4.9. The p-dimensional SHO and SU(p)

Finally, before leaving the SHO, let's consider the isotropic SHO in p-dimensions. This system is equivalent to p-uncoupled 1-dimensional simple harmonic oscillators. The Hamiltonian is

$$H = \sum_{i=1}^{p} H_i$$

$$H_i = \frac{1}{2m} P_i^2 + \frac{1}{2} m\omega^2 X_i^2,$$

and $[H_i, H_j] = 0$ for all $i, j = 1 \dots p$.
Since the one dimensional Number operators $N_i = a_i^\dagger a_i$, or equivalently, Hamiltonian H_i have non-degenerate eigenvalues they can be used to uniquely label a complete set of simultaneous eigenvectors. That is $\{N_1, \dots, N_p\}$ are a CSCO. The eigenvectors are given by the direct (tensor) product of the p one-dimensional SHO eigenvectors

$$|n_1, n_2, \dots, n_p\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_p\rangle$$

shorthand we write $= |n_1\rangle |n_2\rangle \cdots |n_p\rangle$

where $H_i |n_i\rangle = \hbar\omega(n_i + \frac{1}{2}) |n_i\rangle$
or equivalently

$N_i|n_i\rangle = n_i|n_i\rangle$ for $i=1,\dots,p$ and $n_i = 0,1,2,\dots$.
Hence the Hilbert space of states has the tensor product structure $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_p$.

The p-dimensional Hamiltonian has eigenvalues E_n given by

$$H|n_1,\dots,n_p\rangle = (H_1+H_2+\cdots+H_p)|n_1\rangle\cdots|n_p\rangle$$
$$= \underbrace{(H_1|n_1\rangle)}_{=\hbar\omega(n_1+\frac{1}{2})|n_1\rangle}|n_2\rangle\cdots|n_p\rangle$$
$$+|n_1\rangle\underbrace{(H_2|n_2\rangle)}_{=\hbar\omega(n_2+\frac{1}{2})|n_2\rangle}|n_3\rangle\cdots|n_p\rangle$$
$$+\cdots+|n_1\rangle\cdots|n_{p-1}\rangle\underbrace{(H_p|n_p\rangle)}_{=\hbar\omega(n_p+\frac{1}{2})|n_p\rangle}$$

$$= \hbar\omega\left(n_1+n_2+\cdots+n_p+\frac{p}{2}\right)|n_1,\dots,n_p\rangle$$

$$\equiv E_n|n_1,\dots,n_p\rangle$$

That is $E_n = \hbar\omega\left(n_1+n_2+\cdots+n_p+\frac{p}{2}\right)$
$$= \hbar\omega\left(n+\frac{p}{2}\right),$$

$$n = n_1+n_2+\cdots+n_p = 0,1,2,\dots$$

The basis vectors of $\mathcal{H}$ are uniquely labelled by the p-integers $(n_1, \ldots, n_p)$, each of which range from 0 to ∞. The energy, on the other hand, depends only on the sum

$n = n_1 + n_2 + \cdots + n_p$. For a given integer $n \geq 0$, there exist

$$C^n_{n+p-1} = \frac{(n+p-1)!}{n!(p-1)!}$$

distinct values for $(n_1, \ldots, n_p)$ such that their sum is n. (i.e. the number of ways for $(n_1, \ldots, n_p)$ to add up to n = the number of different ways of putting n identical objects into p boxes.) Hence E_n is C^n_{n+p-1}-fold degenerate.

The creation and annihilation operators for each 1dimensional SHO are defined as usual as

$$a_i \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\, X_i + \frac{i}{\sqrt{m\hbar\omega}} P_i\right)$$

$$a_i^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\, X_i - \frac{i}{\sqrt{m\hbar\omega}} P_i\right)$$

The coordinate-momentum operator canonical commutation relations are

$$[X_i, P_j] = i\hbar\delta_{ij}$$

$$[X_i, X_j] = 0 = [P_i, P_j] ,$$

This implies the CCR for the creation and annihilation operators

$$[a_i, a_j^\dagger] = \delta_{ij}$$

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] .$$

The ground state of the system is non-degenerate and is defined by

$$|0\rangle = |\underbrace{0, 0, \ldots, 0}_{p \text{ entries}}\rangle$$

and

$$a_1|0\rangle = a_2|0\rangle = \cdots = a_p|0\rangle = 0 .$$

The excited states are given by

$$|n_1, \ldots, n_p\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots n_p!}} a_1^{\dagger n_1} \cdots a_p^{\dagger n_p} |0\rangle .$$

They are orthonormal

$$\langle n_1', \ldots, n_p' | n_1, \ldots, n_p\rangle$$
$$= \langle n_1' | n_1\rangle \langle n_2' | n_2\rangle \cdots \langle n_p' | n_p\rangle$$
$$= \delta_{n_1 n_1'} \cdots \delta_{n_p n_p'} ,$$

and complete

$$\underline{1} = \sum_{\substack{n_1, \ldots, n_p \\ =0}}^{\infty} |n_1, \ldots, n_p\rangle\langle n_1, \ldots, n_p| .$$

The individual 1-dimensional SHO number operators are given, as indicated, by

$$N_i = a_i^\dagger a_i .$$

The set $\{N_1, \ldots, N_p\}$ form a CSCO and in particular $[N_i, H] = 0$, they are constants of motion. The total number operator N can be defined as the sum of N_i

$$N = \sum_{i=1}^{p} N_i \quad .$$ Its eigenvalues are the integers $n = 0, 1, 2, \dots$.

In fact, any product $a_i^\dagger a_j$ of creation and annihilation operators commutes with H

$$[H, a_i^\dagger a_j] = 0 \qquad i,j = 1, \dots, p \, .$$

Further $(a_i^\dagger a_j)^\dagger = a_j^\dagger a_i$, so we can make p^2 Hermitian operators from them that commute with H.

1) $a_i^\dagger a_j + a_j^\dagger a_i$ for $i \neq j$, these are $\frac{p(p-1)}{2}$ different Hermitian operators

2) $i(a_i^\dagger a_j - a_j^\dagger a_i)$ for $i \neq j$, these are $\frac{p(p-1)}{2}$ different Hermitian operators

3) $a_i^\dagger a_i$, these are p different Hermitian operators,

Indeed this totals to $p + 2 \times \frac{p(p-1)}{2} = p^2$ different Hermitian operators.

