Finally, before leaving the STHO, let's consider the isotropic STHO in p-dimensions. This system is equivalent to p uncoupled 1-dimensional simple harmonic oscillators. The Hamiltonian is:

\[ H = \sum_{i=1}^{p} H_i \]

where

\[ H_i = \frac{1}{2m} \dot{P}_i^2 + \frac{1}{2} m \omega^2 \dot{R}_i^2 \]

and

\[ [H_i, H_j] = 0 \quad \text{for all} \quad i, j = 1, \ldots, p. \]

Since the one-dimensional number operators \(N_i = a_i a_i^\dagger\), or equivalently, Hamiltonian \(H_i\) have non-degenerate eigenvalues, they can be used to uniquely label a complete set of simple harmonic eigenvectors, that is \(|\mathbf{N}_i| = N_i, \ldots, N_p\) and a CSCO. The eigenvectors are given by the direct (tensor) product of the p one-dimensional STHO eigenvectors:

\[ |n_1, n_2, \ldots, n_p\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_p\rangle \]

short-hand we write \(= |n_1n_2\cdots n_p\rangle\)

where \(H_i |n_i\rangle = \hbar \omega (n_i + \frac{1}{2}) |n_i\rangle\)

or equivalently
\[ N_i |n_i\rangle = n_i |n_i\rangle \quad \text{for } i = 1, \ldots, P \text{ and } n_i = 0, 1, 2, \ldots \]

Hence the Hilbert space of states has the tensor product structure \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_P \).

The \( p \)-dimensional Hamiltonian has eigenvalues \( E_n \) given by

\[ H |n_1, \ldots, n_p\rangle = (H_1 + H_2 + \cdots + H_P) |n_1, \ldots, n_p\rangle \]

\[ = (H_1 |n_1\rangle \langle n_1| + H_2 |n_2\rangle \langle n_2| + \cdots + H_P |n_p\rangle \langle n_p|) |n_1, \ldots, n_p\rangle \]

\[ = \hbar \omega (n_1 + \frac{P}{2}) |n_1\rangle \langle n_1| + \hbar \omega (n_2 + \frac{P}{2}) |n_2\rangle \langle n_2| + \cdots + \hbar \omega (n_p + \frac{P}{2}) |n_p\rangle \langle n_p| \]

\[ = \hbar \omega (\frac{n_1 + n_2 + \cdots + n_p + P}{2}) |n_1\rangle \langle n_1| + \cdots + |n_p\rangle \langle n_p| \]

\[ = E_n |n_1, \ldots, n_p\rangle \]

Thus \( E_n = \hbar \omega (n_1 + n_2 + \cdots + n_p + \frac{P}{2}) \)

\[ = \hbar \omega (n + \frac{P}{2}) , \]

\( n = n_1 + n_2 + \cdots + n_P = 0, 1, 2, \ldots \).
The basis vectors of \( \mathcal{H} \) are labelled by the \( p \)-integers \((n_1, \ldots, n_p)\), each of which range from 0 to \( \infty \). The energy, on the other hand, depends only on the sum

\[ n = n_1 + n_2 + \cdots + n_p. \]

For a given integer \( n \geq 0 \), there exist distinct values for \((n_1, \ldots, n_p)\) such that their sum is \( n \) (i.e., the number of ways for \((n_1, \ldots, n_p)\) to add up to \( n \) = the number of different ways of putting \( n \) identical objects into \( p \) boxes). Hence \( E_n \) is \( C_n \) \(-p\)-fold degenerate.

The creation and annihilation operators for each \( 1 \)-dimensional STO are defined as usual as

\[ \alpha_i = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\hbar \omega}{\mathcal{F}}} X_i + \frac{i}{\hbar m \omega} P_i \right) \]

\[ \alpha^\dagger_i = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\hbar \omega}{\mathcal{F}}} X_i - \frac{i}{\hbar m \omega} P_i \right) \]
The coordinate-momentum operators

\[ [\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij} \]

\[ [\hat{X}_i, \hat{X}_j] = 0 = [\hat{P}_i, \hat{P}_j] \, . \]

This implies the CCR for the creation and annihilation operators

\[ [\hat{a}_i, \hat{a}^+_j] = \delta_{ij} \]

\[ [\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}^+_i, \hat{a}^+_j] \, . \]

The ground state of the system is non-degenerate and is defined by

\[ |0\rangle = |0, 0, \ldots, 0\rangle \]

and

\[ \hat{a}_i |0\rangle = \hat{a}_2 |0\rangle = \cdots = \hat{a}_p |0\rangle = 0 \, . \]
The excited states are given by
\[ |n_1, \ldots, n_p\rangle = \frac{1}{\sqrt{n_1! \cdot n_2! \cdot \ldots \cdot n_p!}} a_1^{n_1} \cdot a_p^{n_p} |0\rangle. \]

They are orthonormal
\[ \langle n'_1, \ldots, n'_p | n_1, \ldots, n_p \rangle = \langle n'_1|n_1\rangle \langle n'_2|n_2\rangle \cdots \langle n'_p|n_p\rangle = \delta_{n_1,n'_1} \cdot \delta_{n_2,n'_2} \cdots \delta_{n_p,n'_p}, \]

and complete
\[ 1 = \sum_{n_1, \ldots, n_p} \sum_{n'_1, \ldots, n'_p} |n_1, \ldots, n_p\rangle \langle n'_1, \ldots, n'_p| \]

The individual 1-dimensional SSH number operators are given, as indicated, by
\[ N_i = a_i^+ a_i. \]

The set \( \mathcal{S} = \{N_1, \ldots, N_p\} \) form a CSSO and in particular, \( [N_i, H] = 0 \), they are
Constants of motion. The total number operator $N$ can be defined as the sum of $N_i$

$$N = \sum_{i=1}^{\infty} N_i$$

Its eigenvalues are the integers $n = 0, 1, \ldots$.

