3.4. The Radial Equation and Bound States

Once again the energy eigenfunctions for the central potential problem are simultaneous eigenfunctions of $\vec{L}^2$ and L_z also ($[H,\vec{L}^2]=0=[H,L_z]=[\vec{L}^2,L_z]$)

$$\psi_{\ell m}(r,\theta,\varphi) = R(r)\, Y_\ell^m(\theta,\varphi),$$

with

$$\vec{L}^2\psi_{\ell m} = \hbar^2\ell(\ell+1)\psi_{\ell m}$$

$$L_z\psi_{\ell m} = m\hbar\,\psi_{\ell m}$$

and $\quad H\psi_{\ell m} = E\psi_{\ell m}$ reduces to the radial equation for $R(r)$

$$0=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\left(\frac{2m}{\hbar^2}(E-V(r))-\frac{\ell(\ell+1)}{r^2}\right)R(r).$$

Before specifying the potential $V(r)$, let's discuss general properties of the radial eigenfunctions $R(r)$. If we assume that the force between the 2 particles decreases with their distance of separation, $V(r)\to 0$ as $r\to\infty$. Then asymptotically the radial equation becomes

$$\left(\frac{d^2}{dr^2}+\frac{2mE}{\hbar^2}\right)R(r)\sim 0 \quad \text{as } r\to\infty.$$

Hence for large r the eigenfunctions behave like

$$R(r)\sim f(r)e^{\pm\chi r}$$

where $\chi\equiv\sqrt{-\frac{2mE}{\hbar^2}}$ and $f(r)$ is a function of r (like a power of r) for which $\frac{1}{f}\frac{df}{dr}\to 0$ as $r\to\infty$. This follows from the radial equation again

$$\frac{dR}{dr}=\frac{f'}{f}R\pm\chi R$$

$$\frac{d^2R}{dr^2}=f''e^{\pm\chi r}\pm 2\chi f'e^{\pm\chi r}+\chi^2R$$

$$=\frac{f''}{f}R\pm 2\chi\frac{f'}{f}R+\chi^2R$$

So

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{l(l+1)}{r^2}R+\frac{2mV(r)}{\hbar^2}R$$

$$=-\frac{f''}{f}R\mp 2\chi\frac{f'}{f}R-\chi^2R-\frac{2}{r}\frac{f'}{f}R\mp\frac{2\chi}{r}R+\frac{l(l+1)}{r^2}R$$

$$+\frac{2mV}{\hbar^2}R$$

$= -\kappa^2 R$

So as $r \to \infty$ this yields

$$-\frac{f''}{f} \mp 2\kappa\frac{f'}{f} - \cancel{\kappa^2} = -\cancel{\kappa^2}$$

$\Rightarrow \frac{f'}{f} \sim 0$ and hence $\frac{f''}{f} \sim 0$ as $r \to \infty$.

Further we desire a $R(r)$ that is square integrable. Thus as $r \to \infty$ $R(r)$ must decrease sufficiently quickly. Hence κ must be real. Thus we must have energy eigenvalues that are negative since

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}}$$

. Since $V(r) \to 0$ as $r \to \infty$, $E < 0$ corresponds to bound states of the 2 particles. The square integrable wavefunction describes a bound state. Thus we have asymptotically

$$R(r) \sim f(r)e^{-\kappa r} \quad \text{and} \quad \frac{d}{dr}\ln f \to 0$$

as $r \to \infty$.

Next consider r close to the origin, $r \to 0$, if $V(r)$ does not diverge as fast as $\frac{1}{r^2}$ as $r \to 0$, then the radial equation close to the origin becomes

$$-\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \ell(\ell+1)R \sim 0 \quad \text{as } r \to 0.$$

This has two solutions

$$R \sim r^\ell \quad \text{or} \quad R \sim \frac{1}{r^{\ell+1}}.$$

Again for the wavefunction to be well defined at the origin we exclude the $\frac{1}{r^{\ell+1}}$ solution. Hence

as $r \to 0$, $R \sim r^\ell$. Thus we can

write the radial wavefunction as

$$R(r) \equiv u(r)\, r^\ell e^{-\kappa r}$$

where $u(0)$ is a finite number and, as $r \to \infty$, $u(r)$ grows at most like a power (i.e. $\frac{f'}{f} \to 0$).

In practice we will fix the

behavior of $u(r)$ at one extreme, say as $r \to 0$, then we will study the solution for $u(r)$ when $r \to \infty$. In general we will find both $e^{+\kappa r}$ and $e^{-\kappa r}$ behavior for R, clearly unacceptable since it will not be square integrable. Likewise if we fix only $R \sim e^{-\kappa r}$ as $r \to \infty$, then when we investigate those solutions at the origin we will find both $R \sim r^{l}$ and $\frac{1}{r^{l+1}}$ behavior, again unacceptable. Only for certain discrete values of κ and hence the bound state energies will the unacceptable solutions disappear. Thus we will search for those allowed values of κ for which there is a solution which behaves like r^{l} as $r \to 0$ <u>and</u> $e^{-\kappa r}$ as $r \to \infty$. That is, the square integrability of the wave function implies bound state energies.

With the above form for $R(r)$,

$$R(r) = u(r)\, r^{l} e^{-\kappa r},$$

The Radial equation becomes

$$\frac{d^2}{dr^2}u(r) + 2\left(\frac{l+1}{r} - \kappa\right)\frac{d}{dr}u(r) - \left[\frac{2(l+1)\kappa}{r} + \frac{2m}{\hbar^2}V(r)\right]u(r) = 0$$

3.5. The Hydrogen Atom

The hydrogen atom is a two body bound state composed of a proton of mass $m_1 = 1.7 \times 10^{-27}$ Kg ($m_1c^2 = 935$ MeV) and an electron of mass $m_2 = 0.91 \times 10^{-30}$ Kg ($m_2c^2 = 0.511$ MeV). The proton carries a positive charge $e > 0$ while the electron has the opposite charge $-e$. The 2 particles are bound by the Coulomb potential

$$V(|\vec{r}_1 - \vec{r}_2|) = \frac{-e^2}{|\vec{r}_1 - \vec{r}_2|}.$$

Hence the hydrogen atom potential is the form of the 2-body central potential we have ~~been~~ analyzing.