Any choice of p mutually commuting Hermitian operators from this set of p^2 Hermitian operators is a CSCO. So far we have chosen the p-number operators $a_i^\dagger a_i$ as our CSCO, let's look at a few specific examples in which we explicitly choose other sets as our CSCO.

In general we must know how the different operators commute with each other in order to find the commuting set. That is we must know the operator algebra.

Since

$$N = \sum_{i=1}^{p} a_i^\dagger a_i$$

commutes with all of the operators

let's separate it from the rest of the operators $a_i^\dagger a_j$; calling the remainder p^2-1 operators

$$T_i{}^j \equiv a_i^\dagger a_j - \delta_i^j \frac{1}{p} N \,, \quad i,j = 1, \dots p.$$

The commutator of the $T_i{}^j$ is $\left(\text{Note } \sum_{i=1}^{p} T_i{}^i = 0\right)$

$$[T_i{}^j, T_k{}^l] = [a_i^\dagger a_j, a_k^\dagger a_l]$$

$\left(\text{Using } [AB,C] = A[B,C] + [A,C]B\right)$

$$= a_i^\dagger [a_j, a_k^\dagger a_l] + [a_i^\dagger, a_k^\dagger a_l] a_j$$

$$= a_i^\dagger [a_j, a_k^\dagger] a_l + a_k^\dagger [a_i^\dagger, a_l] a_j$$

$\left(\text{Using } [a_i, a_j^\dagger] = \delta_{ij}\right)$

$$= \delta_k^j a_i^\dagger a_l - \delta_i^l a_k^\dagger a_j$$

$$= \delta_k^j \left(a_i^\dagger a_l - \delta_i^l \tfrac{1}{p} N\right) + \cancel{\delta_k^j \delta_i^l \tfrac{1}{p} N}$$

$$- \delta_i^l \left(a_k^\dagger a_j - \delta_k^j \tfrac{1}{p} N\right) - \cancel{\delta_i^l \delta_k^j \tfrac{1}{p} N}$$

$$\boxed{[T_i{}^j, T_k{}^l] = \delta_k^j T_i{}^l - \delta_i^l T_k{}^j}$$

from this commutator we see that the $T_i{}^j$

The $T_i{}^j$ are called the generators of the group.

obey the $SU(p)$ Lie algebra. $\hat{SU}(p)$ is a rank $(p-1)$ group, thus only $(p-1)$ operators can be mutually commuting out of the (p^2-1) $T_i{}^j$. Along with the number operator

$$N = \sum_{i=1}^{p} a_i^{+} a_i \quad , \text{ they}$$

will form a CSCO. Since

$$[N, T_i{}^j] = 0 = [H, T_i{}^j]$$

The $T_i{}^j$ are constants of motion, as we shall see this implies H is $SU(p)$ invariant. The a_i^{+} & a_i creation and annihilation operators are tensor operators under the action of the group

$$
\begin{aligned}
[T_i{}^j, a_k] &= [a_i^{+} a_j - \tfrac{\delta_i{}^j}{p} N, a_k] \\
&= -a_j \delta_i{}^k + \frac{\delta_i{}^j}{p} a_k \\
&= -(\delta_i{}^k \delta_l{}^j - \tfrac{1}{p} \delta_i{}^j \delta_l{}^k) a_l \\
&\equiv -a_l (\hat{T}_i{}^j)_l{}^k
\end{aligned}
$$

$$[T_i{}^j, a_k^\dagger] = +(\hat{T}_i{}^j)_k{}^l a_l^\dagger$$

The $p \times p$ matrices $\hat{T}_i{}^j$ are the <u>fundamental representation</u> matrices of $SU(p)$. The $a_l^\dagger$ operators transform as the p-dimensional representation of $SU(p)$ called simply the p of $SU(p)$. The a_l are the conjugate representation called the $\bar{p}$ of $SU(p)$.

($\bar{p}$ is contravariant, p is covariant)

Since the ground state is defined by $a_i|0\rangle = 0$ for all $i = 1, \ldots, p$ we have

$$T_i{}^j|0\rangle = 0$$, the trivial

one-dimensional representation of $SU(p)$, i.e. the matrices representing $T_i{}^j$ on the state $|0\rangle$ are all zero.

Since the excited states are all made by the action of creation

Notice that the operator $T_i{}^j$ and the matrix $\hat{T}_i{}^j$ are further related by

$$T_i{}^j = a^\dagger_k (\hat{T}_i{}^j)_\ell{}^k a_\ell$$

operators on $|0\rangle$ i.e.

$$|n_1, \ldots, n_p\rangle = a_1^{\dagger n_1} \cdots a_p^{\dagger n_p} |0\rangle ,$$

They transform as the symmetric product of $n_1 + \cdots + n_p \equiv n$ fundamental p-representations of SU(p).

Thus the n^{th}-energy eigenvectors transform as this symmetric product under the SU(p) group, (just like n^{th} rank symmetric covariant tensors).

For example, we have that

The first excited states, there are p-of them,

$$|1, 0, \ldots, 0\rangle, |0, 1, 0, \ldots, 0\rangle, \ldots, |0, \ldots, 1\rangle$$

form the fundamental representation — the p of SU(p); grouping them as a column vector

$$\begin{pmatrix} |e_1\rangle \\ |e_2\rangle \\ \vdots \\ |e_p\rangle \end{pmatrix} \equiv \begin{pmatrix} |1, 0, \ldots\rangle \\ |0, 1, 0, \ldots\rangle \\ \vdots \\ |0, \ldots, 0, 1\rangle \end{pmatrix}$$

we have

$$T_i{}^j|e_k\rangle = T_i{}^j a_k^\dagger|0\rangle$$

$$= [T_i{}^j, a_k^\dagger]|0\rangle \text{ since}$$

$$T_i{}^j|0\rangle = 0$$

$$= (\hat{T}_i{}^j)_k{}^l a_l^\dagger|0\rangle$$

$$= (\hat{T}_i{}^j)_k{}^l |e_l\rangle$$

So, in the $\{|e_i\rangle\}$ basis $T_i{}^j$ has matrix elements $\hat{T}_i{}^j$.

