4.8. SHO in an external time dependent potential

Indeed, the use of different pictures can prove convenient when determining the time evolution of a system.

For example, consider the case of the one-dimensional SHO acted upon by an externally controlled force $F = F(t)$. Let the force be time dependent but position independent, and, as well, let it be even in time

$F(-t) = F(t)$ and vanishing

for early and late times; $F(t) = 0$ for $|t| > T$. ($F(t)$ could be tempered rather than have compact support!) Since $F(t)$ is $X$ independent, the potential energy of a particle in such a force field is $-F(t)X$. Hence

The SHO with linear driving force has the Hamiltonian

$$H(t) = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 X^2 - F(t) X.$$

As we saw if $F=0$, the probability that a state of the system, $|\psi(t)\rangle$, is in the energy eigenstate $|n\rangle$ is independent of time

$$P(n;t) = |\langle n|\psi(t)\rangle|^2$$
$$= |\langle n|\psi(t_0)\rangle|^2 .$$

If $|\psi(t_0)\rangle = |0\rangle$, for instance, that is the system is in the ground state at time $t_0$; then $\langle n|\psi(t_0)\rangle = \langle n|0\rangle = \delta_{n0}$ and $P(n;t) = \delta_{n0}$. The oscillator has probability 1 to remain in the ground state at any time.

However, when the external force is applied, we can ask a similar question. If the oscillator is initially in the ground state, that is at $t_0 \to -\infty$

$|\psi(t_0 \to -\infty)\rangle = |0\rangle$, what is the probability of finding it in its $n^{th}$ excited state as $t \to +\infty$. This probability is

$$P(0 \to n) \equiv |\langle n|\psi(t \to +\infty)\rangle|^2 .$$

Now $|\psi(t)\rangle$ evolves in time according to the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$

with $H(t)$ given above. Since $H$ depends on time the time evolution operator $U(t,t_0)$, with

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle ,$$

is not simply the exponential of $H$.

However, we know how to solve the SHO time evolution and it is useful to remove this time dependence from the states. That's we shall consider this problem in the interaction picture with

$$H = H_0^s + H_I^s \quad \text{(in the Schrödinger picture)}$$

where $H_0^s = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$

and the interaction Hamiltonian in the Schrödinger picture given by

$$H_I^s = -F(t)X \; .$$

The interaction picture states and operators are defined by

$$|\psi(t)\rangle_{IP} = e^{iH_0^s t/\hbar} |\psi(t)\rangle_s$$

$$A_{IP}(t) = e^{iH_0^s t/\hbar} A_s(t) e^{-iH_0^s t/\hbar} \; .$$

Introducing the creation and annihilation operators as before

$$a = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,X + i\sqrt{\frac{1}{m\hbar\omega}}\,P\right]$$

$$a^{+} = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,X - i\sqrt{\frac{1}{m\hbar\omega}}\,P\right]$$

The Hamiltonian is

$$H_0^S = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)$$

$$H_I^S = -F(t)\sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+})$$

$$\equiv -J(t)\,(a + a^{+})$$

with $J(t) \equiv \sqrt{\frac{\hbar}{2m\omega}}\,F(t)$. In the IP the $a, a^{+}$ operators become

$$a(t) \equiv a_{IP}(t) = e^{+iH_0^S t/\hbar}\, a\, e^{-iH_0^S t/\hbar}$$

$$a^{+}(t) \equiv a^{+}_{IP}(t) = e^{+iH_0^S t/\hbar}\, a^{+}\, e^{-iH_0^S t/\hbar}$$

where

$$[a(t), a^{+}(t)] = [a, a^{+}] = 1.$$

These are just the Heisenberg equations of motion for the IP operators $a(t), a^\dagger(t)$ with the SHO Hamiltonian

$$H_0^S = H_0^{IP} = \hbar\omega\left(a^\dagger(t)a(t) + \tfrac{1}{2}\right).$$

As we have seen, their time evolution is simple harmonic (p. -412- to -413-)

$$a(t) = e^{-i\omega t}a$$
$$a^\dagger(t) = e^{+i\omega t}a^\dagger .$$

(i.e.

