## 4.10. Feynman's Path Integral Formulation of Quantum Mechanics Revisited

As we have found earlier, Feynman was able to represent the time evolution of a quantum mechanical system by means of a kernel or Green function which was written as the sum over all possible paths in configuration space each weighted by $\exp(\frac{i}{\hbar}\text{Action})$ for the path. That is the wavefunction at time $t$ and position $\vec{r}$, $\psi(\vec{r},t)$, for the system was related to $\psi(\vec{r}_0,t_0)$, the wavefunction at earlier time $t_0$ and all other positions $\vec{r}_0$ by the Green function equation

$$\psi(\vec{r},t) = \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\,\psi(\vec{r}_0,t_0)$$

with the kernel given as the sum over energy eigenstates as

$$K(\vec{r},t;\vec{r}_0,t_0) = \Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\,e^{-i\frac{E_n(t-t_0)}{\hbar}}.$$

With repeated application of this formula for smaller and smaller time slices, Feynman was able to represent $K$ as an integral of a new type

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dp][dx]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\,[\vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})]}$$

with $H(\vec{r},\vec{p})$ the classical Hamiltonian, for instance

$$H(\vec{r},\vec{p}) = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r}) .$$

The Gaussian momentum integrals can be performed in such a case to obtain an integral over all paths

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dr]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\, L(\vec{r}(t),\dot{\vec{r}}(t))}$$

where $L$ is the classical Lagrangian

$$L(\vec{r}(t),\dot{\vec{r}}(t)) = \frac{1}{2} m\dot{\vec{r}}(t)^2 - V(\vec{r}(t)) .$$
$$= \vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p}) .$$

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Since the wavefunction $\psi(\vec{r},t)$ is simply the inner product

$$\psi(\vec{r},t) = \langle\vec{r}|\psi(t)\rangle$$ we

as well can re-formulate the dynamical laws of quantum mechanics (Postulate 4) to be that of Feynman in our abstract approach.

For simplicity we work in one-dimension with the position operator denoted by $Q\ (=X)$ and its eigenstates $|q\rangle$ with eigenvalues $q$; $Q|q\rangle = q|q\rangle$.

In the Schrödinger picture formulation of quantum mechanics the states depend on time while the operators are time independent; thus $|\psi(t)\rangle_S$ obeys the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_S = H(Q,P)\,|\psi(t)\rangle_S$$

and, for instance, $\frac{d}{dt}Q_S = 0$. In the coordinate basis, $Q_S|q\rangle = q|q\rangle$, the vectors $|q\rangle$ are time independent eigenvectors of $Q_S$, normalized so that

$$\langle q'|q\rangle = \delta(q'-q) \quad \text{and}$$

complete

$$\int_{-\infty}^{+\infty} dq\, |q\rangle\langle q| = 1 \,.$$

The solution to Schrödinger's equation is simply

$$|\psi(t)\rangle_S = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle_S \,.$$

The initial Schrödinger state $|\psi(0)\rangle_S$ is time independent and provides the bridge to the Heisenberg picture formulation of quantum mechanics. The Heisenberg state $|\psi\rangle_H$ is simply

$$|\psi\rangle_H \equiv |\psi(0)\rangle_S = e^{+\frac{i}{\hbar}Ht} |\psi(t)\rangle_S \,.$$

Since $e^{+\frac{i}{\hbar}Ht}$ is unitary we have

$${}_S\langle\psi(t)| Q_S |\phi(t)\rangle_S = {}_H\langle\psi| e^{+\frac{iHt}{\hbar}} Q_S e^{-\frac{i}{\hbar}Ht} |\phi\rangle_H$$

$$\equiv {}_H\langle\psi| Q_H(t) |\phi\rangle_H \,,$$

The Heisenberg picture operators depend on time

$$Q_H(t) = e^{+\frac{i}{\hbar}Ht} \underbrace{Q_S}_{=Q_H(0)} e^{-\frac{i}{\hbar}Ht}.$$

The eigenstates of the position operator $Q_H(t)$ at time $t$ are given in terms of the moving coordinate vectors

$$|q,t\rangle \equiv e^{+\frac{i}{\hbar}Ht} |q\rangle .$$

Hence

$$Q_H(t)|q,t\rangle = e^{+\frac{i}{\hbar}Ht} Q_S e^{-\frac{i}{\hbar}Ht} e^{+\frac{i}{\hbar}Ht} |q\rangle$$

$$= e^{+\frac{i}{\hbar}Ht} \underbrace{Q_S|q\rangle}_{=q|q\rangle}$$

$$= q\, e^{+\frac{i}{\hbar}Ht} |q\rangle$$

$$= q|q,t\rangle .$$

$|q,t\rangle$ is a Heisenberg picture state, $t$ is just a label for the moving axes.