In fact, any product $a_i^+ a_j$ of creation and annihilation operators commutes with $H$,

$$[H, a_i^+ a_j] = 0 \quad i, j = 1, \ldots, \infty$$

Further $(a_i^+ a_j)^+ = a_j^+ a_i^+$, so we can make $p^2$-Hermitian operators from them that commute with $H$.

1) $a_i^+ a_j + a_j^+ a_i$ for $i \neq j$; there are $P(p-1)$ different operators $p^2$-Hermitian.

2) $i(a_i^+ a_j - a_j^+ a_i)$ for $i \neq j$; there are $P(p-1)$ different operators $p^2$-Hermitian.
3) $a_i a_i$, these are $p$ different Hermitian operators.

Indeed this totals to $p + 2 \times \frac{p(p-1)}{2} = p^2$ different Hermitian operators.

Any choice of $p$ mutually commuting Hermitian operators from this set of $p^2$ Hermitian operators is a CSCO. So far we have chosen the $p$-number operators $a_i a_i$ as our CSCO. Let's look at a few specific examples in which we explicitly choose other sets as our CSCO.

In general we must know how the different operators commute with each other in order to find the commuting set. That is, we must know the operator algebra.

Since

$$N = \sum_{i=1}^{p^2} a_i^+ a_i$$

commutes with all of the operators
Let separately it from the rest of the operators $a^i a_j$; calling the remainder $P^{2-1}$ operators

$$T_i^j = a^i a_j - \delta^i_j \frac{1}{P} N,$$

The commutator of the $T_i^j$ is

$$\left[ T_i^j, T_h^k \right] = [a^i a_j, a^k a_l] = a^i [a^j a_k a_l] + [a^i, a^k a_l a_j] = a^i [a^j, a_k a_l] a_j + a^k [a^i, a_l a_j]$$

(Using $[AB, C] = ABC + ACB$)

(Using $[a^i a^j] = \delta^i_j$)

$$= \delta^j_k a^i a_l a_j - \delta^i_j a^k a_l a_j = \delta^j_k (a^i a_l - \delta^i_j \frac{1}{P} N) + \delta^i_j \delta^k_l \frac{1}{P} N$$

$$- \delta^j_l (a^k a_j - \delta^k_l \frac{1}{P} N) - \delta^k_l \delta^j_l \frac{1}{P} N$$

$$\left[ T_i^j, T_h^k \right] = \delta^j_k T_i^h - \delta^i_j T_h^k$$

From this commutator we see that the $T_i^j$
The $T_{i,j}$ are called the generators of the group.

They obey the $SU(p)$ Lie algebra. $SU(p)$ is a reducible $(p-1)$-group, thus only $(p-1)$ operators can be mutually commuting out of the $p^2-1$ $T_{i,j}$. Along with the number operator

$$ N = \frac{p}{2} \sum_i a^+_i a_i, $$

they will form a CS CO. Since

$$ [N, T_{i,j}] = 0 = [H, T_{i,j}] $$

The $T_{i,j}$ are constants of motion as we shall see their implies $H$ is $SU(p)$ invariant. The $a^+_i$, creation and annihilation operators, are traces under the action of the group

$$ [T_{i,j}, a_k] = [a^+_i a^+_j - \delta_{i,j} N, a_k] $$

$$ = -a^+_i \delta_{i,k} + \delta_{i,j} a_k $$

$$ = -\left(\delta_{i,k} \delta_{i,j} - \frac{1}{p} \delta_{i,j} S_{i,k}\right) a_k $$

$$ = -a_k (T_{i,j})^k_{i,j} $$
\[ [T_{ij}, \alpha_k^+] = +(\check{T}_{ij})_{k} \alpha_{l}^{+} \]

The \( p \times p \) matrices \( \check{T}_{ij} \) are the fundamental representation matrices of \( SU(p) \). The \( \alpha_{l}^{+} \) operators transform as the \( p \)-dimensional representation of \( SU(p) \) called simply the \( F \) of \( SU(p) \).

The \( \alpha_{l} \) are the conjugate representation, called the \( F^{\dagger} \) of \( SU(p) \).

Since the ground state is defined by \( \alpha_{i} |0\rangle = 0 \) for all \( i = 1, ..., p \) we have

\[ T_{ij} |0\rangle = 0 ; \text{ the trivial one-dimensional representation of } SU(p), \]

i.e., the matrices representing \( T_{ij} \) are the state \( |0\rangle \) are all zero.

Since the excited states are all made by the action of creation
Notice that the operators $T_{i:j}$ and the matrix $\hat{T}_{i:j}$ are further related by

$$T_{i:j} = a^+_k (\hat{T}_{i:j})_k a_l$$
operators on $|0\rangle$ i.e. $|n_1, ..., n_p\rangle = a_{1}^{+n_1} ... a_{p}^{+n_p} |0\rangle$, 

They transform as the symmetric product of $n_1 + ... + n_p = \alpha$ fundamental $\rho$-representations of $SU(p)$. 

Thus the $n^{th}$-energy eigenvectors transform as field symmetric product under the $SU(p)$ group (just like $n^{th}$ rank symmetric contravariant tensor).

For example, we have that the first excited states, there are $p$-of them $|1, 0, ..., 0\rangle, |10, 0, ..., 0\rangle, ... , |10, ... 1\rangle$ form the fundamental representation -- the $p$ of $SU(p)$; grouping them as a column vector:

\begin{pmatrix}
|1e_1\rangle \\
|1e_2\rangle \\
... \\
|1e_p\rangle \\
\end{pmatrix} = \begin{pmatrix}
|1, 0, ... \rangle \\
|10, 1, ... \rangle \\
... \\
|10, ... 0, 1\rangle \\
\end{pmatrix}
we have
\[ T_{ij} |e_k\rangle = T_{ij} a_k^+ |10\rangle \]
\[ = [T_{ij} a_k^+] |10\rangle \text{ since } T_{i0} |10\rangle = 0 \]
\[ = (T_{ij})_{kl}^k a_k^+ |10\rangle \]
\[ = (T_{ij})_{kl}^k |e_k\rangle \]
So in the \{ |e_i\rangle \} basis \( T_{ij} \) has matrix elements \( T_{ij} \).
And so on for the higher excited states.

