1.3.9. Feynman's Path Integral Formulation of Quantum Mechanics

As we have seen for time independent potentials, the stationary states of the Schrödinger equation form a complete set of functions. The time independent Schrödinger equation is the energy eigenvalue equation

$$H\psi_n(\vec{r}) = E_n \psi_n(\vec{r})$$

with $\{\psi_n\}$ orthonormal: $\int d^3r\, \psi_m^*(\vec{r})\psi_n(\vec{r}) = \delta_{nm}$ and complete $\sum_n \psi_n^*(\vec{r}\,')\psi_n(\vec{r}) = \delta^3(\vec{r}-\vec{r}\,')$.

Hence any solution to the time dependent Schrödinger equation at time t, $\psi(\vec{r},t)$, may be expanded in terms of $\psi_n(\vec{r})$

$$\psi(\vec{r},t) = \sum_n C_n(t)\psi_n(\vec{r})$$

where

$$C_n(t) = \int d^3r\, \psi_n^*(\vec{r})\psi(\vec{r},t)$$

from orthonormality of $\{\psi_n\}$. Since $\psi(\vec{r},t)$ obeys the time dependent Schrödinger equation, we have

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = H\psi(\vec{r},t)$$

$$\Rightarrow$$

$$\sum_n i\hbar \frac{d}{dt} C_n(t)\psi_n(\vec{r}) = \sum_n C_n(t) H\psi_n(\vec{r})$$

$$= \sum_n E_n C_n(t)\psi_n(\vec{r}).$$

Applying orthonormality yields $i\hbar \frac{d}{dt} C_n(t) = E_n C_n(t)$

This has the solution

$$C_n(t) = e^{-i\frac{E_n(t-t_0)}{\hbar}} C_n(t_0),$$

where t_0 is some initial condition time, previously we chose $t_0=0$, but in general it is arbitrary. Thus

$$\psi(\vec{r},t) = \sum_n C_n(t_0) e^{-i\frac{E_n(t-t_0)}{\hbar}} \psi_n(\vec{r}).$$

Alternately stated, given $\psi(\vec{r},t_0)$, then

$$C_n(t_0) = \int d^3r_0\, \psi_n^*(\vec{r}_0)\psi(\vec{r}_0,t_0)$$

and $\psi(\vec{r},t)$ at all other times is given by

$$\psi(\vec{r},t) = \sum_n{}' C_n(t_0)\, e^{-i\frac{E_n(t-t_0)}{\hbar}} \psi_n(\vec{r})$$

$$= \sum_n{}' \int d^3r_0\, \psi_n^*(\vec{r}_0)\psi(\vec{r}_0,t_0)\, e^{-i\frac{E_n(t-t_0)}{\hbar}} \psi_n(\vec{r})$$

$$= \int d^3r_0 \left[\sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-i\frac{E_n(t-t_0)}{\hbar}}\right]\psi(\vec{r}_0,t_0)$$

For times $t > t_0$, that is t later than t_0, we define this as

$$\equiv \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\,\psi(\vec{r}_0,t_0)$$

The quantity in brackets, K, is the kernel or the Green function for the Schrödinger equation. It is given by

$$K(\vec{r},t;\vec{r}_0,t_0) \equiv \Theta(t-t_0) \sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

where the step function $\Theta(t-t_0)$ is defined as

$$\Theta(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

it ensures that $t > t_0$ and that $K=0$ for $t < t_0$. The interpretation of the above equation for $\psi(\vec{r},t)$ is that given the state everywhere in space $(\vec{r}_0)$ at an earlier time t_0, we can find the state at any point in space ($\vec{r}$) at a later time t by letting $\psi(\vec{r}_0, t_0)$ propagate via K through space and time via Huygens principle. K is also known as the propagator. To see that K is the Schrödinger Green function consider

$$\left(i\hbar\frac{\partial}{\partial t} - H(\vec{r})\right) K(\vec{r}, t; \vec{r}_0, t_0)$$

$$= i\hbar\left(\frac{\partial}{\partial t}\Theta(t-t_0)\right)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$+\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, i\hbar\frac{\partial}{\partial t}e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$-\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\left(H(\vec{r})\psi_n(\vec{r})\right)e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$= i\hbar\delta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$+\cancel{\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{\frac{-iE_n(t-t_0)}{\hbar}}(E_n-E_n)} \rightarrow 0$$