And so on for the higher excited states.

Rather than continue on in this general fashion, let's apply these observations to the 2 and 3 dimensional isotropic SHO cases explicitly.

Example 1: $p=2$; the 2-dimensional Isotropic SHO

$$H = H_1 + H_2 \text{ with}$$

$$H_1 = \frac{1}{2m}P_1^2 + \frac{1}{2}m\omega^2 X_1^2$$

and

$$H_2 = \frac{1}{2m}P_2^2 + \frac{1}{2}m\omega^2 X_2^2$$

where X_i, P_i obey the CCR

$$[X_i, P_j] = i\hbar\delta_{ij}$$

$$[X_i, X_j] = 0 = [P_i, P_j] .$$

Introducing creation and annihilation operators

$$a_1 = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}X_1 + \frac{i}{\sqrt{m\hbar\omega}}P_1\right)$$

$$a_2 = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}X_2 + \frac{i}{\sqrt{m\hbar\omega}}P_2\right)$$

and

$$a_1^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}X_1 - \frac{i}{\sqrt{m\hbar\omega}}P_1\right)$$

$$a_2^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\, X_2 - \frac{i}{\sqrt{m\hbar\omega}} P_2\right).$$

They obey the creation and annihilation operator CCR

$$[a_i, a_j^\dagger] = \delta_{ij}$$

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger].$$

The Hamiltonian becomes

$$H = \hbar\omega(N+1)$$

with $N = N_1 + N_2$ and $N_1 = a_1^\dagger a_1$, $N_2 = a_2^\dagger a_2$.

The Hilbert space of states is spanned by the tensor product of N_1 and N_2 eigenstates. That is choosing $\{N_1, N_2\}$ as our CSCO, the basis vectors of $\mathcal{H}$ are given by $\{|n_1, n_2\rangle\}$ with

$$|n_1, n_2\rangle = |n_1\rangle |n_2\rangle$$

and

$$N_1 |n_1\rangle = n_1 |n_1\rangle \; ; \; n_1 = 0, 1, 2, \ldots$$
$$N_2 |n_2\rangle = n_2 |n_2\rangle \; ; \; n_2 = 0, 1, 2, \ldots$$

The energy eigenvalues E_n are simply

$$H|n_1,n_2\rangle = \hbar\omega(N_1+N_2)|n_1,n_2\rangle + \hbar\omega|n_1n_2\rangle$$

$$= (n_1+n_2+1)\hbar\omega\,|n_1,n_2\rangle$$

that is

$$E_n = (n_1+n_2+1)\hbar\omega = (n+1)\hbar\omega,$$

they are $(n+1)$-fold degenerate

$n=n_1+n_2$	H-eigenvalue E_n	E_n-eigenvectors
0	$\hbar\omega$	$\|0,0\rangle$
1	$2\hbar\omega$	$\|1,0\rangle, \|0,1\rangle$
2	$3\hbar\omega$	$\|2,0\rangle, \|1,1\rangle, \|0,2\rangle$
⋮	⋮	
n	$(n+1)\hbar\omega$	$\|n,0\rangle, \ldots, \|n-l,l\rangle \ldots \|0,n\rangle$
⋮	⋮	⋮

In addition to $N=\sum_{i=1}^{2} a_i^\dagger a_i$, there are 3 other bi-linear products of $a_i^\dagger$ and a_j

$$T_i{}^j \equiv a_i^\dagger a_j - \frac{\delta_{ij}}{2}N .$$

They are the generators of $SU(2)$ as can be seen from their commutation relations

$$[T_i{}^j, T_k{}^l] = \delta_k{}^j T_i{}^l - \delta_i{}^l T_k{}^j .$$

This can be made to look like the more familiar $SU(2)$ algebra by considering the 3 Hermitian operators made from $T_i{}^j$

$$T^1 \equiv \tfrac{1}{2}(T_1{}^2 + T_2{}^1) = \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

$$T^2 \equiv -\tfrac{i}{2}(T_1{}^2 - T_2{}^1) = -\tfrac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)$$

$$T^3 \equiv \tfrac{1}{2}(T_1{}^1 - T_2{}^2) = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$$

Then

$$[T^1, T^2] = -\tfrac{i}{4}[T_1{}^2 + T_2{}^1, T_1{}^2 - T_2{}^1]$$

$$= -\tfrac{i}{4}\Big(\cancelto{0}{[T_1{}^2, T_1{}^2]} - [T_1{}^2, T_2{}^1] + [T_2{}^1, T_1{}^2] - \cancelto{0}{[T_2{}^1, T_2{}^1]}\Big)$$

$$= \tfrac{i}{2}[T_1{}^2, T_2{}^1]$$

$$= \tfrac{i}{2}(\delta_2^2 T_1{}^1 - \delta_1{}^1 T_2{}^2) = \tfrac{i}{2}(T_1{}^1 - T_2{}^2)$$

$$= i T^3$$

So $[T^1, T^2] = iT^3$, similarly for cyclic permutation of the indices,

that's in general

$$\boxed{[T^i, T^j] = i\epsilon^{ijk}\,T^k \;\; ; \text{ the}}$$

familiar SU(2) algebra.

Now since $[N, T^i] = 0$ we can choose any of the T^i so that $\{N, T^i\}$ are a CSCO. Let's choose T^3, so $\{N, T^3\}$ form our CSCO our eigenstates are labelled by the eigenvalues of N and T^3, that is

$$|n, l\rangle \,, \quad \text{for}$$

a given n we have

$$N|n,l\rangle = n|n,l\rangle \quad, n = 0, 1, 2, \ldots$$

$$T^3|n,l\rangle = l|n,l\rangle \quad ; \; l = \tfrac{1}{2}(n - 2m)$$

with $m = 0, 1, 2, \ldots, n$

So that l takes on $(n+1)$ values, and of course $n = 0, 1, 2, \ldots$

That is on the $\{|n_1,n_2\rangle\}$ basis we have

$$N|n_1,n_2\rangle = (n_1+n_2)|n_1,n_2\rangle$$

$$T^3|n_1,n_2\rangle = \tfrac{1}{2}(n_1-n_2)\,|n_1,n_2\rangle$$

Hence $|n,l\rangle \equiv |n_1,n_2\rangle$ where

$$n_1 = \tfrac{1}{2}(n+2l)\ ;\ n_2 = \tfrac{1}{2}(n-2l)$$

that is $n = n_1+n_2$; $2l = n_1-n_2$ and

$$n = 0,1,2,\ldots$$

and

$$l = \tfrac{1}{2}n, \tfrac{1}{2}(n-2), \tfrac{1}{2}(n-4), \ldots, -\tfrac{1}{2}(n-2), -\tfrac{1}{2}n$$

Now within each n-subspace, i.e. for all states with energy E_n, ~~the~~ matrices representing T^i can be found.