Rather than continue on in their general fashion, let's apply these observations to the 1D and 2D dimensional isotropic STO cases explicitly.
Example 1: $p=2$; the 2-dimensional Isotropic SHO

$H = H_1 + H_2$ with

$H_1 = \frac{1}{2m} \mathbf{P}_1^2 + \frac{1}{2} m \omega^2 \mathbf{X}_1^2$

and

$H_2 = \frac{1}{2m} \mathbf{P}_2^2 + \frac{1}{2} m \omega^2 \mathbf{X}_2^2$

where $X_i, P_i$ obey the CCR

$[X_i, P_j] = i \hbar \delta_{ij}$

$[X_i, X_j] = 0 = [P_i, P_j]$.

Introducing creation and annihilation operators $\mathbf{a}_i$

$\mathbf{a}_1 = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m \omega}{\hbar}} \mathbf{X}_1 + \frac{i}{\sqrt{m \hbar \omega}} \mathbf{P}_1 \right)$

$\mathbf{a}_2 = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m \omega}{\hbar}} \mathbf{X}_2 + \frac{i}{\sqrt{m \hbar \omega}} \mathbf{P}_2 \right)$

and

$\mathbf{a}_1^* = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m \omega}{\hbar}} \mathbf{X}_1 - \frac{i}{\sqrt{m \hbar \omega}} \mathbf{P}_1 \right)$
\[ a_2^+ = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m}{\hbar}} x_2 - \sqrt{\frac{\hbar}{m}} p_2 \right) , \]

They obey the creation and annihilation operators CCR

\[ [a_i^+, a_j^+] = \delta_{ij} \]
\[ [a_i, a_j] = 0 = [a_i, a_j^+] \]

The Hamiltonian becomes

\[ H = \hbar \omega (N+1) \]

with \( N = N_1 + N_2 \) and \( N_i = a_i^+ a_i \), \( N_2 = a_2^+ a_2 \).

The Hilbert space of states is spanned by the tensor product of \( N_1 \) and \( N_2 \) eigenstates. That is, choosing \( \{ N_1, N_2 \} \) as our CS CO, the basis vectors of \( H \) are given by \( \{ n_1, n_2 \} \) with

\[ | n_1, n_2 \rangle = | n_1 \rangle | n_2 \rangle \]

and

\[ N_1 | n_1 \rangle = n_1 | n_1 \rangle , \quad n_1 = 0, 1, 2, \ldots \]
\[ N_2 | n_2 \rangle = n_2 | n_2 \rangle , \quad n_2 = 0, 1, 2, \ldots \]
The energy eigenvalues $E_n$ are simply
\[ H |n_1, n_2\rangle = \hbar \omega (n_1 + n_2) |n_1, n_2\rangle + \hbar \omega |n_2\rangle \]

Hence
\[ E_n = (n_1 + n_2 + 1) \hbar \omega = (n + 1) \hbar \omega, \]
they are $(n+1)$-fold degenerate.

<table>
<thead>
<tr>
<th>$N = n_1 + n_2$</th>
<th>$H$-eigenvalue $E_n$</th>
<th>$E_n$-eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\hbar \omega$</td>
<td>$10,0\rangle$</td>
</tr>
<tr>
<td>1</td>
<td>$2 \hbar \omega$</td>
<td>$11,0\rangle, 10,1\rangle$</td>
</tr>
<tr>
<td>2</td>
<td>$3 \hbar \omega$</td>
<td>$12,0\rangle, 11,1\rangle, 10,2\rangle$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$N$</td>
<td>$(n+1) \hbar \omega$</td>
<td>$1n,0\rangle, \ldots, 1n-e, e\rangle, \ldots, 1n, N\rangle$</td>
</tr>
</tbody>
</table>

In addition to $N = \frac{1}{2} \sum_i a_i^+ a_i$, there are 3 other bi-linear products of $\dot{a}_i$ and $a_i$:
\[ T_{i,j} = a_i^+ a_j - \frac{\delta_{ij}}{2} N \]
They are the generators of SU(2) as can be seen from their commutation relations

\[ [T_i, T_k] = \delta_{jk} T_i - \delta_{ik} T_j. \]

This can be made to look like the more familiar SU(2) algebra by considering the 3 Hermitian operators made from \( T_i \):

\[
T_1^\dagger = \frac{1}{2} (T_1^2 + T_2^2) = \frac{1}{2} (a_1^+ a_2 + a_2^+ a_1)
\]
\[
T_2^\dagger = -\frac{i}{2} (T_1^2 - T_2^2) = -\frac{i}{2} (a_1^+ a_2 - a_2^+ a_1)
\]
\[
T_3^\dagger = \frac{1}{2} (T_1^1 - T_2^2) = \frac{1}{2} (a_1^+ a_1 - a_2^+ a_2)
\]

Then

\[
[T_1^\dagger, T_2^\dagger] = -\frac{i}{4} \left[ T_1^2 + T_2^2, T_1^2 + T_2^2 \right]
\]

\[
= -\frac{i}{4} \left( [T_1^2, T_1^2] - [T_1^2, T_2^2] + [T_2^2, T_1^2] - [T_2^2, T_2^2] \right)
\]

\[
= \frac{i}{2} \left[ T_1^2, T_2^2 \right]
\]

\[
= \frac{i}{2} \left( \delta_{2}^1 T_1^1 - \delta_{1}^1 T_2^2 \right) = \frac{i}{2} (T_1^1 - T_2^2)
\]

\[
= i T_3^\dagger
\]
So \[ [T^1, T^2] = i T^3 \]
similarly for cyclic permutation of the indices.

That's in general
\[ [T_i, T_j] = i \epsilon_{ijk} T^k \]

the familiar SU(2) algebra.