Since $t=t_0$ we get in the first term

$$= i\hbar\delta(t-t_0)\underbrace{\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})}_{=\delta^3(\vec{r}-\vec{r}_0) \text{ by completeness}}$$

$$= i\hbar\delta(t-t_0)\delta^3(\vec{r}-\vec{r}_0)$$

Thus K is the Green function of $\left[i\hbar\frac{\partial}{\partial t} - H(\vec{r})\right]$. Further K obeys the boundary condition $K=0$ for $t<t_0$ by definition. See page -90c-

We can apply this time evolution over and over again to go from $\psi(\vec{r}_0,t_0) \longrightarrow \psi(\vec{r}_1,t_1) \longrightarrow \psi(\vec{r}_2,t_2) \rightarrow \dots \rightarrow \psi(\vec{r}_n,t_n) \rightarrow \psi(\vec{r},t)$ with $t > t_n > t_{n-1} > \dots > t_1 > t_0$,

thus

$$\psi(\vec{r},t) = \int d^3r_0\, d^3r_1\, d^3r_2 \cdots d^3r_n \times$$

$$\times K(\vec{r},t;\vec{r}_n,t_n)K(\vec{r}_n,t_n;\vec{r}_{n-1},t_{n-1}) \times \dots \times$$

$$\dots \times K(\vec{r}_2,t_2;\vec{r}_1,t_1)K(\vec{r}_1,t_1;\vec{r}_0,t_0)\psi(\vec{r}_0,t_0).$$

For very short time intervals we can find an integral representation for the Green function. Let $t_k - t_{k-1} = \epsilon$ be small, then

The energy eigenfunction expansion of the wavefunction is valid for times earlier and later than t_0

$$\psi(\vec{r},t) = \sum_n C_n(t_0)\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\, \psi_n(\vec{r}) ,$$

When we use orthonormality of the $\psi_n(\vec{r})$ to find $C_n(t_0)$

$$C_n(t_0) = \int d^3r_0\, \psi_n^*(\vec{r}_0)\,\psi(\vec{r}_0,t_0),$$

we have for all times t

$$\psi(\vec{r},t) = \int d^3r_0 \left[\sum_n \psi_n^*(\vec{r}_0)\,\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\right] \times \psi(\vec{r}_0,t_0).$$

The Kernel or Green function K is defined by

$$K(\vec{r},t;\vec{r}_0,t_0) \equiv \Theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0)\,\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}},$$

That is for times $t > t_0$, it gives directly the time evolution of the wavefunction. By definition for $t_0 > t$, $K = 0$.

That is for $t \to t_0^+$, $K(\vec{r},t_0,\vec{r}_0,t_0) = \delta^3(\vec{r}-\vec{r}_0)$, K initially begins as a Dirac delta-function in position, it then propagates according to the Schrödinger equation for $t > t_0$ since

$$\left(i\hbar\frac{\partial}{\partial t} - H_{\vec{r}}\right)K = i\hbar\,\delta(t-t_0)\,\delta^3(\vec{r}-\vec{r}_0)$$

$$(= 0 \text{ for } t > t_0).$$

Even though K directly describes forward time evolution, $t > t_0$, we can use it to evolve backwards in time before t_0. In particular if $t < t_0$ $\psi(\vec{r},t)$ is still given by

$$\psi(\vec{r},t) = \int d^3r_0 \left[\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\right]\psi(\vec{r}_0,t_0)$$

The term in brackets is nothing but K^*, that is for $t < t_0$ we have simply

$$K^*(\vec{r}_0,t_0;\vec{r},t) = \Theta(t_0-t)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r}) \times \\ \times\, e^{-\frac{iE_n(t-t_0)}{\hbar}}.$$

Thus for $t<t_0$

$$\psi(\vec{r},t)=\int d^3r_0\, K^*(\vec{r}_0,t_0;\vec{r},t)\,\psi(\vec{r}_0,t_0),$$

that is K^* gives the time evolution backwards in time for $t<t_0$.