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix} = \frac{i}{\hbar}\left[H_0(t), \begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix}\right]$$

but

$$[H_0, a(t)] = -\hbar\omega a(t)$$
$$[H_0, a^\dagger(t)] = +\hbar\omega a^\dagger(t)$$

So

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix} = \begin{pmatrix} -i\omega & 0 \\ 0 & +i\omega \end{pmatrix}\begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix}$$

$$\Rightarrow a(t) = e^{-i\omega t}a(0) = e^{-i\omega t}a$$
$$a^\dagger(t) = e^{+i\omega t}a^\dagger(0) = e^{+i\omega t}a^\dagger \quad )$$

Hence the IP interaction Hamiltonian becomes

$$H_I(t) \equiv H_I^{IP}(t) = e^{iH_0 t/\hbar} H_I^S e^{-iH_0 t/\hbar}$$

$$= -J(t) e^{iH_0 t/\hbar} (a + a^\dagger) e^{-iH_0 t/\hbar}$$

$$= -J(t) (a(t) + a^\dagger(t))$$

$$\boxed{H_I(t) = -J(t) \left[ a e^{-i\omega t} + a^\dagger e^{+i\omega t} \right]}$$

The IP states evolve according to the Schrödinger equation with $H_I(t)$

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_{IP} = H_I(t) |\psi(t)\rangle_{IP} ,$$

Since $H_I(t)$ is only linear in $a$ and $a^\dagger$ we can solve this differential equation, although it is not trivial since $[a, a^\dagger] = 1$.

Since each $a$, $a^\dagger$ commutes with itself, let's remove this individual time dependence from the state $|\psi(t)\rangle_{IP}$.

Towards this, separate $H_I$ into a positive frequency, $H_+$, and a negative frequency, $H_-$, component

$$H_I(t) = H_+(t) + H_-(t)$$

with

$$H_+(t) \equiv -J(t)e^{-i\omega t}a$$

$$H_-(t) \equiv -J(t)e^{+i\omega t}a^\dagger$$

Note $H_-(t) = H_+^\dagger(t)$ and while

$$[H_\pm(t_1), H_\pm(t_2)] = 0$$

$$[H_+(t_1), H_-(t_2)] = J(t_1)J(t_2) \times$$
$$\times\, e^{-i\omega(t_1-t_2)} \underbrace{[a, a^\dagger]}_{=1}$$

$$= \left(J(t_1)e^{-i\omega t_1}\right)\left(J(t_2)e^{+i\omega t_2}\right).$$

So we define a new state $|X(t)\rangle$ by

$$|\Psi(t)\rangle_{IP} = e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_-(t_1)} e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_+(t_2)} |X(t)\rangle$$

that is

$$|X(t)\rangle = e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_-(t_2)} |\Psi(t)\rangle_{IP}$$

Note that $|X(t_0)\rangle = |\Psi(t_0)\rangle_{IP}$.

Now the Schrödinger equation for $|\Psi(t)\rangle_{IP}$ implies

$$i\hbar \frac{d}{dt}|X(t)\rangle = -H_+(t)|X(t)\rangle$$

$$+ e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} (-H_-(t)) e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} |X(t)\rangle$$

$$+ e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_-(t_2)} H_I(t) \times$$

$$\times e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_-(t_2)} e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} |X(t)\rangle$$

Putting this together yields

$$i\hbar\frac{d}{dt}|\chi(t)\rangle = \Big[-H_+(t) - e^{\frac{i}{\hbar}\int_{t_0}^{t}dt_1 H_+(t_1)} H_-(t)\, e^{-\frac{i}{\hbar}\int_{t_0}^{t}dt_1 H_+(t_1)}$$

$$+ e^{\frac{i}{\hbar}\int_{t_0}^{t}dt_1 H_+} e^{\frac{i}{\hbar}\int_{t_0}^{t}dt_2 H_-} \big(H_+(t) + H_-(t)\big) \times$$

$$\times e^{-\frac{i}{\hbar}\int_{t_0}^{t}dt_2 H_-} e^{-\frac{i}{\hbar}\int_{t_0}^{t}dt_1 H_+}\Big] |\chi(t)\rangle,$$

quite messy — however we can use the Baker-Campbell-Hausdorff formula to simplify the operators. If A and B are two operators whose commutator is a number (not another operator) times the identity

$$[A, B] = c\mathbb{1}$$

then

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \cdots$$

but $[A,[A,B]] = [A, c\mathbb{1}] = 0$ so all higher than first order in $\lambda$ terms vanish and we have