Now since \[ [N, T_i] = 0 \]
we can choose any of the \( T_i \) so that \( EN T^3 \) are a CSCO. Let's choose \( T^3 \); so \( EN T^3 \) form our CSCO. Our eigenvalues are labelled by the eigenvalues of \( N \) and \( T^3 \): that's
\[ |n, l\rangle \]
for a given \( n \) we have
\[ N |n, l\rangle = n |n, l\rangle \quad n = 0, 1, 2, \ldots \]
\[ T^3 |n, l\rangle = l |n, l\rangle \quad l = \frac{1}{2}(n - 2m) \]
with \( m = 0, 1, 2, \ldots n \)

so that \( l \) takes on \((n+1)\) values, and of course \( n = 0, 1, 2, \ldots \).
That is on the $|n_1, n_2 >^3$ basis we have

$|n_1, n_2 > = (n_1 + n_2) |n_1, n_2 >$

$T^3 |n_1, n_2 > = \frac{1}{2} (n_1 - n_2) |n_1, n_2 >$

Hence $|n, l > \equiv |n_1, n_2 >$ where

$n_1 = \frac{1}{2} (n + 2l)$; $n_2 = \frac{1}{2} (n - 2l)$

That is $n = n_1 + n_2$; $2l = n_1 - n_2$ and

$n = 0, 1, 2, ...$

and

$l = \frac{1}{2} n, \frac{1}{2} (n - 2), \frac{1}{2} (n - 4), ..., \frac{1}{2} (n - 2) - \frac{1}{2} n$

Now within each $N$-subspace, i.e. for all states with energy $E_N$, the matrices representing $T^i$ can be found.

$n = 0, l = 0$ \hspace{1cm} \langle 0, 0 | T^i | 0, 0 > = 0$

The trivial representation.
2) \( n=1; \quad l=\frac{-1}{2}, \frac{1}{2} \)

The states are

\[
\begin{align*}
|n=1, l=-\frac{1}{2}\rangle &= |n=0, n_z=1\rangle = a_2^+ 10 \\
|n=1, l=+\frac{1}{2}\rangle &= |n=1, n_z=0\rangle = a_1^+ 10
\end{align*}
\]

So we desire the matrix elements of \( T^i \) denoted \( \langle n, l | T^i | n', l' \rangle \) in the \( \{|n, l\rangle\}_{n=1}^{3} \) basis

\[
\langle n, l | T^i | n', l' \rangle = \left( \frac{\lambda}{2} \right)^{l+\frac{1}{2}} \left( \begin{array}{c}
\frac{1}{2} \\
-\frac{1}{2}
\end{array} \right) \left( \begin{array}{c}
\langle 01a_1 T^i a_1^+ 10 \rangle \\
\langle 01a_2 T^i a_2^+ 10 \rangle
\end{array} \right)
\]

For \( T^3 \) we have

\[
\left( \frac{\lambda}{2} \right)^{l+\frac{1}{2}} \left( \begin{array}{c}
\frac{1}{2} \\
-\frac{1}{2}
\end{array} \right) \left( \begin{array}{c}
\langle 01a_1 (a_1^+ a_1 - a_2 a_2^+) 10 \rangle \\
\langle 01a_2 (a_1^+ a_1 - a_2 a_2^+) 10 \rangle \\
\langle 01 (a_1^+ a_1^+ a_2^+ a_2) 10 \rangle \\
\langle 01 (a_2^+ a_2^+ a_2^+ a_2) 10 \rangle
\end{array} \right)
\]
The vanishing terms are such since $a_i |10\rangle = 0 = \langle 10 | a_i$.

\[
T^3 = \frac{1}{2} \begin{pmatrix}
\langle 01a_i a_i^+ a_i^+ |10\rangle & 0 \\
0 & \langle 01a_2 a_2^+ a_2^+ |10\rangle
\end{pmatrix}
\]

But $\langle 01a_i a_i^+ a_i^+ |10\rangle = \langle 01[ a_i a_i^+ ] a_i^+ |10\rangle = \langle 10 | = 1$

Similarly for $\langle 01a_2 a_2^+ a_2^+ |10\rangle = 1$

So

\[
T^3 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Likewise we find

\[
T^1 = \frac{1}{2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

\[
T^2 = \frac{1}{2} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}
\]
Since these matrices occur frequently in the theory of SU(2) as well as SU(3),
they are given a special symbol and are called the Pauli matrices:

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Thus in the 11, \ell \geq space \text{T}_\ell \text{i} is represented by 2 \times 2 Hermitian matrices:

\[ \hat{T}_\ell \text{i} = \frac{i}{2} \sigma^\ell \text{i}. \]

while the components of the ket \text{k}_\ell \text{i} in this subspace are bispinors \( |\ell\rangle \).

3) Similarly we can continue building the matrix representations on the \( N \)-subspaces. For \( N = 2 \) we have the Pauli vectors for \( \ell = 1, 0, -1 \).
\[ |n=2, l=1\rangle = \frac{-2460}{52} a_1^2 10\rangle \]
\[ |n=2, l=0\rangle = a_1^+ a_2^+ 10\rangle \]
\[ |n=2, l=-1\rangle = \frac{1}{12} a_2^+ 10\rangle \]

Any ket in this energy eigenvalue \( E_2 = 3hw \)

Sub-space of \( H \) has the expansion

\[ |\Psi_{2>0}\rangle = \sum_{l=-1}^{+1} \psi_{2,2l} |n=2, l\rangle \]

and so is represented by the column vectors

\[
\begin{pmatrix}
\psi_{2,-1} \\
\psi_{2,0} \\
\psi_{2,1}
\end{pmatrix}
\]