Since $K(\vec{r},t;\vec{r}_0,t_0)$ takes the system forward in time from t_0 to $t>t_0$ and $K^*(\vec{r}_0,t_0;\vec{r}',t')$ evolves the system back from t_0 to $t'<t_0$ we have that to go from $t_0 \to t \to t'$ with $t>t'>t_0$ is accomplished by

$$\int d^3r\, K^*(\vec{r},t;\vec{r}',t')\,K(\vec{r},t;\vec{r}_0,t_0)=K(\vec{r}',t';\vec{r}_0,t_0)$$

as compared to evolving forward in time from $t_0 \to t \to t'$ when $t'>t>t_0$

$$\int d^3r\, K(\vec{r}',t';\vec{r},t)\,K(\vec{r},t;\vec{r}_0,t_0)=K(\vec{r}',t';\vec{r}_0,t_0).$$

To reinforce this interpretation, we show that it follows directly from the conservation of probability

$$\int d^3r\, |\psi(\vec{r},t)|^2 = \int d^3r_0\, |\psi(\vec{r}_0,t_0)|^2,$$

substituting on the LHS

$$\psi(\vec{r},t) = \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\,\psi(\vec{r}_0,t_0)$$

we get

$$\int d^3r\, K^*(\vec{r},t;\vec{r}_0{}',t_0)\,K(\vec{r},t;\vec{r}_0,t_0) = \delta^3(\vec{r}_0{}'-\vec{r}_0)$$

Again it is seen that $K^*(\vec{r}_0,t_0;\vec{r},t)$ just undoes the work of $K(\vec{r},t;\vec{r}_0,t_0)$.

Note multiplying the above by $K(\vec{r}_0,t_0;\vec{r}',t')$ and integrating over $\vec{r}_0$ yields $(t > t_0 > t')$

$$K(\vec{r}_0{}',t_0;\vec{r}',t') = \int d^3r \int d^3r_0\, K^*(\vec{r},t;\vec{r}_0{}',t_0) \times$$

$$\times K(\vec{r},t;\vec{r}_0,t_0)\,K(\vec{r}_0,t_0;\vec{r}',t')$$

$$= \int d^3r\, K^*(\vec{r},t;\vec{r}_0{}',t_0)\,K(\vec{r},t;\vec{r}',t')$$

which is just our previous formula on page —90'''—.

$$K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) = \sum_n{}' \psi_n^*(\vec{r}_{k-1})\psi_n(\vec{r}_k) \times$$
$$\times \underbrace{e^{-iE_n(t_k-t_{k-1})/\hbar}}$$
$$= e^{-i\epsilon E_n/\hbar} \simeq 1 - i\frac{\epsilon}{\hbar}E_n$$

$$= \sum_n{}' \psi_n^*(\vec{r}_{k-1})\left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\psi_n(\vec{r}_k)$$

Since $H(\vec{r}_k)\psi_n(\vec{r}_k) = E_n\psi_n(\vec{r}_k)$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\sum_n{}' \psi_n^*(\vec{r}_{k-1})\psi_n(\vec{r}_k)$$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\delta^3(\vec{r}_k - \vec{r}_{k-1})$$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+i\frac{\vec{p}_k\cdot(\vec{r}_k-\vec{r}_{k-1})}{\hbar}}$$

$$= \int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+\frac{i}{\hbar}\vec{p}_k\cdot(\vec{r}_k-\vec{r}_{k-1})}\left[1 - \frac{i\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k)\right)\right]$$

$$\boxed{\begin{aligned} &= \int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+\frac{i}{\hbar}\vec{p}_k\cdot(\vec{r}_k-\vec{r}_{k-1})}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k)\right)} \\ &= K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) \quad \text{with } t_k - t_{k-1} = \epsilon \end{aligned}}$$

Putting all these factors together, we find

$$K(\vec{r},t;\vec{r}_0,t_0) \approx \int d^3r_1 \cdots d^3r_n \int \frac{d^3p_1}{(2\pi\hbar)^3} \cdots \frac{d^3p_{n+1}}{(2\pi\hbar)^3} \times$$

$$\times\, e^{-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right]}$$

where we let $\vec{r} = r_{n+1}$, $t = t_{n+1}$.