$n=0,\ l=0$ $\quad\langle 0,0|T^i|0,0\rangle = 0,$

The trivial representation.

2) $n=1$; $\ell = -\frac{1}{2}, +\frac{1}{2}$ The states are

$$|n=1, \ell=-\tfrac{1}{2}\rangle = |n_1=0, n_2=1\rangle = a_2^{\dagger}|0\rangle$$

$$|n=1, \ell=+\tfrac{1}{2}\rangle = |n_1=1, n_2=0\rangle = a_1^{\dagger}|0\rangle$$

So we desire the matrix elements of T^i denoted $\hat{T}^i$ in the $\{|n,\ell\rangle\}_{n=1}$ basis

$$\langle n, \ell | T^i | n, \ell' \rangle \equiv (\hat{T}^i)_{\ell\ell'}$$

$$= \begin{array}{c|cc} \ell \backslash \ell' & +\frac{1}{2} & -\frac{1}{2} \\ \hline +\frac{1}{2} & \langle 0|a_1 T^i a_1^{\dagger}|0\rangle & \langle 0|a_1 T^i a_2^{\dagger}|0\rangle \\ -\frac{1}{2} & \langle 0|a_2 T^i a_1^{\dagger}|0\rangle & \langle 0|a_2 T^i a_2^{\dagger}|0\rangle \end{array}$$

For T^3 we have

$$\hat{T}^3 = \frac{1}{2}\begin{pmatrix} \langle 0|a_1(a_1^{\dagger}a_1 - \cancelto{0}{a_2^{\dagger}a_2})a_1^{\dagger}|0\rangle & \langle 0|a_1(\cancelto{0}{a_1^{\dagger}a_1} - \cancelto{0}{a_2^{\dagger}a_2})a_2^{\dagger}|0\rangle \\ \langle 0|a_2(\cancelto{0}{a_1^{\dagger}a_1} - \cancelto{0}{a_2^{\dagger}a_2})a_1^{\dagger}|0\rangle & \langle 0|a_2(\cancelto{0}{a_1^{\dagger}a_1} - a_2^{\dagger}a_2)a_2^{\dagger}|0\rangle \end{pmatrix}$$

The vanishing terms are such since
$a_i|0\rangle = 0 \Rightarrow \langle 0| a_i^\dagger$

$$\hat{T}^3 = \frac{1}{2}\begin{pmatrix} \langle 0|a_1 a_1^\dagger a_1 a_1^\dagger|0\rangle & 0 \\ 0 & -\langle 0|a_2 a_2^\dagger a_2 a_2^\dagger|0\rangle \end{pmatrix}$$

but $\langle 0|a_1 a_1^\dagger a_1 a_1^\dagger|0\rangle = \langle 0|\underbrace{[a_1, a_1^\dagger]}_{=1}\underbrace{[a_1, a_1^\dagger]}_{=1}|0\rangle$

$$= \langle 0|0\rangle = 1$$

similarly for $\langle 0|a_2 a_2^\dagger a_2 a_2^\dagger|0\rangle = 1$

So
$$\hat{T}^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Likewise we find

$$\hat{T}^1 = \frac{1}{2}\begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$$

$$\hat{T}^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

Since these matrices occur frequently in the theory of SU(p) as well as SU(2) they are given a special symbol and are called the Pauli-matrices

$$\sigma^1 \equiv \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} ; \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} ;$$

$$\sigma^3 \equiv \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Thus in the $|1,l\rangle$ space T^i is represented by 2×2 Hermitian matrices

$$\hat{T}^i = \tfrac{1}{2}\sigma^i$$

while the components of the kets in this subspace are spinors $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

3) Similarly one can continue building the matrix representations on the n-subspaces: for $n=2$ we have the basis vectors for $l = 1, 0, -1$

$$|n=2, \ell=1\rangle = \frac{1}{\sqrt{2}} a_1^{\dagger 2} |0\rangle$$

$$|n=2, \ell=0\rangle = a_1^\dagger a_2^\dagger |0\rangle$$

$$|n=2, \ell=-1\rangle = \frac{1}{\sqrt{2}} a_2^{\dagger 2} |0\rangle$$

Any ket in this energy eigenvalue $E_2 = 3\hbar\omega$ subspace of $\mathcal{H}$ has the expansion

$$|\psi_2\rangle = \sum_{\ell=-1}^{+1} \psi_{2\ell} \, |n=2, \ell\rangle$$

and so is represented by the column vectors

$$\begin{pmatrix} \psi_{2+1} \\ \psi_{20} \\ \psi_{2-1} \end{pmatrix}.$$

The matrices $\hat{T}^i$ representing the T^i operators in this space are 3×3 Hermitian matrices given by

$$\left(\hat{T}^i\right)_{\ell\ell'} = \langle n, \ell | T^i | n, \ell' \rangle .$$

As before they can be found to be

$$\hat{T}^1 = \begin{array}{c|ccc} \ell \backslash \ell' & +1 & 0 & -1 \\ \hline +1 & & & \\ 0 & & & \\ -1 & & & \end{array}\! $$

$$\hat{T}^1 = \begin{array}{c} \ell \backslash \ell' \\ +1 \\ 0 \\ -1 \end{array} \begin{array}{c} \begin{array}{ccc} +1 & 0 & -1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \frac{1}{\sqrt{2}}$$

$$\hat{T}^2 = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \frac{i}{\sqrt{2}}$$

$$\hat{T}^3 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

---

4) The higher n-matrix representations can all be built up as tensor products of these matrices. We will study them in great detail when we investigate the theory of angular momentum.