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda C .$$

Further one can show that

$$e^{\lambda A} e^{\lambda B} = e^{\lambda(A+B+\frac{1}{2}\lambda[A,B])}$$

$$= e^{\lambda(A+B)} e^{\frac{1}{2}\lambda^2 C} .$$

Applying this to the above operators we find

$$i\hbar \frac{d}{dt}|\chi(t)\rangle = \Big[ -H_+(t) - H_-(t) - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 [H_+(t_1), H_-(t)]$$

$$+ H_-(t) + \frac{i}{\hbar}\int_{t_0}^{t} dt_1 [H_+(t_1), H_-(t)]$$

$$+ e^{\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+} \left( H_+(t) + \frac{i}{\hbar}\int_{t_0}^{t} dt_2 [H_-(t_2), H_+(t)] \right) \times$$

$$\times e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+} \Big] |\chi(t)\rangle .$$

Simplifying further, this becomes

$$i\hbar \frac{d}{dt}|\chi(t)\rangle = \Big[ -\frac{i}{\hbar}\int_{t_0}^{t} \cancel{dt_1 [H_+(t_1), H_-(t)]}$$

$$+ \frac{i}{\hbar}\int_{t_0}^{t} \cancel{dt_1 [H_+(t_1), H_-(t)]}$$

$$+ \frac{i}{\hbar}\int_{t_0}^{t} dt_2 [H_-(t_2), H_+(t)] \Big] |\chi(t)\rangle$$

where $[H_+, H_-]$ is just a number

$$i\hbar \frac{d}{dt}|\chi(t)\rangle = \Big( \underbrace{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 [H_+(t), H_-(t_1)]}_{\text{c-number}} \Big) |\chi(t)\rangle$$

Thus $|\chi(t)\rangle$ just evolves with a c-number phase

$$|\chi(t)\rangle = e^{-\frac{1}{\hbar^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [H_+(t_1), H_-(t_2)]} |\psi(t_0)\rangle_I$$

So we find the time evolution operator in the IP:

$$\boxed{\begin{aligned} |\psi(t)\rangle_{IP} &= U_{IP}(t,t_0)\,|\psi(t_0)\rangle_{IP} \\ U_{IP}(t,t_0) &= e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_-(t_2)}\, e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)} \times \\ &\times e^{\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \left[H_+(t_1), H_-(t_2)\right]} \end{aligned}}$$

(But recall the Baker-Campbell-Hausdorf identity again

$$e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_2 H_-(t_2)}\, e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_+(t_1)}$$

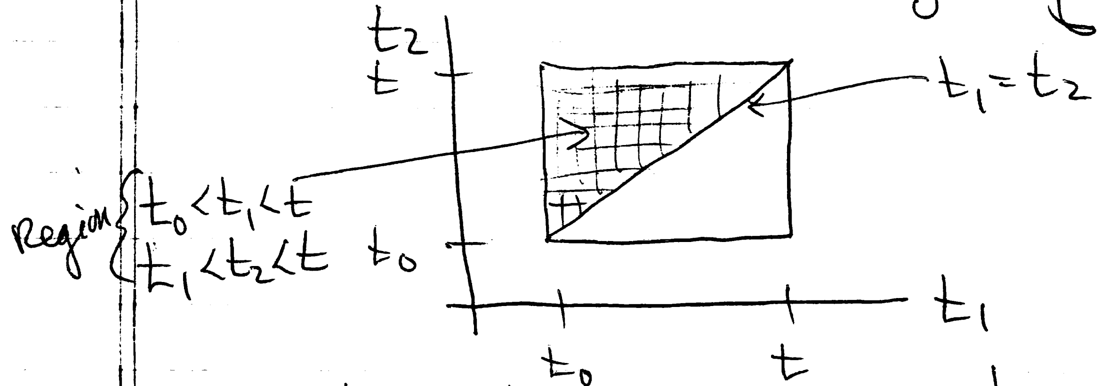
$$= e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 \overbrace{(H_+(t_1)+H_-(t_1))}^{=H_I(t_1)}} \times$$

$$\times e^{-\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left[H_+(t_1), H_-(t_2)\right]}$$