In the limit $n \to \infty$, $\epsilon \to 0$ this becomes exact

$$\boxed{K(\vec{r},t;\vec{r}_0,t_0) = \lim_{\substack{n\to\infty\\ \epsilon\to 0}} \int d^3r_1 \cdots d^3r_n \int \frac{d^3p_1}{(2\pi\hbar)^3} \cdots \frac{d^3p_{n+1}}{(2\pi\hbar)^3} \times e^{-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right]}}$$

with $(n+1)\epsilon = (t-t_0)$

This infinite product of integrals is called a Feynman path integral, or functional integral. It is a new type of integral. This infinite product of integrals is often denoted by the symbol

$$\int [dp][dx] \quad \text{or} \quad \int \mathcal{D}p\,\mathcal{D}x$$

As $\epsilon \to 0$ and $n \to \infty$ we see that $\frac{\vec{r}_k - \vec{r}_{k-1}}{\epsilon} = \dot{\vec{r}}(t_k)$ while $\vec{p}_k = \vec{p}(t_k)$ so that the exponent becomes

$$-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon}\right]$$

$$= +\frac{i}{\hbar}\int_{t_0}^{t} dt\left[\vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})\right]$$

with $H(\vec{r},\vec{p}) = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$ the classical Hamiltonian.

Hence we write the functional integral representation for the Green function as

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dp][dx]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\left[\vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})\right]}$$

where it is understood that this stands for the infinite product of integrals on page -92-.

Returning to this latticized integral, we perform the momentum integrals

Consider

$$\int \frac{d^3p_k}{(2\pi\hbar)^3}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right)}$$

$\vec{p}_k = p_{kx}\hat{i} + p_{ky}\hat{j} + p_{kz}\hat{k}$ (so $p = p_{kx}$)

$$= \left[\int \frac{dp}{(2\pi\hbar)}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{p^2}{2m} - p\frac{(x_k-x_{k-1})}{\epsilon}\right)}\right]_x \left[\;\right]_y \left[\;\right]_z$$

But recall

$$\int \frac{dp}{(2\pi\hbar)}\, e^{\frac{-i\epsilon}{\hbar 2m}(p-a)^2} = \frac{1}{(2\pi\hbar)}\sqrt{\frac{2\pi\hbar m}{i\epsilon}} = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}$$

$$= \int \frac{dp}{(2\pi\hbar)}\, e^{-\frac{i\epsilon}{\hbar}\left(\frac{p^2}{2m} - \frac{ap}{m} + \frac{a^2}{2m}\right)}$$

$$= e^{-i\frac{\epsilon a^2}{2m\hbar}} \int \frac{dp}{(2\pi\hbar)}\, e^{-\frac{i\epsilon}{\hbar}\left(\frac{p^2}{2m} - \frac{a}{m}p\right)}$$

Hence

$$\int \frac{d^3p_k}{(2\pi\hbar)^3}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right)}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{3/2} e^{+\frac{i\epsilon}{\hbar}\frac{m}{2}\frac{(\vec{r}_k-\vec{r}_{k-1})^2}{\epsilon^2}}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{3/2} e^{+\frac{i\epsilon}{\hbar}\frac{1}{2}m\dot{\vec{r}}_k^{\,2}}$$

Substituting this into the functional integral yields

$$K(\vec{r},t;\vec{r}_0,t_0) = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int \frac{d^3r_1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \cdots \frac{d^3r_n}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}}$$

with $(n+1)\epsilon = (t-t_0)$

$$\times \frac{1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \; e^{+\frac{i}{\hbar}\epsilon\sum_{k=1}^{n+1}\left[\frac{1}{2}m\frac{(\vec{r}_k-\vec{r}_{k-1})^2}{\epsilon^2} - V(\vec{r}_k)\right]}$$

The bracketed terms in the exponent is just the Lagrangian for the classical trajectory $\vec{r}(t)$, $\dot{\vec{r}}(t)$ beginning at t_0 at $\vec{r}_0$, $\dot{\vec{r}}_0$ and ending at t at $\vec{r}$, $\dot{\vec{r}}$

$$L(\vec{r}(t),\dot{\vec{r}}(t)) = \frac{1}{2}m\dot{\vec{r}}(t)^2 - V(\vec{r}(t)) .$$