Finally we have that the operators obey $[T^i, T^j] = i\epsilon_{ijk} T^k$. The matrix representations obey the same commutation relations

$$(\hat{T}^i)_{\ell m} (\hat{T}^j)_{m\ell'} - (\hat{T}^j)_{\ell m} (\hat{T}^i)_{m\ell'}$$

$$= i\epsilon_{ijk} (\hat{T}^k)_{\ell\ell'}$$

i.e. $[\hat{T}^i, \hat{T}^j] = i\epsilon_{ijk} \hat{T}^k$.

on each E_n-subspace.

Example 2: $p = 3$: The 3-dimensional isotropic SHO

$$H = \frac{1}{2m}\vec{P}^2 + \frac{1}{2}m\omega^2\vec{R}^2 = H_1 + H_2 + H_3$$

with $H_i = \frac{1}{2m}P_i^2 + \frac{1}{2}m\omega^2 X_i^2$ and the X_i, P_i obey the CCR

$$[X_i, P_j] = i\hbar\delta_{ij}$$

$$[X_i, X_j] = 0 = [P_i, P_j].$$

As before the creation and annihilation operators can be introduced with

$$a_i = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,X_i + \frac{i}{\sqrt{m\hbar\omega}}P_i\right]$$

$$a_i^\dagger = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,X_i - \frac{i}{\sqrt{m\hbar\omega}}P_i\right]$$

obeying the CCR

$$[a_i, a_j^\dagger] = \delta_{ij}$$

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger].$$

The number operators are $N_i = a_i^+ a_i$ with $N = \sum_{i=1}^{3} N_i$; the Hamiltonian becomes

$$H = \hbar\omega\left(N + \tfrac{3}{2}\right) .$$

Since $[N_i, a_i] = -a_i$; $[N_i, a_i^+] = +a_i^+$

the eigenstates of N_i are $|n_i\rangle$ with

$$N_i|n_i\rangle = n_i|n_i\rangle \; ; \; n_i = 0, 1, 2, \ldots$$

The $\{N_1, N_2, N_3\}$ can be taken as our CSCO with their simultaneous eigenvectors given by the tensor product basis vectors

$$|n_1, n_2, n_3\rangle = \frac{1}{\sqrt{n_1! n_2! n_3!}} a_1^{+n_1} a_2^{+n_2} a_3^{+n_3} |0,0,0\rangle$$

with the ground state

defined by

$$a_1|0,0,0\rangle = a_2|0,0,0\rangle = a_3|0,0,0\rangle = 0$$

The energy eigenvalues E_n are

$$H|n_1,n_2,n_3\rangle = E_n|n_1,n_2,n_3\rangle$$
$$= \hbar\omega(n_1+n_2+n_3+\tfrac{3}{2})|n_1,n_2,n_3\rangle,$$

they are $\frac{(n+2)(n+1)}{2}$ - fold degenerate.

In addition to N, there are $3^2-1=8$ other bilinear products of a_i and $a_j^\dagger$: the generators of $SU(3)$

$$T_i{}^j = a_i^\dagger a_j - \tfrac{1}{3}\delta_{ij}N.$$

They obey the algebra

$$[T_i{}^j, T_k{}^l] = \delta_k{}^j T_i{}^l - \delta_i{}^l T_k{}^j.$$

Introducing the 8 Hermitian operators

$$T^1 \equiv \tfrac{1}{2}(T_1{}^2 + T_2{}^1) \;;\; T^2 \equiv \tfrac{-i}{2}(T_1{}^2 - T_2{}^1)$$

$$T^4 \equiv \tfrac{1}{2}(T_1{}^3 + T_3{}^1) \;;\; T^5 \equiv \tfrac{-i}{2}(T_1{}^3 - T_3{}^1)$$

$$T^6 \equiv \tfrac{1}{2}(T_2{}^3 + T_3{}^2) \;;\; T^7 \equiv \tfrac{-i}{2}(T_2{}^3 - T_3{}^2)$$

$$T^3 \equiv \tfrac{1}{2}(T_1{}^1 - T_2{}^2)$$

$$T^8 \equiv -\sqrt{3}\, T_3{}^3$$

we find that they obey the SU(3) commutation relations

$$[T^i, T^j] = i f_{ijk} T^k \qquad ; i,j,k = 1, \ldots, 8$$

with the SU(3) <u>structure constants</u> f_{ijk} given by (f_{ijk}'s completely anti-symmetric)

$$f_{123} = +1$$

$$f_{147} = f_{246} = f_{257} = f_{345} = +\tfrac{1}{2}$$

$$f_{156} = f_{367} = -\tfrac{1}{2}$$

$$f_{458} = f_{678} = \tfrac{1}{2}\sqrt{3}$$

all others not related by permutations $= 0$.

From the algebra we see only 2 out of the 8 operators can be taken to mutually commute (i.e. SU(p) is Rank(p-1))

Let's choose T^3 and T^8. Thus our CSCO is $\{N, T^3, T^8\}$. In terms of a_i and $a_i^\dagger$ the t^i are

$$T^1 = \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) \; ; \; T^2 = -\tfrac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)$$

$$T^4 = \tfrac{1}{2}(a_1^\dagger a_3 + a_3^\dagger a_1) \; ; \; T^5 = -\tfrac{i}{2}(a_1^\dagger a_3 - a_3^\dagger a_1)$$

$$T^6 = \tfrac{1}{2}(a_2^\dagger a_3 + a_3^\dagger a_2) \; ; \; T^7 = -\tfrac{i}{2}(a_2^\dagger a_3 - a_3^\dagger a_2)$$

$$T^3 = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$$

$$T^8 = -\sqrt{3}\,(a_3^\dagger a_3 - \tfrac{1}{3} N)$$

The eigenvectors of the CSCO $\{N, T^3, T^8\}$ can be found directly from the SU(3) algebra, or by noting the eigenvectors in the $\{N_1, N_2, N_3\}$ basis had the form

$$|n_1, n_2, n_3\rangle = |n_1\rangle |n_2\rangle |n_3\rangle$$

so that we can list all states which are eigenstates of the same n as we did in the wavefunction case, page -152-