The matrices \( \hat{T} \) representing the \( \hat{T} \) operators in this space are \( 3 \times 3 \) Hermitian matrices given by

\[
\left( \hat{T}^{\dagger} \right)_{l l'} = \langle n, l | T^\dagger | n, l' \rangle .
\]
As before they can be found to be

\[ \frac{\hat{T}}{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \]

\[ \frac{\hat{T}}{2} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{i}{\sqrt{2}} \]

\[ \frac{\hat{T}}{3} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

4) The higher \( n \)-matrix representations can all be built up as tensor products of these matrices. We will study them in great detail when we investigate the theory of angular momentum.
Finally we have that the operators obey
\[ [T^i, T^j] = i \epsilon_{ijk} T^k. \]

The matrix representations obey the same commutation relation
\[
(\hat{T}^i)_{lm} (\hat{T}^j)_{m'l'} - (\hat{T}^j)_{lm} (\hat{T}^i)_{m'l'} = i \epsilon_{ijk} (\hat{T}^k)_{ll'}
\]

i.e., \[ [\hat{T}^i, \hat{T}^j] = i \epsilon_{ijk} \hat{T}^k. \]

On each \( E_n \)-subspace.
Example 2: $p = 3$. The 3-dimensional isotropic ETH

$$H = \frac{1}{2m} \dot{P}^2 + \frac{1}{2} m \omega^2 \vec{R}^2 = H_1 + H_2 + H_3$$

with $H_i = \frac{1}{2m} P_i^2 + \frac{1}{2} m \omega^2 \vec{X}_i^2$ and the $\vec{X}_i, \vec{P}_i$ obey the CCR

$$[\vec{X}_i, \vec{P}_j] = i \hbar \delta_{ij}$$

$$[\vec{X}_i, \vec{X}_j] = 0 = [\vec{P}_i, \vec{P}_j].$$

As before, the creation and annihilation operators can be introduced with

$$a_i = \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{m \omega}{\hbar}} \vec{X}_i + \frac{i}{\sqrt{m \hbar \omega}} \vec{P}_i \right]$$

$$a^+_i = \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{m \omega}{\hbar}} \vec{X}_i - \frac{i}{\sqrt{m \hbar \omega}} \vec{P}_i \right]$$

obeying the CCR

$$[a_i, a^+_j] = \delta_{ij}$$

$$[a_i, a_j] = 0 = [a^+_i, a^+_j].$$
The number operators are $N_i = a_i^+ a_i$ with $N = \sum_i N_i$; the Hamiltonian becomes

$$H = \hbar \omega (N + \frac{3}{2})$$

Since $[N_i, a_i^+] = -a_i$; $[N_i, a_i] = +a_i^+$

the eigenstates of $N_i$ are $|n_i\rangle$ with

$$N_i |n_i\rangle = n_i |n_i\rangle; n_i = 0, 1, 2, \ldots$$

The $3N_1, N_2, N_3$ can be taken as our CSCO with their simultaneous eigenvectors given by the tensor product basis vectors

$$|n_1, n_2, n_3\rangle = \frac{1}{\sqrt{n_1! n_2! n_3!}} a_1^{+ n_1} a_2^{+ n_2} a_3^{+ n_3} |0, 0, 0\rangle$$

with the ground state defined by

$$a_1^{+ 10, 0, 0} = a_2^{+ 10, 0, 0} = a_3^{+ 10, 0, 0} = 0$$
The energy eigenvalues $E_n$ are

$$H | n_1, n_2, n_3 \rangle = E_n | n_1, n_2, n_3 \rangle = \hbar \omega (n_1 + n_2 + n_3 + \frac{3}{2}) | n_1, n_2, n_3 \rangle$$

They are $\frac{(n+2)(n+1)}{2}$-fold degenerate.

In addition to $N$ there are $3^2 - 1 = 8$ other bilinear products of $a_i$ and $a_i^\dagger$; the generators of $SU(3)$

$$T_{ij} = a_i^\dagger a_j - \frac{1}{3} \delta_{ij} N.$$ They obey the algebra

$$[T_{ij}, T_{k\ell}] = 8 \delta_{ij} T_{k\ell} - 8 \delta_{k\ell} T_{ij}.$$ Introducing the 8 Hermitian operators
\[-466-\]

\[
T^1 = \frac{1}{2} (T_1^2 + T_2^2) \quad T^2 = \frac{i}{2} (T_1^2 - T_2^2)
\]

\[
T^4 = \frac{1}{2} (T_1^3 + T_3^3) \quad T^5 = \frac{i}{2} (T_1^3 - T_3^3)
\]

\[
T^6 = \frac{1}{2} (T_2^3 + T_3^3) \quad T^7 = \frac{i}{2} (T_2^3 - T_3^3)
\]

\[
T^3 = \frac{1}{2} (T_1^2 - T_2^2)
\]

\[
T^8 = -\sqrt{3} \quad T_3^3
\]

We find that these obey the SU(3) commutation relations:

\[
[T^i, T^j] = i f^{ijk} T^k \quad ; \quad i, j, k = 1, \ldots, 8
\]

with the SU(3) structure constants \( f^{ijk} \), given by (\( f^{ijk} \) is completely anti-symmetric)

\[
f_{123} = +1
\]

\[
f_{147} = f_{246} = f_{257} = f_{345} = +\frac{1}{2}
\]

\[
f_{156} = f_{367} = -\frac{1}{2}
\]

\[
f_{458} = f_{678} = \frac{1}{2} \sqrt{3} \quad \text{all others not related by permutations} = 0.
\]
From the algebra we see only 2 out of the 8 operators can be taken to mutually commute (i.e. $SU(p) \times SU(2)$ rank $(p-1)$)

Let's choose $T^3$ and $T^8$. Thus our CSCO is $EN, T^3, T^8$. In terms of $a_i$ and $a_i^+$ the $T^i$ are:

\[
T^1 = \frac{1}{2}(a_1 a_2 + a_2 a_1) \quad T^2 = \frac{i}{2}(a_1 a_2 - a_2 a_1) \\
T^4 = \frac{1}{2}(a_1 a_3 + a_3 a_1) \quad T^5 = \frac{i}{2}(a_1 a_3 - a_3 a_1) \\
T^6 = \frac{1}{2}(a_2 a_3 + a_3 a_2) \quad T^7 = \frac{i}{2}(a_2 a_3 - a_3 a_2) \\
T^3 = \frac{1}{2}(a_1 a_1 - a_2 a_2) \\
T^8 = -\sqrt{3}(a_3 a_3 - \frac{1}{3} N)
\]