Thus in the limit we write this integral as

$$\boxed{K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dr]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\, L(\vec{r}(t),\dot{\vec{r}}(t))}}$$

Again the Feynman path integral

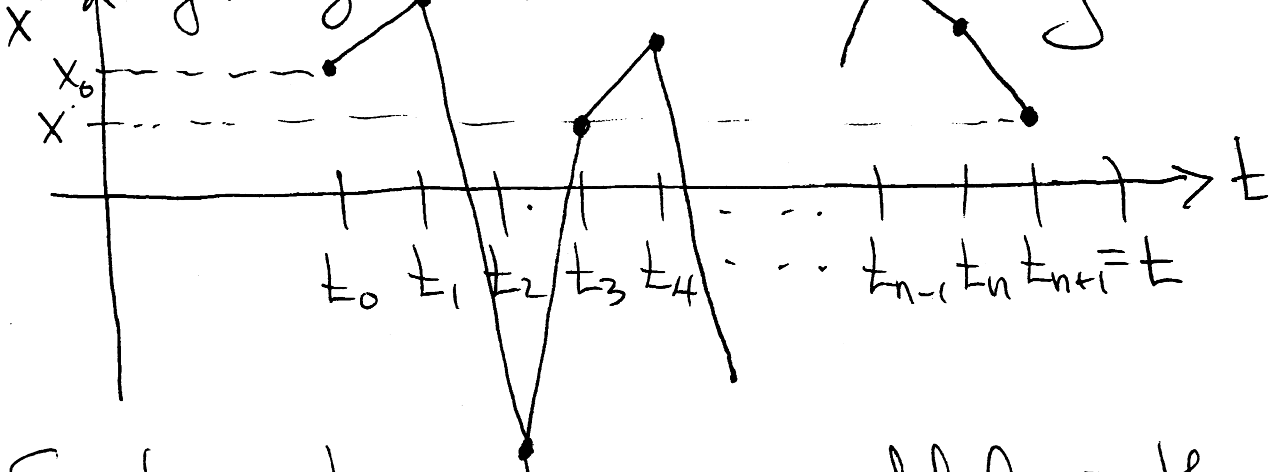
$\int[dr]$ stands for the infinite product of integrals

$$\int[dr] = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int \frac{d^3r_1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \cdots \frac{d^3r_n}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \left(\frac{1}{\frac{2\pi\hbar i\epsilon}{m}}\right)^{3/2}.$$

Hence the Green function is obtained by summing over all classical paths between $\vec{r}_0$ at t_0 and $\vec{r}$ at t weighted by the factor

$$e^{\frac{i}{\hbar}(\text{Action})} \quad \text{for that path.}$$

That is we can break up the classical trajectory into time slots between t_0 and t. Thus for $(n+1)$ divisions we have $t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = t$ and we sum over all possible values of the coordinate $\vec{r}(t)$ in each interval beginning with $\vec{r}_0$ at t_0 and ending with $\vec{r}$ at t



Such a trajectory is weighted in the

sum by $e^{+\frac{i}{\hbar}\int dt\, L}$. This gives a Lagrangian formulation of Q.M.

Thus Feynman replaces the Schrödinger equation with the Green function represented as a path integral as the dynamical starting point of quantum mechanics (Postulate 4). Thus Feynman showed that

$$\psi(\vec{r},t) = \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\, \psi(\vec{r}_0,t_0)$$

with

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dr]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\, L(\vec{r}(t),\dot{\vec{r}}(t))}$$

with $L(\vec{r},\dot{\vec{r}}) = \vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})$, the classical Lagrangian.

