_k | \phi \rangle U|\phi_k\rangle$$

or

$$|\phi'\rangle = U|\phi\rangle = \sum_k {}' \langle \phi_k | \phi \rangle^* U|\phi_k\rangle.$$

(anti-unitary case)

Assume that $|\psi'\rangle = \sum_k' \langle\phi_k|\psi\rangle U|\phi_k\rangle$, the unitary case 1) for $|\psi\rangle$. Then if $|\phi'\rangle = \sum_k' \langle\phi_k|\phi\rangle U|\phi_k\rangle$, the unitary case also, we have

$$\langle\phi'|\psi'\rangle = \sum_{k,l}' \langle\phi_k|\phi\rangle^* \langle\phi_l|\psi\rangle \underbrace{\langle\phi_k'|\phi_l'\rangle}_{=\delta_{kl}}$$

$$= \sum_k' \langle\phi_k|\phi\rangle^* \langle\phi_k|\psi\rangle$$

But

$$\langle\phi|\psi\rangle = \sum_{k,l}' \langle\phi_k|\phi\rangle^* \langle\phi_l|\psi\rangle \underbrace{\langle\phi_k|\phi_l\rangle}_{=\delta_{kl}}$$

$$= \sum_k' \langle\phi_k|\phi\rangle^* \langle\phi_k|\psi\rangle$$

So

$$\underline{\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle}$$

and hence $|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle|$.

But if $|\phi'\rangle = \sum_k' \langle\phi_k|\phi\rangle^* |\phi_k'\rangle$, the anti-unitary case we have

$$\langle\phi'|\psi'\rangle = \sum_k' (\langle\phi_k|\phi\rangle^*)^* \langle\phi_k|\psi\rangle$$

$$= \sum_k' \langle\phi_k|\phi\rangle \langle\phi_k|\psi\rangle .$$

Thus $$|\langle\phi'|\psi'\rangle| = \left|\sum_k' \langle\phi_k|\phi\rangle\langle\phi_k|\psi\rangle\right|$$

$$\neq |\langle\phi|\psi\rangle| ,$$

the equivalence relation is violated. Hence for a particular transformation $O \to O'$, <u>all</u> <u>vectors</u> are transformed according to a <u>unitary</u> or <u>anti-unitary</u> transformation.

In more detail for case 1) Unitary we have $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$

Since $|\psi'\rangle = U|\psi\rangle$; $|\phi'\rangle = U|\phi\rangle$,

this is

$$\langle\phi'|\psi'\rangle = \langle\phi|U^\dagger U|\psi\rangle = \langle\phi|\psi\rangle \Rightarrow$$

$U^\dagger U = \mathbb{1}$, the unitary case.

Further, for $|\chi\rangle = |\psi\rangle + \lambda|\phi\rangle$, $\lambda \in \mathbb{C}$,

$$U|\chi\rangle = \sum_k{}' \langle\phi_k'|\chi'\rangle |\phi_k'\rangle$$

by case 1)
$$= \sum_k{}' \langle\phi_k|\chi\rangle |\phi_k'\rangle$$

$$= \sum_k{}' \left(\langle\phi_k|\psi\rangle + \lambda\langle\phi_k|\phi\rangle\right)|\phi_k'\rangle$$

$$= \sum_k{}' \langle\phi_k|\psi\rangle|\phi_k'\rangle + \lambda\sum_k{}' \langle\phi_k|\phi\rangle|\phi_k'\rangle$$

$$= \overbrace{\sum_k{}' \langle\phi_k'|\psi'\rangle|\phi_k'\rangle}^{=U|\psi\rangle} + \lambda\underbrace{\sum_k{}' \langle\phi_k'|\phi'\rangle|\phi_k'\rangle}_{U|\phi\rangle}$$

So
$$\mathcal{U}(|\psi\rangle + \lambda|\phi\rangle) = \mathcal{U}|\psi\rangle + \lambda\,\mathcal{U}|\phi\rangle$$
$\mathcal{U}$ is a <u>linear</u> operator.

Case 2) $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle^*$, $\mathcal{U}$ is anti-unitary.

Indeed
$$\mathcal{U}|\chi\rangle = \sum_k{}' \langle\phi_k'|\chi'\rangle |\phi_k'\rangle$$
by case 2)
$$= \sum_k{}' \langle\phi_k|\chi\rangle^* |\phi_k'\rangle$$
$$= \sum_k{}' \left(\langle\phi_k|\psi\rangle + \lambda\langle\phi_k|\phi\rangle\right)^* |\phi_k'\rangle$$
$$= \sum_k{}' \langle\phi_k|\psi\rangle^* |\phi_k'\rangle + \lambda^* \sum_k{}' \langle\phi_k|\phi\rangle^* |\phi_k'\rangle$$
$$= \sum_k{}' \langle\phi_k'|\psi'\rangle |\phi_k'\rangle + \lambda^* \sum_k{}' \langle\phi_k'|\phi'\rangle |\phi_k'\rangle$$
$$= \mathcal{U}|\psi\rangle + \lambda^* \mathcal{U}|\phi\rangle$$

Thus

$$\boxed{U\left(|\psi\rangle + \lambda|\phi\rangle\right) = U|\psi\rangle + \lambda^* U|\phi\rangle}$$

U is an anti-linear operator.