Thus the IP time evolution operator becomes

$$U_{IP}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)} \times e^{\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, [H_+(t_1), H_-(t_2)]} \times e^{-\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, [H_+(t_1), H_-(t_2)]}$$

Now consider the $(t_1, t_2)$ region of integration




$$\text{Thus } \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, f(t_1,t_2) = \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, f(t_1,t_2)$$

Re-label $t_1 \Leftrightarrow t_2$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, f(t_2,t_1)$$

Thus

$$U_{IP}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)} \times$$

$$\times\, e^{\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \left( [H_+(t_1), H_-(t_2)] - [H_+(t_2), H_-(t_1)] \right)}$$

$$\underbrace{[H_+(t_1), H_-(t_2)] - [H_+(t_2), H_-(t_1)]}_{= J(t_1) J(t_2)\left(e^{i\omega(t_2-t_1)} - e^{-i\omega(t_2-t_1)}\right)}$$

$$\underbrace{J(t_1) J(t_2)\left(e^{i\omega(t_2-t_1)} - e^{-i\omega(t_2-t_1)}\right)}_{= 2i\, J(t_1) J(t_2) \sin\omega(t_2-t_1)}$$

So

$$\boxed{U_{IP}(t,t_0) = e^{-i\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, J(t_1) \sin\omega(t_1-t_2) J(t_2)} \times e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)}}$$

$$\boxed{= e^{\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, J(t_1)\, e^{-i\omega(t_1-t_2)} J(t_2)} \times e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_2\, H_-(t_2)}\; e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_+(t_1)}}$$

So in detail we have found

$$\boxed{U_{IP}(t,t_0) = e^{-\frac{1}{\hbar^2}\int_{t_0}^{t} dt_1\, J(t_1) e^{-i\omega t_1} \int_{t_0}^{t_1} dt_2\, J(t_2) e^{+i\omega t_2}} \times e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_2\, J(t_2) e^{+i\omega t_2} a^\dagger} \times e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, J(t_1) e^{-i\omega t_1} a}}$$

and $|\psi(t)\rangle_{IP} = U_{IP}(t,t_0)\,|\psi(t_0)\rangle_{IP}$ .

At early and late times the force goes to zero and the system is in one of the SHO states. If, for instance, initially the system is in the SHO ground state $|0\rangle$ this state evolves into $|\psi(t)\rangle$ at time $t$. Thus we have

$$|\psi(-T)\rangle = |0\rangle$$ as our

initial state. In the IP this becomes

$$|\psi(-T)\rangle_{IP} = e^{\frac{i}{\hbar}H_0^S(-T)}|\psi(-T)\rangle$$
$$= e^{-\frac{i}{\hbar}H_0 T}|0\rangle$$

That is

$$|\psi(-T)\rangle_{IP} = e^{-\frac{i}{\hbar}\frac{1}{2}\hbar\omega T}|0\rangle$$

$$= e^{-i\frac{\omega}{2}T}|0\rangle .$$

The state $|\psi(-T)\rangle_{IP}$ then evolves in time according to $\mathcal{U}_{IP}(t,-T)$. Again at late time $t \geq T$ the force is zero and the state of the system is given by, at time $t = T$,

$$|\psi(T)\rangle_{IP} = \mathcal{U}_{IP}(T,-T)|\psi(-T)\rangle_{IP}$$

$$= e^{-i\frac{\omega}{2}T}\,\mathcal{U}_{IP}(T,-T)|0\rangle .$$

Further, since $a|0\rangle = 0 \Rightarrow$

$$e^{\frac{i}{\hbar}\int_{-T}^{T} dt_1 J(t_1) e^{-i\omega t_1} a}|0\rangle = |0\rangle .$$

