where $R(r)$ obeys the radial equation

$$\left(\frac{-\hbar^2}{2m}\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right)+\left[V(r)+\frac{\hbar^2 l(l+1)}{2mr^2}\right]R(r)$$

$$= E R(r).$$

3.3. Orbital Angular Momentum

In classical mechanics the Hamiltonian for the central potential problem can be written as

$$H = \frac{p_r^2}{2m} + \frac{\vec{L}^2}{2mr^2} + V(r)$$

with p_r the momentum conjugate to r, $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$, while $\vec{L} = \vec{r}\times\vec{p}$ is the orbital angular momentum. Comparing this to the quantum mechanical Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)$$

$$-\frac{\hbar^2}{2mr^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

we identify the radial component of the momentum squared with

$$p_r^2 = -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \quad \text{and}$$

with the orbital angular momentum squared given by

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

which is just $(-\hbar^2 r^2)$ times the angular part of the Laplacian ∇^2. Further, since

$$H\psi_{\ell m}(r,\theta,\varphi) = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi_{\ell m}(r,\theta,\varphi) + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right]\psi_{\ell m}(r,\theta,\varphi)$$

$$= E\psi_{\ell m}(r,\theta,\varphi),$$

we see that the $Y_\ell^m(\theta,\varphi)$ are eigenfunctions of $\vec{L}^2$

$$\vec{L}^2 Y_\ell^m(\theta,\varphi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\varphi),$$

with the orbital angular momentum eigenvalues given by $\hbar^2\ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$.

To show that the spherical harmonics are indeed the eigenfunctions of $\vec{L}^2$ and L_z more rigorously. Recall the definition of the orbital angular momentum to

$$\vec{L} = \vec{r} \times \vec{p} = -i\hbar \vec{r} \times \vec{\nabla},$$

thus in components we have

$$L_i = \epsilon_{ijk} x_j p_k = \frac{\hbar}{i} \epsilon_{ijk} x_j \frac{\partial}{\partial x_k}$$

with ϵ_{ijk} the anti-symmetric (Levi-Civita) tensor (permutation tensor)

$$\epsilon_{ijk} = \begin{cases} +1 & , i,j,k \text{ is an even permutation of } 1,2,3 \\ -1 & , i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{otherwise i.e. any } i,j \text{ or } k \text{ are equal.} \end{cases}$$

that is

$$L_1 = x_2 p_3 - x_3 p_2 = \frac{\hbar}{i}\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right)$$

$$L_2 = x_3 p_1 - x_1 p_3 = \frac{\hbar}{i}\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right)$$

$$L_3 = x_1 p_2 - x_2 p_1 = \frac{\hbar}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right).$$

(with $x_1 = x$, $x_2 = y$, $x_3 = z$ and $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$)

Using the canonical commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij}$$

$$[x_i, x_j] = 0 = [p_i, p_j]$$

we find that

$$[L_1, L_2] = \left(\frac{\hbar}{i}\right)^2 \left[x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}, x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\right]$$

$$= -\hbar^2\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right)$$

$$= -\hbar^2 \frac{-i}{\hbar} L_3 = i\hbar L_3$$

Thus

$$[L_1, L_2] = i\hbar L_3$$ plus cyclic permutations of the components 1, 2, 3.

In general

$$[L_i, L_j] = [\epsilon_{ik\ell} x_k p_\ell, \epsilon_{jmn} x_m p_n]$$

$$= \epsilon_{ik\ell}\epsilon_{jmn}[x_k p_\ell, x_m p_n]$$

Now use the identity

$$[A, BC] = ABC - BCA + BAC - BAC$$

$$= B[A, C] + [A, B]C$$

to write

$$[x_k p_l, x_m p_n] = x_m[x_k p_l, p_n] + [x_k p_l, x_m] p_n$$

$$= x_m x_k \cancel{[p_l, p_n]}^{0} + x_m [x_k, p_n] p_l$$

$$+ x_k [p_l, x_m] p_n + \cancel{[x_k, x_m]}^{0} p_l p_n$$

$$= i\hbar (x_m p_l \delta_{kn} - x_k p_n \delta_{lm})$$

So

$$[L_i, L_j] = i\hbar(\epsilon_{ikl}\epsilon_{jmk} x_m p_l - \epsilon_{ikl}\epsilon_{jln} x_k p_n)$$

(relabelling dummy indices) $$= -i\hbar \epsilon_{ilk}\epsilon_{jmk}(x_m p_l - x_l p_m)$$

Now $\epsilon_{ilk}\epsilon_{jmk} = \delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}$

Thus

$$[L_i, L_j] = -i\hbar(\cancel{\delta_{ij} x_m p_m} - x_i p_j - \cancel{\delta_{ij} x_m p_m} + x_j p_i)$$

$$= +i\hbar(x_i p_j - x_j p_i)$$

But $\epsilon_{ijk} L_k = \epsilon_{ijk}\epsilon_{klm} x_l p_m$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_l p_m$$

$$= x_i p_j - x_j p_i$$

So $$\boxed{[L_i, L_j] = i\hbar\epsilon_{ijk}L_k}$$

The components of $\vec{L}$ do not commute with each other. Hence they are not simultaneously diagonalizable i.e. $\Delta L_1 \Delta L_2 \geq \frac{\hbar}{2}|L_3|$