The eigenvectors of the CSCO $EN, T^3, T^8$ can be found directly from the $SU(C)$ algebra or by noting the eigenvectors in the $EN, a_i, a_i^+$ basis take the form

\[
|n_1, n_2, n_3> = |n_1> |n_2> |n_3>
\]

so that we can list all states which have eigenstates of the same $n$.
as we did in the azeotropic case, page 152.

\[
\begin{array}{c|c}
\text{N} & \text{E}_n \\
\hline
0 & \frac{3}{2} \hbar \omega \\
1 & \frac{5}{2} \hbar \omega \\
2 & \frac{7}{2} \hbar \omega \\
3 & \frac{9}{2} \hbar \omega \\
\vdots & \vdots \\
N & (N+\frac{3}{2}) \hbar \omega \\
\end{array}
\]

\[
\prod_{i=1}^{n} \text{degenerate states} \quad \text{with} \quad n = n_1 + n_2 + n_3
\]

\[
\begin{align*}
10,0,0 > & \quad \text{10, 0, 0> } \\
11,0,0 > & \quad \text{11, 0, 0> } \\
10,1,0 > & \quad \text{10, 1, 0> } \\
10,0,1 > & \quad \text{10, 0, 1> } \\
12,0,0 > & \quad \text{12, 0, 0> } \\
10,2,0 > & \quad \text{10, 2, 0> } \\
10,0,2 > & \quad \text{10, 0, 2> } \\
11,0,0 > & \quad \text{11, 0, 0> } \\
11,0,1 > & \quad \text{11, 0, 1> } \\
10,1,1 > & \quad \text{10, 1, 1> } \\
\end{align*}
\]

The operators $T^3$ and $T^8$ on the $|n_1,n_2,n_3\rangle$ vectors yield

\[
T^3 |n_1,n_2,n_3\rangle = \frac{1}{2}(n_1-n_2) |n_1,n_2,n_3\rangle
\]

\[
T^8 |n_1,n_2,n_3\rangle = -\sqrt{3} n_3 |n_1,n_2,n_3\rangle + \frac{1}{\sqrt{3}} (n_1+n_2+n_3) |n_1,n_2,n_3\rangle
\]
Thus the states are uniquely specified by their $T^3$ and $T^8$ (and $N$) quantum numbers. Indeed we can expand each vector in terms of the $\{N, l, m\} > 3$ set of $\{N, T^3, T^8\}$ eigenvectors.

For example in the $N=1$ subspace any $n=1$ vector

$$|\text{e}_1\rangle = |n=1, l=\frac{1}{2}, m=\frac{1}{\sqrt{3}}\rangle = |n_1=1, n_2=0, n_3=0\rangle$$

$$+ 2|n_{1/3} \mid 1, 0, -\frac{2}{\sqrt{3}}\rangle$$

Similarly

$$|\text{e}_2\rangle = |n=1, l=-\frac{1}{2}, m=\frac{1}{\sqrt{3}}\rangle = |n_1=0, n_2=1, n_3=0\rangle$$

$$|\text{e}_3\rangle = |n=1, l=0, m=-\frac{2}{\sqrt{3}}\rangle = |n_1=0, n_2=0, n_3=-1\rangle$$

The $T^i$ operators are represented by the Gell-Mann matrices $\frac{\lambda_i}{2}$ in this subspace ([these are like the Pauli matrices for $SU(3)$]).

$$\langle \text{e}_a \mid T^i \mid \text{e}_b \rangle = (\frac{i}{2})_{ab}$$

$i = 1, 2, \ldots, 8$ but $a, b = 1, 2, 3$
we find
\[ x^1 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad x^2 = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ x^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad x^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \]
\[ x^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad x^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \]
\[ x^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ x^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]

All other subspaces, \( n=2,3,\ldots \), can be treated similarly.

Note that the \([T^i, T^j] = i\epsilon_{ijk} T^k\), the same as the \(T^i\) operators for each subspace.
The $T_i$ are called generators of the group of $SU(p)$ (operators) matrices since every unitary matrix can be written as an exponential of these matrices.

For example, every $2 \times 2$ unitary matrix with determinant $1$, that is, every $SU(2)$ group element in the 2 dimensional subspace, can be written as

$$U(\theta) = e^{i \theta \cdot \frac{\pi}{2}} = e^{i \frac{\cos \theta}{2} \frac{\pi}{2}}$$

Every $3 \times 3$ special unitary matrix can be written as

$$U(\psi) = e^{i \phi \cdot \frac{\pi}{2}} = e^{i \frac{\psi}{2} \frac{\pi}{2}}$$

and so on.

Before leaving the harmonic oscillator, however, let's look at this case one more way. In particular we know that the 3 dimensional isotropic $\text{SHO}$ potential $V(R) = \frac{1}{2} \omega^2 R^2$
In a central potential, hence the orbital angular momentum operator can be used to describe the degenerate eigenstates. That is, consider

\[ \vec{L} = \vec{R} \times \vec{P} \]

That is in terms of the creation and annihilation operators

\[ L_1 = X_2 P_3 - X_3 P_2 = -\frac{i\hbar}{2} (a_2^+ a_3 - a_3^+ a_2) \]
\[ L_2 = X_3 P_1 - X_1 P_3 = +\frac{i\hbar}{2} (a_1^+ a_3 - a_3^+ a_1) \]
\[ L_3 = X_1 P_2 - X_2 P_1 = -\frac{i\hbar}{2} (a_1^+ a_2 - a_2^+ a_1) \]

Then \( L_1 = \hbar T^7, L_2 = -\hbar T^5, L_3 = \hbar T^2 \).