Thus we obtain

$$|\psi(T)\rangle_{IP} = e^{-i\frac{\omega}{2}T}\, e^{-\frac{1}{\hbar^2}\int_{-T}^{T} dt_1 J(t_1) e^{-i\omega t_1} \int_{-T}^{t_1} dt_2 J(t_2) e^{+i\omega t_2}}$$

$$\times\, e^{\frac{i}{\hbar}\left(\int_{-T}^{+T} dt\, J(t) e^{+i\omega t}\right) a^\dagger} |0\rangle$$

---

Denoting $\boxed{\tilde{J}(\omega,T) \equiv \int_{-T}^{+T} dt\, J(t) e^{+i\omega t}}$

we have

$$e^{\frac{i}{\hbar}\tilde{J}(\omega,T)a^\dagger}|0\rangle = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{\hbar}\tilde{J}(\omega,T)\right)^m \underbrace{(a^\dagger)^m|0\rangle}_{=\sqrt{m!}\,|m\rangle}$$

$$= \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left(\frac{i}{\hbar}\tilde{J}(\omega,T)\right)^m |m\rangle$$

So

$$|\psi(T)\rangle_{IP} = e^{-i\frac{\omega}{2}T}\, e^{-\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 J(t_1) e^{-i\omega t_1} \int_{-T}^{t_1} dt_2 J(t_2) e^{+i\omega t_2}} \times$$

$$\times \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left(\frac{i}{\hbar}\tilde{J}(\omega,T)\right)^m |m\rangle,$$

---

a sum over all SHO energy eigenstates $|m\rangle$.

The probability of finding this state at time T in the $n^{th}$ SHO eigenstate is

$$P(0\to n;T) = |\langle n|\psi(T)\rangle|^2$$

$$= |\langle n|e^{-iH_0T/\hbar}|\psi(T)\rangle_{IP}|^2$$

$$= |\langle n|\psi(T)\rangle_{IP}|^2\underbrace{|e^{-iE_nT/\hbar}|^2}_{=1}$$

$$= |\langle n|\psi(T)\rangle_{IP}|^2$$

$$= |\langle n|U_{IP}(T,-T)|0\rangle|^2\underbrace{|e^{-i\frac{\omega}{2}T}|^2}_{=1}$$

So

$$\boxed{P(0\to n;T) = |\langle n|U_{IP}(T,-T)|0\rangle|^2}$$

for $T\to\infty$, $U_{IP}(+\infty,-\infty)\equiv S_{IP}$ the Scattering or S-operator in the IP, and the transition probability is given by the square of the S-matrix elements.

Using the expansion for $|\psi(T)\rangle_{IP}$ we find

$$P(0\to n;T) = \left| e^{-\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 J(t_1) e^{-i\omega t_1} \int_{-T}^{t_1} dt_2 J(t_2) e^{+i\omega t_2}} \times \right.$$

$$\left. \times \frac{1}{\sqrt{n!}} \left(\frac{1}{\hbar}\tilde{J}(\omega,T)\right)^n \right|^2 .$$

Now the exponential squared can be simplified calling

$$\mathcal{J}(\omega,T) \equiv -\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 J(t_1) e^{-i\omega t_1} \int_{-T}^{t_1} dt_2 J(t_2) e^{+i\omega t_2}$$

$$P(0\to n;T) = e^{\mathcal{J}(\omega,T)+\mathcal{J}^*(\omega,T)} \frac{1}{n!} \left|\left(\frac{1}{\hbar}\tilde{J}(\omega,T)\right)^n\right|^2$$

But

$$\mathcal{J}(\omega,T)+\mathcal{J}^*(\omega,T) = -\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 J(t_1) e^{-i\omega t_1} \int_{-T}^{t_1} dt_2 J(t_2) e^{+i\omega t_2}$$

$$-\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 \underbrace{J^*(t_1)}_{=J(t_1)} e^{+i\omega t_1} \int_{-T}^{t_1} dt_2 \underbrace{J^*(t_2)}_{J(t_2)} e^{-i\omega t_2}$$

since $J(t)$ is real.

