4.3. The Postulates of Quantum Mechanics

1) The set of all possible states of a physical system stand in a one-to-one correspondence with the vector directions (rays) in a Hilbert space $\mathcal{H}$. ($\mathcal{H}$ can be finite or infinite depending upon the system. To allow for continuous basis vectors we will extend the system and space to $\hat{\mathcal{H}}$.)

Since the state is described by the entire ray, we have that $|\psi\rangle$ and $\lambda|\psi\rangle$ with $\lambda \in \mathbb{C}$ describe the same state.

2) The physical observables of a system stand in a one-to-one correspondence with the set of Hermitian operators on the state space $\mathcal{H}$. That is, for each measurable quantity $\mathcal{A}$, there corresponds a Hermitian operator A acting on $\mathcal{H}$. Out of the set of all Hermitian operators, there is a subset which consists of mutually commuting operators and are assumed to be complete (they form a CSCO).

(each A is assumed an observable)

3) Spectral Decomposition

a) The only possible result of the measurement of a physical observable $\mathcal{A}$ is one of the (real) eigenvalues of the corresponding Hermitian operator A.

b) Let $\{|\phi_k\rangle\}$ be the simultaneous eigenstates of a CSCO so that

$$A|\phi_k\rangle = a_k|\phi_k\rangle \text{ etc.}$$

These states form an orthonormal basis for the state space $\mathcal{H}$.

For a system in state $|\psi\rangle$ (with $\langle\psi|\psi\rangle = 1$) the probability of measuring the value a_k for the physical observable $\mathcal{A}$ is

$$P_k = |\langle\phi_k|\psi\rangle|^2 .$$

Immediately following the measurement, the system is in the state $|\phi_k\rangle$.

To be more explicit:

If the orthonormal basis is a discrete basis, then the simultaneous eigenvectors, $|\phi_{ab\ldots}\rangle$, of the CSCO $\{A, B, \ldots\}$ obey

$$\langle\phi_{a'b'\ldots}|\phi_{ab\ldots}\rangle = \delta_{a'a}\delta_{b'b}\cdots$$

$$\mathbb{1} = \sum_{a,b,\ldots}' |\phi_{ab\ldots}\rangle\langle\phi_{ab\ldots}|$$

where

$$A|\phi_{ab\ldots}\rangle = a|\phi_{ab\ldots}\rangle$$

$$B|\phi_{ab\ldots}\rangle = b|\phi_{ab\ldots}\rangle \text{ etc.}$$

with the eigenvalues $\{a, b, \ldots\}$ taking discrete values (in 1-1 correspondence with the integers)

For an arbitrary state vector $|\psi\rangle$, we have the expansion in terms of the $\{|\phi_{ab\ldots}\rangle\}$ basis

$$|\psi\rangle = \sum_{a,b\ldots}' \psi_{ab\ldots}|\phi_{ab\ldots}\rangle$$

with $\psi_{ab\ldots} = \langle\phi_{ab\ldots}|\psi\rangle$.

This implies

$$A|\psi\rangle = \sum_{a,b,\ldots} a\, \psi_{ab\ldots} |\phi_{ab\ldots}\rangle$$

$$B|\psi\rangle = \sum_{a,b,\ldots} b\, \psi_{ab\ldots} |\phi_{ab\ldots}\rangle,$$

etc.

The probability of finding the values $a, b, \ldots$ for the system state $|\psi\rangle$ when $\mathcal{A}, \mathcal{B}, \ldots$ are measured is

$$P_{ab\ldots} = |\langle\phi_{ab\ldots}|\psi\rangle|^2 .$$

Note that

$$\sum_{a,b,\ldots} P_{ab\ldots} = \sum_{a,b\ldots} |\langle\phi_{ab\ldots}|\psi\rangle|^2$$

$$= \sum_{a,b,\ldots} \langle\psi|\phi_{ab\ldots}\rangle\langle\phi_{ab\ldots}|\psi\rangle$$

$$= \langle\psi| \underbrace{\sum_{a,b,\ldots} |\phi_{ab\ldots}\rangle\langle\phi_{ab\ldots}|}_{=1} |\psi\rangle$$

$$= \langle\psi|\psi\rangle$$

$$= 1$$ as required of a probability.