Similarly the normalization of the states $|q,t\rangle$ is

$$\langle q',t|q,t\rangle = \langle q'|q\rangle = \delta(q'-q)$$

and Completeness becomes

$$e^{\frac{i}{\hbar}Ht}\left[1 = \int_{-\infty}^{+\infty} dq\, |q\rangle\langle q|\right] e^{-\frac{i}{\hbar}Ht}$$

$$\Rightarrow \quad 1 = \int_{-\infty}^{+\infty} dq\, |q,t\rangle\langle q,t|$$

(i.e. ${}_S\langle \psi(t)|\phi(t)\rangle_S = {}_H\langle \psi|\phi\rangle_H$

$$= \int dq\, {}_S\langle \psi(t)|q\rangle\langle q|\phi(t)\rangle_S$$

$$= \int dq\, {}_H\langle \psi| e^{\frac{iHt}{\hbar}}|q\rangle\langle q| e^{-\frac{iHt}{\hbar}}|\phi\rangle_H$$

$$= \int dq\, {}_H\langle \psi|q,t\rangle\langle q,t|\phi\rangle_H$$

$\Rightarrow \int dq\, |q,t\rangle\langle q,t| = 1.$ )

The kernel or Green function defined by

$$K(q,t;q_0,t_0) \equiv \langle q,t | q_0,t_0 \rangle$$

controls the time evolution of the states, all dynamics is contained in K.

$$\psi(q,t) = \langle q | \psi(t) \rangle_S$$

$$= \langle q,t | \psi \rangle_H$$

$$= \int dq_0 \langle q,t | q_0,t_0 \rangle \underbrace{\langle q_0,t_0 | \psi \rangle_H}_{= \langle q_0 | \psi(t_0) \rangle_S}$$

$$= \int dq_0 \langle q,t | q_0,t_0 \rangle \psi(q_0,t_0)$$

$$\boxed{\psi(q,t) = \int dq_0 \, K(q,t;q_0,t_0) \, \psi(q_0,t_0)} .$$

Note for $t \to t_0$ $K(q,t;q_0,t) = \langle q t | q_0,t \rangle$

$$= \delta(q_0 - q)$$

by the normalization of the $|q,t\rangle$ states.

Also

$$K(q,t;q_0 t_0) = \langle q,t | q_0, t_0 \rangle$$

$$= \langle q | e^{-\frac{i}{\hbar}H(t-t_0)} | q_0 \rangle$$

$$= \langle q | U(t,t_0) | q_0 \rangle$$

The Kernel is the $\{|q\rangle\}$ basis matrix elements of the time evolution operator $U$. For early times $t_0 \to -\infty$ $U$ relates the in-coming states to the late time $t \to +\infty$ out-going states, it is the scattering operator $S = U(+\infty, -\infty)$, and $K$ is just its position matrix elements.

As before if we know the energy eigenvectors

$$H|n\rangle = E_n |n\rangle$$

with normalization $\langle m|n\rangle = \delta_{mn}$ and

completeness $\quad \sum\!\!\!\!\!\!\int_n |n\rangle\langle n| = 1$

we can represent the time evolution as an energy sum

$$\psi(q,t) = \langle q | \psi(t) \rangle_s$$
$$= \sum_n \langle q|n\rangle\langle n|\psi(t)\rangle_s$$
$$= \sum_n \langle q|n\rangle\langle n| e^{-\frac{i}{\hbar}H(t-t_0)} |\psi(t_0)\rangle_s$$
$$= \int dq_0 \sum_n \langle q|n\rangle \underbrace{\langle n| e^{-\frac{i}{\hbar}H(t-t_0)} |q_0\rangle}_{= e^{-\frac{i}{\hbar}E_n(t-t_0)}\langle n|q_0\rangle} \langle q_0|\psi(t_0)\rangle_s$$
$$= \int dq_0 \left[\sum_n \langle q|n\rangle e^{-\frac{i}{\hbar}E_n(t-t_0)} \langle n|q_0\rangle\right]\psi(q_0,t_0)$$

denoting the energy eigenwavefunctions as

$\psi_n(q) \equiv \langle q|n\rangle$ we have

$$\psi(q,t) = \int dq_0 \left(\sum_n \psi_n(q) e^{-\frac{i}{\hbar}E_n(t-t_0)} \psi_n^*(q_0)\right)\psi(q_0,t_0)$$