Now since \([X_i, P_j] = i\hbar \delta_{ij}\) we have

\[ [L_i, L_j] = i\hbar \varepsilon_{ijk} L_k \]

The components of \( \vec{L} \) obey the SU(2) commutation relations (as can be seen from the SU(3) algebra; \( ST^7 T^5 T^3 \) form a SU(2) subalgebra in SU(3)).
If we also consider higher powers of $\hat{a}_i^\dagger$, we have that

$$\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$$

with the $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_3$

$$[\hat{L}_i^2, \hat{L}_j] = 0.$$

Since $[\hat{H}, \hat{T}_j^2] = 0 \Rightarrow [\hat{H}, \hat{L}_j^2] = 0 = [\hat{H}, \hat{L}_j^2]$,

some can choose as a CSO the set $\{\hat{N}, \hat{L}_1^2, \hat{L}_3^2\}$ with eigenvectors $\{\mid n, l, m \rangle \}$ defined by

$$\hat{N} \mid n, l, m \rangle = n \mid n, l, m \rangle, \quad n = 0, 1, 2, \ldots$$

$$\hat{L}_1^2 \mid n, l, m \rangle = (l(l+1)) \mid n, l, m \rangle$$

$$\hat{L}_3 \mid n, l, m \rangle = m \mid n, l, m \rangle$$

we must determine the spectrum of ladder given $n$. 

We could appeal to wave mechanics, since after all, the eigenstates of the position operators are the tensor product of the 1-dimensional eigenstates:

\[ R |\vec{r}\rangle = \vec{r} |\vec{r}\rangle \]

\[ |\vec{r}\rangle = |x_{1}\rangle |x_{2}\rangle |x_{3}\rangle = |x_{1}\rangle |y_{1}\rangle |z_{1}\rangle \]

and we have in Cartesian coordinates:

\[ 4_{n_{1}n_{2}n_{3}}(\vec{r}) = \langle \vec{r} | n_{1}, n_{2}, n_{3} \rangle \]

\[ = \langle x_{1} | n_{1} \rangle \langle x_{2} | n_{2} \rangle \langle x_{3} | n_{3} \rangle \]

\[ = 4_{n_{1}}(x_{1}) 4_{n_{2}}(x_{2}) 4_{n_{3}}(x_{3}) \]

with

\[ 4_{n_{i}}(x_{i}) = \left[ \frac{m_{0}}{\hbar} \right]^{\frac{1}{2}} \frac{1}{\sqrt{2^{n_{i}} n_{i}!}} H_{n_{i}} \left( \sqrt{\frac{m_{0}}{\hbar}} x_{i} \right) \times \]

\[ - \frac{1}{2} \frac{m_{0}}{\hbar} x_{i}^{2} \]

\[ x_{i} \in C \]

But we can also expand these as well as the in-field states in spherical polar coordinates.
\[ A_{n\ell m}(r) = \langle \tilde{r} | n, \ell, m \rangle \]

\[ = \frac{\mathcal{U}_{n\ell}(r)}{r} Y^m_{\ell} (\theta, \phi) \]

Where \( \mathcal{U}_{n\ell}(r) \) obeys the radial equation

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\ell (\ell + 1) \hbar^2}{2mr^2} + \frac{1}{2} mw^2 r^2 \right] \mathcal{U}_{n\ell}(r) \]

\[ = (n + \frac{3}{2}) \hbar w \mathcal{U}_{n\ell}(r) \]

with the BCs \( \mathcal{U}_{n\ell}(0) = 0 \); 2) \( \mathcal{U}_{n\ell}(r \to \infty) \) finite

Conditions sufficient for such solutions to exist only for:

\[ l = n, n-2, n-4, \ldots, 0 \] for \( n \) even

\[ l = n, n-2, n-4, \ldots, 1 \] for \( n \) odd

and of course we have already that

\[ m = -l, -l+1, \ldots, l-1, l. \]
Thus we find, for each energy eigenvalue
\[ E_n = \hbar \omega (n + \frac{3}{2}) \quad n = 0, 1, 2, \ldots \]

There are \( g_n \)-degenerate states with \( l^2 \) eigenvalues given by \( l \) and \( L_3 \) eigenvalues given by \( m \).

\[ g_n = \sum_{l=0,2, \ldots, n} (2l+1) \quad \text{for } n = \text{even} \]

\[ g_n = \sum_{l=1,3,5, \ldots, n} (2l+1) \quad \text{for } n = \text{odd} \]

In both cases this adds up to

\[ g_n = \frac{(n+1)(n+2)}{2} \quad \text{as we know.} \]

It must, thus, the set of eigenvalues of \( E_n, L_2, L_3 \) are basis vectors.

This set \( E, n, l, m \) consists of vectors with

\[ n = 0, 1, 2, \ldots \]

\[ l = \{0, 2, 4, \ldots, n \} \quad n, \text{even} \]

\[ l = \{1, 3, 5, \ldots, n \} \quad n, \text{odd} \]
\[ m = -l, -l+1, \ldots, +l \]

and we can plot this spectrum

\[ \text{(En)} \]

\[ \text{N} \]

\[ \frac{1}{2} n \]

\[ \begin{array}{c}
  n = 4 \\
  n = 3 \\
  n = 2 \\
  n = 1 \\
  n = 0 \\
\end{array} \]

\[ \text{4s} \quad \text{4d} \quad \frac{4f}{4g} \]

\[ \begin{array}{c}
  n = 2 \\
  n = 1 \\
  n = 0 \\
\end{array} \]

\[ \frac{1}{2} n \]

\[ \begin{array}{c}
  l = 0 \\
  l = 1 \\
  l = 2 \\
  l = 3 \\
\end{array} \]