However $\vec{L}^2 = \vec{L}\cdot\vec{L}$ commutes with $\vec{L}$ (as we shall see $\vec{L}$ generates spatial rotations, as $\vec{p}$ generates spatial translations, hence, $\vec{L}^2$ is a scalar dot product and invariant under rotations $[\vec{L}^2, \vec{L}] = 0$)

$$[\vec{L}^2, L_j] = [L_iL_i, L_j]$$

$$= L_i[L_i, L_j] + [L_i, L_j]L_i$$

$$= i\hbar\epsilon_{ijk}L_iL_k + i\hbar\epsilon_{ijk}L_kL_i$$

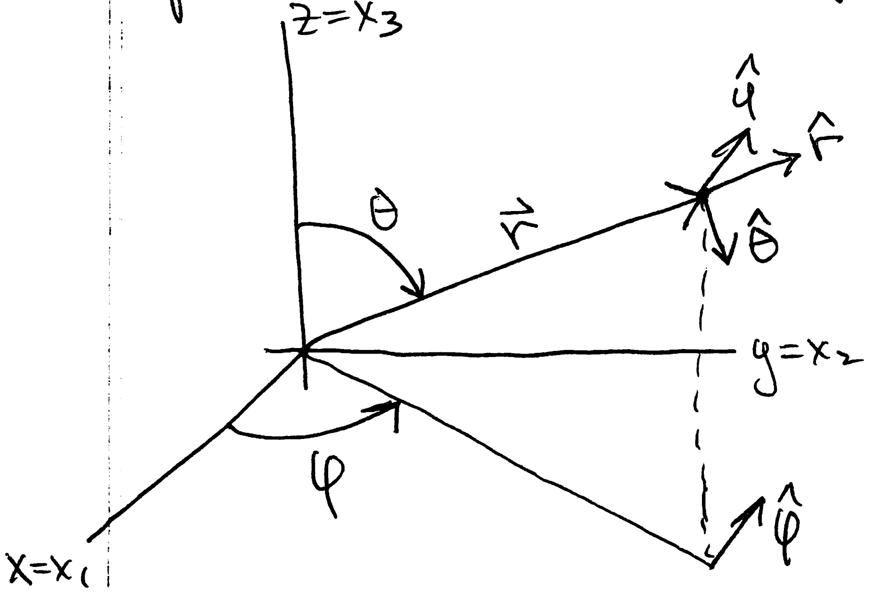
Change dummy i,k indices on second term

$$= i\hbar(\epsilon_{ijk} + \underbrace{\epsilon_{kji}}_{= -\epsilon_{ijk}})L_iL_k$$

$$= 0$$

Hence we can determine the eigenvalues of $\vec{L}^2$ and one of the components L_i, say L_3, simultaneously since $[\vec{L}^2, L_3] = 0$.

The relevance of all this to our earlier work is that the spherical harmonics $Y_\ell^m(\theta,\varphi)$ are the simultaneous eigenfunctions of $\vec{L}^2$ and L_3. To see this we next express the orbital angular momentum operator in spherical polar coordinates.



Note:

$$\hat{r} \times \hat{\theta} = \hat{\varphi}$$

$$\hat{\theta} \times \hat{\varphi} = \hat{r}$$

$$\hat{\varphi} \times \hat{r} = \hat{\theta}.$$

In spherical polar coordinates the Cartesian coordinates are given by

$$x_1 = r \sin\theta \cos\varphi$$
$$x_2 = r \sin\theta \sin\varphi$$
$$x_3 = r \cos\theta$$

while the spherical coordinate basis vectors $(\hat{r}, \hat{\theta}, \hat{\varphi})$ are given by

$$\hat{r} = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \,.$$

The gradient operator in spherical coordinates is given as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

hence the orbital angular momentum operator in spherical polar coordinates is

$$\vec{L} = \vec{r}\times\vec{p} = r\,\hat{r}\times\frac{\hbar}{i}\vec{\nabla}$$

$$= \frac{\hbar}{i}\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right).$$

Thus the Cartesian components, $L_1 = \hat{i}\cdot\vec{L}$, $L_2 = \hat{j}\cdot\vec{L}$, $L_3 = \hat{k}\cdot\vec{L}$, are easily found in polar coordinates to be

$$\boxed{\begin{aligned} L_x = L_1 &= \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \\ L_y = L_2 &= \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\ L_z = L_3 &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\ . \end{aligned}}$$

A direct calculation then gives $\vec{L}^2$ in terms of spherical polar coordinates

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$
$$= -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right].$$

This differential operator is nothing but the angular part of the Laplacian, thus

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\vec{L}^2}$$

Indeed, recalling page -203-, the spherical harmonics are the eigenfunctions of the orbital ~~&~~ momentum squared and it's z-component

$$\vec{L}^2 Y_\ell^m(\theta,\varphi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\varphi)$$

$$L_z Y_\ell^m(\theta,\varphi) = m\hbar\, Y_\ell^m(\theta,\varphi)$$

for $\ell = 0,1,2,\ldots;\ m=-\ell,-\ell+1,\ldots,0,\ldots,\ell-1,\ell$.

The orbital ∡ momentum squared $\vec{L}^2$ has eigenvalues $\ell(\ell+1)\hbar^2$, $\ell=0,1,2,\ldots$ and the third (or z-) component L_z has eigenvalues $m\hbar$, $m=-\ell,-\ell+1,\ldots,\ell-1,\ell$. The angular momentum is said to be "quantized". Each eigenstate of ~~total~~ orbital ∡ momentum squared, i.e. for each ℓ, has $(2\ell+1)$ components, one for each L_z eigenvalue $m\hbar$. The integer $\ell=0,1,2,\ldots$ denotes the orbital ∡ momentum and according to spectroscopic notation is denoted as S, P, D, F, ... for $\ell=0,1,2,3,\ldots$ eigenstates while the magnetic quantum number m takes $(2\ell+1)$ possible values $-\ell,\ldots,+\ell$.

Sharp principal diffuse ~~fine~~ fundamental