Hence the energy representation of the Kernel is

$$K(q,t;q_0,t_0) = \sum_n \psi_n(q)\psi_n^*(q_0)\, e^{-\frac{i}{\hbar}E_n(t-t_0)}$$

as we found earlier. (Note $t \to t_0$

$$K = \sum_n \psi_n(q)\psi_n^*(q_0) = \delta(q_0-q) \text{ by}$$

completeness of the energy eigenstates

$$\langle q|\left(\sum_n |n\rangle\langle n| = 1\right)|q_0\rangle$$

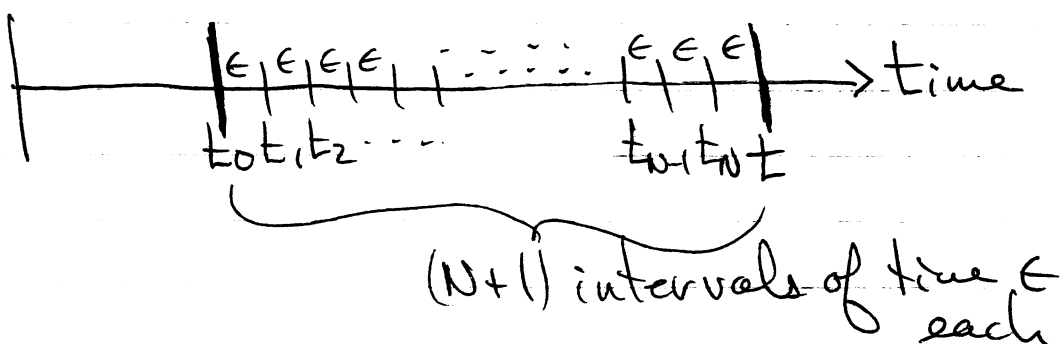
$$\Rightarrow \quad \sum_n \langle q|n\rangle\langle n|q_0\rangle = \langle q|q_0\rangle = \delta(q-q_0)$$

$$= \sum_n \psi_n(q)\psi_n^*(q_0) = \delta(q-q_0).)$$

Of course we could proceed as before to derive the Feynman path integral representation for K from the above energy representation, but let's return to the definition of K as the inner product of position eigenstates at different times and exploit our completeness relation for these states directly,

$K(q,t;q_0,t_0) \equiv \langle q,t|q_0,t_0\rangle$. Let's divide the time interval $(t_0,t)$ into $(N+1)$ short intervals $\epsilon$

$$t > t_N > t_{N-1} > \cdots > t_1 > t_0$$




At each time $t_i$, $i=1\ldots N$ we insert 1 as a sum over the complete set of states $\{|q,t_i\rangle\}$

$$\langle q,t|q_0,t_0\rangle = \int\left(\prod_{i=1}^{N} dq_i\right)\langle q,t|q_N,t_N\rangle\langle q_N,t_N|q_{N-1},t_{N-1}\rangle$$

$$\cdots\langle q_2,t_2|q_1,t_1\rangle\langle q_1,t_1|q_0,t_0\rangle$$

As previously for small time intervals $\epsilon$ we can evaluate the Kernal

$$\langle q_{i+1}, t_{i+1} | q_i, t_i \rangle = \langle q_{i+1} | e^{-\frac{i}{\hbar} H (t_{i+1} - t_i)} | q_i \rangle$$

$$= \langle q_{i+1} | e^{-\frac{i}{\hbar} H \epsilon} | q_i \rangle$$

$$= \langle q_{i+1} | (1 - \tfrac{i}{\hbar} H \epsilon) | q_i \rangle + O(\epsilon^2) .$$

The Hamiltonian has the form

$$H = H(P, Q) = \frac{1}{2m} P^2 + V(Q)$$

so that

$$\langle q_{i+1} | H | q_i \rangle = \frac{1}{2m} \langle q_{i+1} | P^2 | q_i \rangle + \langle q_{i+1} | V(Q) | q_i \rangle$$

$$= \frac{1}{2m} \langle q_{i+1} | P^2 | q_i \rangle + V(q_i) \delta(q_{i+1} - q_i) .$$