Also the expectation value of $A, B, \ldots$ in state $|\psi\rangle$ is

$$\langle\psi|A|\psi\rangle = \sum_{a,b,\ldots}' a \underbrace{\psi_{ab\ldots}}_{=\langle\phi_{ab\ldots}|\psi\rangle} \langle\psi|\phi_{ab\ldots}\rangle$$

$$= \sum_{a,b,\ldots}' a \left|\langle\phi_{ab\ldots}|\psi\rangle\right|^2$$

$$= \sum_{a,b,\ldots}' a P_{ab\ldots} \quad , \quad \text{etc.}$$

On the other hand if the orthonormal basis is continuous, then

$$\langle\phi_{\alpha'\beta'\ldots}|\phi_{\alpha\beta\ldots}\rangle = \delta(\alpha'-\alpha)\,\delta(\beta'-\beta)\cdots$$

$$1 = \int d\alpha\, d\beta \ldots\, |\phi_{\alpha\beta\ldots}\rangle\langle\phi_{\alpha\beta\ldots}|$$

where

$$A|\phi_{\alpha\beta\ldots}\rangle = \alpha|\phi_{\alpha\beta\ldots}\rangle$$

$$B|\phi_{\alpha\beta\ldots}\rangle = \beta|\phi_{\alpha\beta\ldots}\rangle \quad \text{etc,}$$

with the eigenvalues $\{\alpha, \beta, \ldots\}$ taking on a continuum of values.

For an arbitrary ket vector $|\psi\rangle$ the expansion in terms of the continuous basis $\{|\phi_{\alpha\beta\ldots}\rangle\}$ is

$$|\psi\rangle = \int d\alpha\, d\beta \cdots \psi(\alpha,\beta,\ldots)\, |\phi_{\alpha\beta\ldots}\rangle$$

with

$$\psi(\alpha,\beta,\ldots) = \langle \phi_{\alpha\beta\ldots} | \psi\rangle .$$

This implies

$$A|\psi\rangle = \int d\alpha\, d\beta \cdots \alpha\, \psi(\alpha,\beta,\ldots)\, |\phi_{\alpha\beta\ldots}\rangle$$

$$B|\psi\rangle = \int d\alpha\, d\beta \cdots \beta\, \psi(\alpha,\beta,\ldots)\, |\phi_{\alpha\beta\ldots}\rangle, \text{ etc.}$$

The probability of measuring $\mathcal{A}$ in the range α to $\alpha + d\alpha$, $\mathcal{B}$ in the range β to $\beta + d\beta$, etc. is

$$d\mathcal{P}(\alpha,\beta,\ldots) = |\langle \phi_{\alpha\beta\ldots}|\psi\rangle|^2\, d\alpha\, d\beta \cdots .$$

Note that

$$\int d\mathcal{P}(\alpha,\beta,\dots) = \int d\alpha\, d\beta \cdots |\langle \phi_{\alpha\beta\dots} | \psi \rangle|^2$$

$$= \langle \psi | \underbrace{\int d\alpha\, d\beta \cdots |\phi_{\alpha\beta\dots}\rangle\langle\phi_{\alpha\beta\dots}|}_{=\mathbb{1}} |\psi\rangle$$

$$= \langle \psi | \psi \rangle$$

$$= 1$$ as required of a probability.

Also the expectation value of A, B, ... etc. in state $|\psi\rangle$ is

$$\langle \psi | A | \psi \rangle = \int d\alpha\, d\beta \cdots \underbrace{\psi(\alpha,\beta,\dots)}_{=\langle\phi_{\alpha\beta\dots}|\psi\rangle} \langle \psi | \phi_{\alpha\beta\dots} \rangle \alpha$$

$$= \int d\alpha\, d\beta \cdots |\langle \phi_{\alpha\beta\dots} | \psi \rangle|^2 \alpha$$

$$= \int d\mathcal{P}(\alpha,\beta,\dots)\, \alpha \quad , \text{etc.}$$

4) The time evolution of the physical states is given by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t)\,|\psi(t)\rangle,$$

where $H = H(t)$ is the Hermitian <u>Hamiltonian</u> operator. $H(t)$ is the total energy of the system.

For an isolated system, the Hamiltonian is time independent. Schrödinger's equation is simply

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H\,|\psi(t)\rangle,$$

with the eigenvalues of H the possible energies of the system.

In general the observables may also have explicit time dependence $A = A(t)$. (For example a charged particle can interact with an external time varying electromagnetic field). For an isolated system as is the Hamiltonian the observables are time independent, $\frac{dA}{dt} = 0$.

ER (1887-)

WERNER HEISENBERG (19

Interaction picture

ERWIN SCHRODINGER (1887-)

Schrödinger's picture

WERNER HEISENBERG (1901-)

Heisenberg's picture