Note also for $|\psi\rangle = \sum_k{}' \psi_k |\phi_k\rangle$

$$|\omega\rangle = \sum_k{}' \omega_k |\phi_k\rangle$$

the inner products in the transformed frame are

$$\langle\omega'|\psi'\rangle = \langle\omega|\psi\rangle = \sum_k{}' \omega_k^* \psi_k$$

for the unitary case, while

$$\langle\omega'|\psi'\rangle = \langle\omega|\psi\rangle^* = \langle\psi|\omega\rangle$$

$$= \sum_k{}' \psi_k^* \omega_k$$

for the anti-unitary case.

(Wigner's Theorem References

1) Gottfried, "Quantum Mechanics Vol. I: Fundamentals," pages 226-228, section 27.1.
2) Wigner, "Group Theory and its application to the Quantum Mechanics of Atomic Spectra", pages 233-236, appendix to Chapter 20.)

Briefly then to summarize Wigner's Theorem: He showed that if a mapping of the vector space onto itself is such that

$$|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle| \quad \text{when}$$

$|\psi\rangle \xrightarrow{u} |\psi'\rangle$, $|\phi\rangle \xrightarrow{u} |\phi'\rangle$, which as we argued, physically corresponds to equivalent descriptions of the same physical system,

then a second mapping, which is merely a phase change of every vector,

$$|\psi''\rangle = e^{i\theta_{\psi'}}|\psi'\rangle,$$

can be found such that the sum

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$$

is mapped into

$$|\psi''\rangle = |\psi_1''\rangle + |\psi_2''\rangle .$$

This result implies that

$$|\langle\psi_1''|\psi''\rangle| = |\langle\psi_1|\psi\rangle| ,$$

that is

$$\langle\psi_1''|\psi_2''\rangle + \langle\psi_1''|\psi_2''\rangle^* = \langle\psi_1|\psi_2\rangle + \langle\psi_1|\psi_2\rangle^*$$

or $\mathrm{Re}\langle\psi_1''|\psi_2''\rangle = \mathrm{Re}\langle\psi_1|\psi_2\rangle$.

Now since $|\langle\psi_1''|\psi_2''\rangle| = |\langle\psi_1|\psi_2\rangle|$

this yields $(\mathrm{Im}\langle\psi_1''|\psi_2''\rangle)^2 = (\mathrm{Im}\langle\psi_1|\psi_2\rangle)^2$.

Hence case 1) $\mathrm{Im}\langle\psi_1''|\psi_2''\rangle = \mathrm{Im}\langle\psi_1|\psi_2\rangle$

case 2) $\mathrm{Im}\langle\psi_1''|\psi_2''\rangle = -\mathrm{Im}\langle\psi_1|\psi_2\rangle$

Case 1) implies $\langle\psi_1''|\psi_2''\rangle = \langle\psi_1|\psi_2\rangle$

so that the operator U, $|\psi''\rangle = U|\psi\rangle$, is unitary and linear

$$\langle\phi''|(\lambda\psi)''\rangle = \langle\phi''|U(\lambda|\psi\rangle)$$

$$= \langle\phi|\lambda\psi\rangle = \lambda\langle\phi|\psi\rangle = \lambda\langle\phi''|\psi''\rangle$$

$$= \lambda\langle\phi''|U|\psi\rangle$$

$$\Rightarrow \quad U(\lambda|\psi\rangle) = \lambda(U|\psi\rangle).$$

Case 2) implies $\langle\psi_1''|\psi_2''\rangle = \langle\psi_1|\psi_2\rangle^*$

thus U is an anti-linear operator hence anti-unitary since

$$\langle\phi''|(\lambda\psi)''\rangle = \langle\phi''|U(\lambda|\psi\rangle)$$

$$= \langle\phi|\lambda\psi\rangle^* = \lambda^*\langle\phi|\psi\rangle^*$$

$$= \lambda^*\langle\phi''|U|\psi\rangle$$

$$\Rightarrow \quad U(\lambda|\psi\rangle) = \lambda^*(U|\psi\rangle).$$

As we shall see, anti-unitary operators will only be needed when considering transformations that reverse the sign of the time. Otherwise we will only need unitary operators to relate the equivalent descriptions of our system.

5.2 Space-time Symmetries

Of particular importance are the equivalent descriptions that are related to the geometrical symmetries of our physical system. In the most general case we will consider two observers to be related by space & time translations, space rotations and uniform constant velocity boosts. The geometrical transformation equations between such inertial frames of reference will imply the existence of the quantum mechanical operators generating the corresponding symmetry transformations on the states of the system. As we anticipate the momentum operator will be related to space-translations