In order to evaluate the $P^2$ term, consider a complete set of momentum eigenstates $|p\rangle$

$$P|p\rangle = p|p\rangle \quad \text{with}$$

continuum normalization $\langle p' | p \rangle = \delta(p' - p)(2\pi\hbar)$

and completeness $\int_{-\infty}^{+\infty} \frac{dp}{(2\pi\hbar)} |p\rangle\langle p| = 1.$

The momentum operator has the matrix elements

$$\langle q|P = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q| \quad ; \text{ that is}$$

$$\langle q|\underbrace{P|p\rangle}_{=p|p\rangle} = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q|p\rangle$$

$\Rightarrow$ The differential equation

$$p\langle q|p\rangle = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q|p\rangle$$

$$\Rightarrow \quad \langle q|p\rangle = N e^{\frac{i}{\hbar}pq} \quad \text{as we know.}$$

Since we have normalization

$$\langle p'|p\rangle = (2\pi\hbar)\delta(p'-p) = \int dq \langle p'|q\rangle\langle q|p\rangle$$
$$= \int dq\, N^2 e^{\frac{i}{\hbar}(p-p')q}$$
$$= \hbar N^2 \int \frac{dq}{\hbar} e^{\frac{i}{\hbar}(p-p')q} = \hbar N^2 (2\pi)\delta(p-p')$$

$$\Rightarrow N = 1.$$

Hence

$$\langle q_{i+1}|P^2|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp}{(2\pi\hbar)} \langle q_{i+1}|p\rangle\langle p|P^2|q_i\rangle$$

$$= \int \frac{dp}{(2\pi\hbar)} p^2 e^{\frac{i}{\hbar}p(q_{i+1}-q_i)}$$

and

$$\langle q_{i+1}|H|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} \left[\frac{p_i^2}{2m} + V\left(\frac{q_{i+1}+q_i}{2}\right)\right] e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$$

where $\delta(q_{i+1}-q_i) = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)}\, e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$

So

$$\langle q_{i+1}|\underbrace{H(P,Q)}_{\substack{\text{quantum}\\ \text{operator}\\ \text{Hamiltonian}}}|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} \underbrace{H\left(p_i, \frac{q_{i+1}+q_i}{2}\right)}_{\substack{\text{classical}\\ \text{function}\\ \text{Hamiltonian}}} e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$$

and

$$\langle q_{i+1}t_{i+1}|q_i,t_i\rangle$$

$$= \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}\, e^{-\frac{i}{\hbar}H\left(p_i, \frac{q_{i+1}+q_i}{2}\right)\epsilon} + O(\epsilon^2)$$

So for each time slice we plug in a similar expression to obtain

$$\langle q,t|q_0,t_0\rangle = \int_{-\infty}^{+\infty}\left(\prod_{i=1}^{N} dq_i\right)\left(\prod_{j=0}^{N}\frac{dp_j}{(2\pi\hbar)}\right) \times$$
$$\times e^{\frac{i}{\hbar}\left[\sum_{i=0}^{N} p_i(q_{i+1}-q_i) - \epsilon H(p_i,\frac{1}{2}(q_{i+1}+q_i))\right]} + O(\epsilon^2)$$

where we have defined $q = q_{N+1}$ and recall $t-t_0=(N+1)\epsilon$.

As usual we consider more and more $N\to\infty$ shorter time intervals $\epsilon\to 0$ so that $t-t_0=(N+1)\epsilon$ is fixed and write this $\infty$ product of integrals as a Feynman functional integral

$$\langle q,t|q_0,t_0\rangle = \int_{q_0}^{q}[dq][dp]\,e^{\frac{i}{\hbar}\int_{t_0}^{t}dt\,[p\dot{q}-H(p,q)]}$$

$$\equiv \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \int_{-\infty}^{+\infty}dq_1\cdots dq_N\int_{-\infty}^{+\infty}\frac{dp_0}{2\pi\hbar}\cdots\frac{dp_N}{2\pi\hbar}\,e^{\frac{i}{\hbar}\epsilon\sum_{i=0}^{N}\left[p_i\frac{(q_{i+1}-q_i)}{\epsilon}-H(p_i,\frac{q_{i+1}+q_i}{2})\right]}$$