n	E_n	$\|n_1,n_2,n_3\rangle$ States with $n=n_1+n_2+n_3$
0	$\frac{3}{2}\hbar\omega$	$\|0,0,0\rangle$
1	$\frac{5}{2}\hbar\omega$	$\|1,0,0\rangle, \|0,1,0\rangle, \|0,0,1\rangle$
2	$\frac{7}{2}\hbar\omega$	$\|2,0,0\rangle, \|0,2,0\rangle, \|0,0,2\rangle, \|1,1,0\rangle, \|1,0,1\rangle, \|0,1,1\rangle$
3	$\frac{9}{2}\hbar\omega$	$\vdots$
$\vdots$	$\vdots$	
n	$(n+\frac{3}{2})\hbar\omega$ $\frac{(n+1)(n+2)}{2}$ degenerate E_n states	$\|n,0,0\rangle$ $\|n-1,1,0\rangle \quad \|n-1,0,1\rangle$ $\|n-2,2,0\rangle \quad \|n-2,1,1\rangle \quad \|n-2,0,2\rangle$ $\vdots$ $\|0,n,0\rangle, \|0,n-1,1\rangle, \|0,n-2,2\rangle, \ldots, \|0,0,n\rangle$
$\vdots$	$\vdots$	$\vdots$

The operators T^3 and T^8 on the $|n_1,n_2,n_3\rangle$ vectors yield

$$T^3|n_1,n_2,n_3\rangle = \tfrac{1}{2}(n_1-n_2)|n_1,n_2,n_3\rangle$$

$$T^8|n_1,n_2,n_3\rangle = -\sqrt{3}\,n_3|n_1,n_2,n_3\rangle + \frac{1}{\sqrt{3}}(n_1+n_2+n_3)|n_1,n_2,n_3\rangle$$

Thus the states are uniquely specified by their T^3 and T^8 (and N) quantum numbers. Indeed we can expand each vector in terms of the $\{|n, l, m\rangle\}$ set of $\{N, T^3, T^8\}$ eigenvectors

For example in the $n=1$ subspace any $n=1$ vector

$$|\psi_1\rangle = \psi_{1,1}|1, \tfrac{1}{2}, \tfrac{1}{\sqrt{3}}\rangle + \psi_{1,2}|1, -\tfrac{1}{2}, \tfrac{1}{\sqrt{3}}\rangle + \psi_{1,3}|1, 0, -\tfrac{2}{\sqrt{3}}\rangle$$

that is

$$|e_1\rangle \equiv |n=1, l=+\tfrac{1}{2}, m=\tfrac{1}{\sqrt{3}}\rangle = |n_1=1, n_2=0, n_3=0\rangle$$
$$|e_2\rangle \equiv |n=1, l=-\tfrac{1}{2}, m=\tfrac{1}{\sqrt{3}}\rangle = |n_1=0, n_2=1, n_3=0\rangle$$

$$|e_3\rangle \equiv |n=1, l=0, m=-\tfrac{2}{\sqrt{3}}\rangle = |n_1=0, n_2=0, n_3=1\rangle$$

The T^i operators are represented by the Gell-Mann Matrices $\hat{T}^i = \frac{1}{2}\lambda^i$ in this subspace (these are like the Pauli matrices for SU(3))

$$\langle e_a|T^i|e_b\rangle = (\hat{T}^i)_{ab}$$

$$i = 1, 2, \ldots, 8 \quad \text{but} \quad a, b = 1, 2, 3$$

we find

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

All other subspaces, $n = 2, 3, \ldots$, can be treated similarly.

Note that the $[\hat{T}^i, \hat{T}^j] = i f_{ijk} \hat{T}^k$, the same as the T^i operators for each subspace.

The $\hat{T}^i$ are called generators of the group of SU(p) (operators) matrices since Every unitary matrix can be written as an exponential of these matrices.

For example every 2×2 unitary matrix with determinant = 1, that is every SU(2) group element in the 2 dimensional subspace, can be written as

$$U(\theta) = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = e^{i\sum_{i=1}^{3}\theta_i\frac{\sigma^i}{2}}$$

every 3×3 Special unitary matrix can be written as

$$U(\varphi) = e^{i\vec{\varphi}\cdot\frac{\vec{\lambda}}{2}} = e^{i\sum_{i=1}^{8}\varphi_i\frac{\lambda^i}{2}}$$

and so on.

Before leaving the harmonic oscillator, however, let's look at this case one more way. In particular we know that the 3 dimensional isotropic SHO potential $V(\vec{R}) = \frac{1}{2}m\omega^2\vec{R}^2$

is a central potential; hence the orbital angular momentum operator can be used to describe the degenerate energy eigenstates. That's consider

$$\vec{L} \equiv \vec{R} \times \vec{P}$$

That is in terms of the creation and annihilation operators

$$L_1 = X_2 P_3 - X_3 P_2 = -\frac{i\hbar}{2}\left(a_2^\dagger a_3 - a_3^\dagger a_2\right)$$

$$L_2 = X_3 P_1 - X_1 P_3 = +\frac{i\hbar}{2}\left(a_1^\dagger a_3 - a_3^\dagger a_1\right)$$

$$L_3 = X_1 P_2 - X_2 P_1 = -\frac{i\hbar}{2}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)$$

Then $L_1 = \hbar T^7$, $L_2 = -\hbar T^5$, $L_3 = \hbar T^2$.

Now since $[X_i, P_j] = i\hbar \delta_{ij}$ we have

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

The components of $\vec{L}$ obey the SU(2) commutation relations (as can be seen from the SU(3) algebra; $\{T^7, T^5, T^2\}$ form a SU(2) subalgebra in SU(3)).

If we also consider higher powers of $a_i, a_i^\dagger$ we have that

$$\vec{L}^2 = L_1^2 + L_2^2 + L_3^2 \quad \text{commutes}$$

with the L_1, L_2, L_3

$$[\vec{L}^2, L_i] = 0 \ .$$

Since $[H, T^i] = 0 \Rightarrow [H, \vec{L}^2] = 0 = [H, \vec{L}]$

So we can choose as a CSCO the set $\{N, \vec{L}^2, L_3\}$ with eigenvectors $\{|n, l, m\rangle\}$ defined by

$$N|n,l,m\rangle = n|n,l,m\rangle, \ n = 0,1,2,\ldots$$

$$\vec{L}^2|n,l,m\rangle = l(l+1)\hbar^2|n,l,m\rangle$$

$$L_3|n,l,m\rangle = m\hbar|n,l,m\rangle$$

we must determine the spectrum of l and m for given n.