Alternatively, we can determine the values of \( l \) and \( m \) for a given \( n \) by transforming to spherical Coulomb coordinates. Let

\[ A_+ = \frac{1}{\sqrt{2}} (a_1 + ia_2) \quad A_+ = \frac{1}{\sqrt{2}} (a_1 + ia_2) \]

\[ A_0 = a_3 \quad A_0 = a_3 \]

\[ A_- = \frac{1}{\sqrt{2}} (a_1 - ia_2) \quad A_- = \frac{1}{\sqrt{2}} (a_1 - ia_2) \]
The commutation relations $A_{\alpha}$ obey
are given by

\[ [A_{\alpha}, A_{\beta}] = 0 = [A^+_{\alpha}, A^+_{\beta}] \]

\[ [A_{\alpha}, A^+_{\beta}] = \delta_{\alpha \beta} \]

for $\alpha, \beta = +, -, 0$. These are just like creation and annihilation operator
commutation relations. Then we define the number operators
for $(+, -, 0)$-modes

\[ N_{\alpha} = A^+_{\alpha} A_{\alpha} \]

Then $N_+, N_0, N_-$ form a CSCO. The Hamiltonian is

\[ H = [N_+ + N_0 + N_- + \frac{3}{2}] \] has

and the total number operator

\[ N = N_+ + N_0 + N_- \]

The $|N_+, N_0, N_-\rangle$ basis eigenvectors are

\[ |N_+, N_0, N_-\rangle = \frac{1}{\sqrt{N_+! N_0! N_-!}} A^+_{N_+} A_0^{N_0} A^-_{N_-} |0, 0, 0\rangle \]
with $|0,0,0\rangle$ the usual ground state $0 = a_{10,0,0} = a_{20,0,0} = a_{30,0,0}$.

So for a given $n = 0, 1, 2, \ldots$

$$H|n_+, n_0, n_-\rangle = \hbar \omega (n + \frac{3}{2}) |n_+, n_0, n_-\rangle$$

with $N = n_+ + n_0 + n_-$.

Thus the set of $|n_+, n_0, n_-\rangle$ with $N = n_+ + n_0 + n_-$ are the degenerate eigenvectors with energy $E_n$.

We can differentiate between them by their $L^2$ and $L^3$ eigenvalues.

In general though, $|n_+, n_0, n_-\rangle$ are not eigenvectors of $L^2$; but they are eigenvectors of $L^3$, since

$$L^3 = \hbar \left(n_+ - n_-\right)$$
\[ L_3 \mid n_+, n_0, n_- \rangle = (n_+-n_-) \frac{1}{n_+ n_0 n_-} \]

Thus, the \( L_3 \) eigenvalue is \( M = n_+-n_- \). For \( n = n_++n_0+n_- \), it can take on values \( m = +n, n-1, \ldots, -n+1, -n \). The number of vectors corresponding to a given \( m \) is found from the condition that

\[ n = n_++n_0+n_- \quad \text{and} \quad m = n_+-n_- \]

\[ |m| = n, n-1, n-2, \ldots, n-2s, n-(2s+1), n-(2s+2), \ldots \]

\[ C_m = 1, 1, 2, \ldots, s+1, s+1, s+2, \ldots \]

For example, \( m = n-2s = n_+-n_- \)

Now, for this value of \( m \), it is possible for \( (n_+, n_-) \) to take the values...
\[ M = n - 2s = n_+ - n_- \]  with \( n_+, n_- \geq 0 \)

<table>
<thead>
<tr>
<th>( n_+ )</th>
<th>( n_- )</th>
<th>( n_+ + n_- )</th>
<th>( No )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n-2s )</td>
<td>0</td>
<td>( n-2s )</td>
<td>2s</td>
</tr>
<tr>
<td>( n-2s+1 )</td>
<td>1</td>
<td>( n-2s+2 )</td>
<td>2s-2</td>
</tr>
<tr>
<td>( n-2s+2 )</td>
<td>2</td>
<td>( n-2s+4 )</td>
<td>2s-4</td>
</tr>
<tr>
<td>( n-2s+3 )</td>
<td>3</td>
<td>( n-2s+6 )</td>
<td>2s-6</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( n-2s+s )</td>
<td>( s )</td>
<td>( n-2s+2s )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n-2s+s+1 )</td>
<td>( s+1 )</td>
<td>( n-2s+2(s+1) )</td>
<td>( -2 )</td>
</tr>
<tr>
<td>( n-2s+2s )</td>
<td>( 2s )</td>
<td>( n+2s ) ( &gt; )</td>
<td>( -2s )</td>
</tr>
</tbody>
</table>

But we also have the constraint that \( No \geq 0 \) and \( No = n - n_+ - n_- \).

Do we use the fact that for each \( n \), there are \( 2l+1 \) allowed values of \( m = -l, \ldots, +l \)? So for a given \( N \), the number of times that \( n^{th} \) state of \( L_z \) angular momentum can occur is when \( l \geq m \). Call this number of times this happens \( C_l \); thus

\[ C_m = \sum_{l \geq m} C_l \]  \( (= d_m + d_{m+1} + \ldots) \)

Hence

\[ C_l - C_{l+1} = \Delta l \]
From above this is (for \( l = n \); \( \gamma_{n+1} = 0 \))

\[ d l = 1 \quad \text{for} \quad l = n, n-2, \ldots, n-2s, \ldots \]

and \( d l = 0 \) for all other values of \( l \).

So for a given \( N \), \( l = n, n-2, \ldots, 0, n \) even,
\[ n, n-2, \ldots, 1, n \) odd \]

while \( m = -l, -l+1, \ldots, l-1, l \) as found earlier.

Clearly, symmetry considerations and operator algebra's are important in finding the spectrum of a CSOO's eigenvalues, as we have seen with the Site in its various guises.

Before we embark on a systematic discussion of symmetries in quantum mechanics, and in particular angular momentum and \( SU(2) \), let's reformulate quantum mechanics from the Feynman path integral point of view employing more abstract Dirac formulation,
This is an example of Wick's Theorem. Let

\[ \phi(t) = C \rho(a + \rho^* a^*) \]

\[ \phi(t_1) \cdots \phi(t_n) = N [ \phi(t_1) \cdots \phi(t_n) ] \]