Now we showed earlier, page -428-, that

$$\int_{-T}^{+T} dt_1 \int_{t_1}^{+T} dt_2\, J(t_1) e^{-i\omega t_1} J(t_2) e^{+i\omega t_2}$$

$$= \int_{-T}^{+T} dt_1 \int_{-T}^{t_1} dt_2\, J(t_2) e^{-i\omega t_2} J(t_1) e^{+i\omega t_1}$$

So

$$\mathcal{J}(\omega,T) + \mathcal{J}^*(\omega,T) = -\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 \left(\int_{-T}^{t_1} dt_2 + \int_{t_1}^{+T} dt_2\right) \times$$

$$\times J(t_1) e^{-i\omega t_1} J(t_2) e^{+i\omega t_2}$$

$$= -\frac{1}{\hbar^2}\int_{-T}^{+T} dt_1 \int_{-T}^{+T} dt_2\, J(t_1) e^{-i\omega t_1} J(t_2) e^{+i\omega t_2}$$

Now letting $t_2 \to -t_2$ and using $J(t) = J(-t)$

$\Rightarrow$

$$\boxed{\mathcal{J}(\omega,T) + \mathcal{J}^*(\omega,T) = -\frac{1}{\hbar^2}\left(\tilde{J}(\omega,T)\right)^2}$$

Further $J(t)=J(-t)$ implies

$$\tilde{J}(\omega,T)^* = \int_{-T}^{+T} dt\, J(t)\, e^{-i\omega t} \quad ; \text{let } t \to -t$$

$$= \int_{-T}^{+T} dt\, J(t)\, e^{+i\omega t}$$

$$= \tilde{J}(\omega,T) \quad ; \tilde{J} \text{ is real.}$$

So putting this altogether, we find

$$\boxed{P(0\to n;T) = e^{-\frac{1}{\hbar^2}\tilde{J}(\omega,T)^2}\,\frac{1}{n!}\left(\frac{1}{\hbar}\tilde{J}(\omega,T)\right)^{2n}}$$

where

$$\tilde{J}(\omega,T) = \int_{-T}^{+T} dt\, J(t)\, e^{+i\omega t}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \int_{-T}^{+T} dt\, F(t)\, e^{+i\omega t}$$

The probability for the external force to induce a transition from the initial ground state $|0\rangle$ to the $n^{th}$ excited state $|n\rangle$ is given by the above Poisson distribution.

Note: 1) Consistency check

$$\sum_{n=0}^{\infty} P(0\to n;T) = e^{-\frac{1}{\hbar^2}|\tilde{J}(\omega,T)|^2} \times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!}\left(\left|\frac{1}{\hbar}\tilde{J}(\omega,T)\right|^2\right)^n$$

$$= e^{-\frac{1}{\hbar^2}|\tilde{J}(\omega,T)|^2} e^{+\frac{1}{\hbar^2}|\tilde{J}(\omega,T)|^2} = 1$$

as required.

2) For $F(t) \neq 0$,

$$P(0\to 0;T) = e^{-\frac{1}{\hbar^2}|\tilde{J}(\omega,T)|^2} < 1.$$

The probability that the system remains in the ground state is less than 1. Hence, there must be a non-zero probability of finding the system in an excited state at $t=T$.

3) For $T \to \infty$

$$P(0\to n) = P(0\to n; T\to\infty)$$
$$= e^{-\frac{1}{\hbar^2}|\tilde{J}(\omega)|^2} \frac{\left(\frac{1}{\hbar}|\tilde{J}(\omega)|\right)^{2n}}{n!}$$

where

$$\tilde{J}(\omega) \equiv \int_{-\infty}^{+\infty} dt\, J(t)\, e^{+i\omega t}$$; the

Fourier transform of $J$.

4) For $F(t) = f\,\delta(t)$ we find

$$\frac{1}{\hbar}\tilde{J}(\omega, T) = \frac{f}{\sqrt{2m\hbar\omega}}$$

$$\Rightarrow P(0\to n; T) = e^{-\frac{f^2}{2m\hbar\omega}} \frac{1}{n!}\left[\frac{f^2}{2m\hbar\omega}\right]^n$$