We could appeal to wavemechanics, since after all, the eigenstates of the position operator are the tensor product of the 1-dimensional eigenstates also

$$\vec{R}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$$\Rightarrow \quad |\vec{r}\rangle = |x_1\rangle|x_2\rangle|x_3\rangle \; \left(= |x\rangle|y\rangle|z\rangle \right)$$

and we have in Cartesian coordinates

$$\psi_{n_1n_2n_3}(\vec{r}) = \langle \vec{r}\,|n_1, n_2, n_3\rangle$$

$$= \langle x_1|n_1\rangle\langle x_2|n_2\rangle\langle x_3|n_3\rangle$$

$$= \psi_{n_1}(x_1)\,\psi_{n_2}(x_2)\,\psi_{n_3}(x_3)$$

with $$\psi_{n_i}(x_i) = \left[\frac{m\omega}{\hbar\pi}\right]^{1/4} \frac{1}{\sqrt{2^{n_i} n_i!}}\, H_{n_i}\!\left(\sqrt{\frac{m\omega}{\hbar}}\, x_i\right) \times$$

$$\times\, e^{-\frac{1}{2}\frac{m\omega}{\hbar} x_i^2} .$$

But we can also expand these as well as the $|n, l, m\rangle$ states in spherical polar coordinates

$$\psi_{n\ell m}(\vec{r}) = \langle \vec{r} | n, \ell, m \rangle$$

$$= \frac{u_{n\ell}(r)}{r} Y_\ell^m(\theta, \varphi)$$

where $u_{n\ell}(r)$ obeys the radial equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \right] u_{n\ell}(r)$$

$$= \left(n + \tfrac{3}{2}\right)\hbar\omega\, u_{n\ell}(r)$$

with the BC 1) $u_{n\ell}(0) = 0$; 2) $u_{n\ell}(r \to \infty) \sim$ finite

These conditions imply that such solutions exist only for

$$\ell = n, n-2, n-4, \ldots, 0 \quad \text{for } n = \text{even}$$
$$\ell = n, n-2, n-4, \ldots, 1 \quad \text{for } n = \text{odd}$$

and of course we have already that

$$m = -\ell, -\ell+1, \ldots, \ell-1, +\ell .$$

Thus we find for each energy eigenvalue
$E_n = \hbar\omega(n + 3/2) \quad n = 0, 1, 2, \ldots$

There are g_n-degenerate states with L^2 eigenvalues given by ℓ and L_3 eigenvalue given by m

$$g_n = \sum_{\ell = 0, 2, \ldots, n} (2\ell + 1) \quad \text{for } n = \text{even}$$

$$g_n = \sum_{\ell = 1, 3, 5, \ldots, n} (2\ell + 1) \quad \text{for } n = \text{odd}$$

in both cases this adds up to

$$g_n = \frac{(n+1)(n+2)}{2} \quad \text{as we know}$$

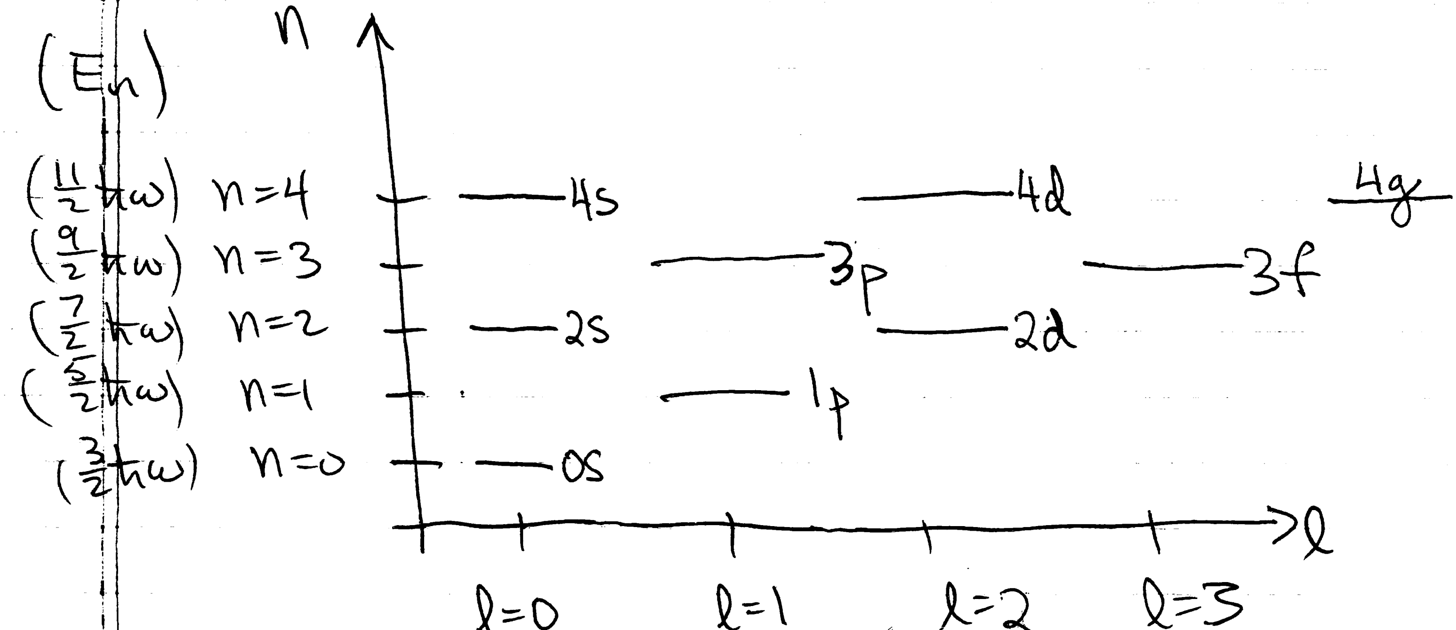
it must. Thus the set of eigenvectors of $\{N, L^2, L_3\}$ are basis ~~states~~ vectors.