Since $p\dot{q}-H$ looks like the Lagrangian $L$, we can then express the functional integral in terms of summing over all classical paths by performing the momentum functional integrals. Since they are Gaussian integrals, we find

$$\cdots \int_{-\infty}^{+\infty} \frac{dp_i}{2\pi\hbar}\, e^{\frac{i}{\hbar} p_i (q_{i+1}-q_i) - \frac{i}{\hbar}\epsilon \frac{p_i^2}{2m}} \cdots$$

$$= \cdots \frac{1}{\sqrt{\frac{2\pi i \epsilon}{m\hbar}}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(q_{i+1}-q_i)^2}{\epsilon}} \cdots .$$

This yields

$$\boxed{\begin{aligned} &\langle q,t|q_0,t_0\rangle \\ &= \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \int_{-\infty}^{+\infty} \frac{dq_1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \cdots \frac{dq_N}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}}\, \frac{1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \times \\ &\qquad \times\, e^{\frac{i}{\hbar}\sum_{i=0}^{N} \epsilon\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_{i+1}+q_i}{2}\right)\right]} \end{aligned}}$$

As before for $\epsilon = dt$ ; $\frac{q_{i+1}-q_i}{\epsilon} = \frac{dq}{dt}$

The timeslice sum in the exponent becomes the time integral

$$\sum_{i=0}^{N-1} dt \to \int_{t_0}^{t} dt$$

and

$$\frac{i}{\hbar}\sum_{i=0}^{N-1} \epsilon \left[ \frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_i+q_{i+1}}{2}\right)\right]$$

$$\longrightarrow \frac{i}{\hbar}\int_{t_0}^{t} dt' \left[\frac{m}{2}\dot{q}(t')^2 - V(q(t'))\right] .$$

This is just the classical action

$$S(q;t,t_0) \equiv \int_{t_0}^{t} dt' \, L(q(t'),\dot{q}(t'))$$

times $\frac{i}{\hbar}$ with classical Lagrangian

$$L(q(t),\dot{q}(t)) = \frac{1}{2}m\dot{q}^2(t) - V(q(t)) .$$

Thus we finally obtain the form of the Green function or Kernel in quantum mechanics as a Feynman path integral

$$\langle q,t|q_0,t_0\rangle = \int_{q(t_0)=q_0}^{q(t)=q} [dq]\, e^{\frac{i}{\hbar}S(q;t,t_0)}$$

where $S(q;t,t_0) = \int_{t_0}^{t} dt'\, L(q(t'),\dot{q}(t'))$.

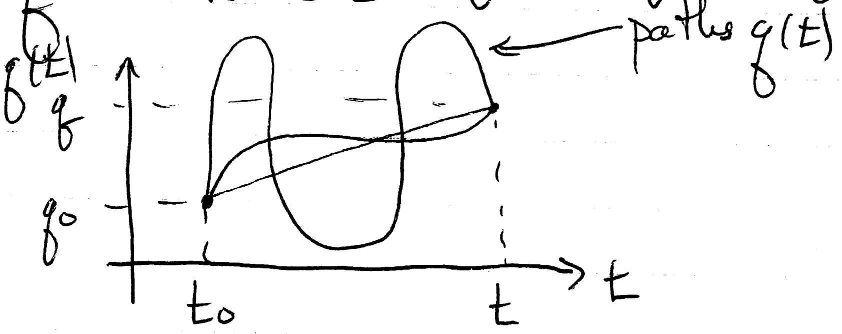
This Feynman path integral is defined as the infinite product of integrals

$$\langle q,t|q_0,t_0\rangle \equiv \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \sqrt{\frac{m\hbar}{2\pi i\epsilon}} \int_{-\infty}^{+\infty} \frac{dq_1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \cdots \frac{dq_N}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \times e^{\frac{i}{\hbar}\sum_{i=0}^{N}\epsilon\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_i+q_{i+1}}{2}\right)\right]}.$$

with the boundary conditions $q_{N+1}=q$ at time $t$ and $q_0=q_0$ at time $t_0$.

Cryptically then, we have that the Green function (Kernel or transformation function as it is sometimes called)

$\langle q,t|q_0,t_0\rangle = K(q,t;q_0,t_0)$ is just a sum over all paths $q(t)$ from $q_0$ at time $t_0$ to $q$ at time $t$




with each path's contribution to the sum weighted by $e^{\frac{i}{\hbar}S}$ where

$S$ is the classical action for such a path. So

$$K(q,t;q_0,t_0) = \sum_{\text{paths}} e^{\frac{i}{\hbar}S} .$$

If $\hbar \to 0$, only one path contributes, the path of stationary phase

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 .$$ This is just the classical Newtonian trajectory which is a solution to the above Euler-Lagrange equation.