The set $\{|n, \ell, m\rangle\}$ consists of vectors with $n = 0, 1, 2, \ldots$

$$\ell = \begin{cases} 0, 2, 4, \ldots, n & n, \text{even} \\ 1, 3, 5, \ldots, n & n, \text{odd} \end{cases}$$

$$m = -l, -l+1, \ldots, +l.$$

and we can plot this spectrum



Alternatively we can determine the allowed values of (l, m) for a given n by transforming to a spherical or charged basis. Let

$$A_+ \equiv \frac{1}{\sqrt{2}}(a_1 - i a_2) \; ; \; A_+^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + i a_2^\dagger)$$

$$A_0 \equiv a_3 \; ; \; A_0^\dagger = a_3^\dagger$$

$$A_- \equiv \frac{1}{\sqrt{2}}(a_1 + i a_2) \; ; \; A_-^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger - i a_2^\dagger)$$

The commutation relations $A_{\pm,0}$ obeys are given by

$$[A_\alpha, A_\beta] = 0 = [A_\alpha^\dagger, A_\beta^\dagger]$$

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta}$$

for $\alpha, \beta = +, -, 0$. These are just like creation and annihilation operator commutation relations. Then we define the number operators for $(+, -, 0)$-modes

$$N_{\pm,0} \equiv A_{\pm,0}^\dagger A_{\pm,0} \; .$$ Then $\{N_+, N_0, N_-\}$ form a CSCO. The Hamiltonian is

$$H = [N_+ + N_0 + N_- + \tfrac{3}{2}]\hbar\omega$$

and the total number operator

$$N = N_+ + N_0 + N_- \, .$$

The $\{N_+, N_0, N_-\}$ basis eigenvectors are

$$|n_+, n_0, n_-\rangle \equiv \frac{1}{\sqrt{n_+! \, n_0! \, n_-!}} A_+^{\dagger n_+} A_0^{\dagger n_0} A_-^{\dagger n_-} |0,0,0\rangle$$

with $|0,0,0\rangle$ the usual ground state

$$0 = a_1|0,0,0\rangle = a_2|0,0,0\rangle = a_3|0,0,0\rangle$$

$$\Longleftrightarrow 0 = A_+|0,0,0\rangle = A_0|0,0,0\rangle = A_-|0,0,0\rangle.$$

So for a given $n = 0, 1, 2, \ldots$

$$H|n_+, n_0, n_-\rangle = \hbar\omega\left(n + \tfrac{3}{2}\right)|n_+, n_0, n_-\rangle$$

with $n = n_+ + n_0 + n_-$.

Thus the set of $|n_+, n_0, n_-\rangle$ with $n = n_+ + n_0 + n_-$ are the degenerate eigenvectors with energy E_n.

We can differentiate between them by their $\vec{L}^2$ and L_3 eigenvalues.

In general though, $|n_+, n_0, n_-\rangle$ are not eigenvectors of $\vec{L}^2$; but they are eigenvectors of L_3, since

$$L_3 = \hbar(N_+ - N_-)$$

$$L_3|n_+,n_0,n_-\rangle = (n_+-n_-)\hbar\,|n_+,n_0,n_-\rangle$$

Thus the L_3 eigenvalue is $m = n_+ - n_-$. For

$$\equiv m\,|n_+,n_0,n_-\rangle$$

$n = n_+ + n_0 + n_-$ it can take on values

$m = +n, n-1, \ldots, -n+1, -n$. The number of vectors corresponding to a given m is found from the conditions that

$$n = n_+ + n_0 + n_- \quad \text{and} \quad m = n_+ - n_-$$

$|m| = n,\ n-1,\ n-2, \ldots, n-2s, n-(2s+1), n-(2s+2), \ldots$

$C_m = 1,\ 1,\ 2, \ldots, s+1,\ s+1,\ s+2, \ldots$

For example $m = n - 2s = n_+ - n_-$

Now for this value of m it is possible for (n_+, n_-) to take the values

$m = n - 2s = n_+ - n_-$ with $n_+, n_- \geq 0$

n_+	n_-	$n_+ + n_-$	n_0
$n-2s$	0	$n-2s$	$2s$
$n-2s+1$	1	$n-2s+2$	$2s-2$
$n-2s+2$	2	$n-2s+4$	$2s-4$
$n-2s+3$	3	$n-2s+6$	$2s-6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2s+s$	s	$n-2s+2s$	0
$n-2s+s+1$	$s+1$	$n-2s+2(s+1) > n$	-2
$\vdots$	$\vdots$		$\vdots$
$n-2s+2s$	$2s$	$n+2s > n$	$-2s$

$s+1$ allowed values of (n_+, n_0, n_-)

not allowed

But we also have the constraint that $n_0 \geq 0$ and $n_0 = n - n_+ - n_-$.

Now we use the fact that for each l there are $(2l+1)$ allowed values of $m = -l, \ldots, +l$. So for a given n the number of times the m^{th} state of L_3 angular momentum can occur is when $l \geq m$, call the number of times this happens, d_l; thus

$$c_m = \sum_{l \geq m} d_l, \quad (= d_m + d_{m+1} + \cdots)$$

hence

$$c_l - c_{l+1} = d_l .$$

From above this is (for $l=n$; $c_{n+1}=0$)

$$d_l = 1 \quad \text{for } l = n, n-2, \ldots, n-2s, \ldots$$

and $d_l = 0$ for all other values of l.

So for a given n, $l = n, n-2, \ldots, 0$, n = even
$n, n-2, \ldots, 1$, n = odd

while $m = -l, -l+1, \ldots, l-1, +l$ as found earlier.

Clearly symmetry considerations and operator algebra's are important in finding the spectrum of a CSCO's eigenvalues, as we have seen with the SHO in its various guises.

Before we embark on a systematic discussion of symmetries in quantum mechanics, and in particular angular momentum and SU(2), let's reformulate quantum mechanics from the Feynman path integral point of view employing our abstract Dirac notation.

This is an example of Wick's Theorem, let

$$\phi(t) = c(t)a + c^*(t)a^\dagger$$

$$\phi(t_1)\cdots\phi(t_n) = N[\phi(t_1)\cdots\phi(t_n)]$$