Examples: Consider a free particle of mass m in one dimension. The classical Hamiltonian is $H = \frac{p^2}{2m}$ with $p = m\dot{x}$. So the classical Lagrangian is

$$L = p\dot{x} - H = \tfrac{1}{2} m \dot{x}^2.$$

The Green function is given by the path integral

$$K(x,t;x_0,t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx]\, e^{\frac{i}{\hbar}\int_{t_0}^{t} dt\, \frac{1}{2} m \dot{x}(t)^2}$$

$$= \lim_{\substack{\epsilon \to 0 \\ n\to\infty}} \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \cdots \int_{-\infty}^{+\infty} \frac{dx_n}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \frac{1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \times$$

$$\times\, e^{+\frac{i\epsilon}{\hbar}\sum_{k=1}^{n+1} \frac{1}{2} m \frac{(x_k - x_{k-1})^2}{\epsilon^2}}$$

These are just Gaussian integrals, we can perform one integration after another just to keep repeating the Gaussian. Start with the x_1 integral

$$\frac{1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}}\int_{-\infty}^{+\infty}\frac{dx_1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}}\, e^{-\frac{m}{2i\hbar\epsilon}\left[(x_1-x_0)^2+(x_2-x_1)^2\right]}$$

$$=\frac{1}{\sqrt{\frac{2\pi\hbar i(2\epsilon)}{m}}}\, e^{-\frac{m}{2i\hbar(2\epsilon)}(x_2-x_0)^2}\,.$$

The x_2-integral becomes

$$\sqrt{\frac{m}{2\pi\hbar i(2\epsilon)}}\int_{-\infty}^{+\infty}dx_2\sqrt{\frac{m}{2\pi\hbar i(\epsilon)}}\, e^{-\frac{m}{2i\hbar\epsilon}\left[(x_3-x_2)^2+\frac{1}{2}(x_2-x_0)^2\right]}$$

$$=\sqrt{\frac{m}{2\pi\hbar i(3\epsilon)}}\, e^{-\frac{m}{2i\hbar(3\epsilon)}(x_3-x_0)^2}\,.$$

Thus we continue until the x_n is performed with $x_{n+1}=x$ we find

$$\sqrt{\frac{m}{2\pi\hbar i(n+1)\epsilon}}\, e^{-\frac{m}{2i\hbar(n+1)\epsilon}(x_{n+1}-x_0)^2}\,.$$

Recall that $(n+1)\epsilon = t-t_0$, so we find the free particle Green function

$$\boxed{K(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\; e^{+\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{(t-t_0)}} \quad \Theta(t-t_0)}$$

Now suppose we have a Gaussian wave packet at $t_0 = 0$

$$\psi(x_0, t_0=0) = \frac{A}{2\sqrt{\pi}\,a}\, e^{ik_0 x_0}\, e^{-\frac{x_0^2}{4a^2}}$$

(recall page -34-), then at t we found that

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}} \sqrt{\frac{1}{a^2 + \frac{i\hbar t}{2m}}} \times e^{i\left[k_0 x - \frac{\hbar k_0^2}{2m} t\right]}\, e^{-\frac{\left(x - \frac{\hbar k_0}{m} t\right)^2}{4\left(a^2 + \frac{i\hbar t}{2m}\right)}} .$$

$\left(\text{Note if } \int dx\, |\psi(x,t)|^2 = 1 = \frac{|A|^2\sqrt{\pi}}{\sqrt{2}\,a} \Rightarrow A = \left(a\sqrt{8\pi}\right)^{1/2}\right)$

We should obtain this same result by using the free particle Green function K, to evolve $\psi(x_0,t_0)$ in time to $\psi(x,t)$.

That is

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{t}}\, \psi(x_0,t_0)$$

$$= \int_{-\infty}^{+\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{t}}\, \frac{A}{2a\sqrt{\pi}}\, e^{ik_0x_0 - \frac{x_0^2}{4a^2}}$$

$$= \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}} \times$$

$$\times \int_{-\infty}^{+\infty} dx_0\, e^{-\left(\frac{1}{4a^2} - \frac{im}{2\hbar t}\right)x_0^2}\, e^{-\left(\frac{im}{\hbar t}x - ik_0\right)x_0} .$$

Once again we have a Gaussian integral to perform. As usual we do it by completing the square

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2 - bx} = \int_{-\infty}^{+\infty} dx\, e^{-a\left(x+\frac{b}{2a}\right)^2 + \frac{b^2}{4a}}$$

$$= e^{+\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} .$$

This gives

$$\psi(x,t) = \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}}\; e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}} \times$$

$$\times \sqrt{\frac{\pi}{\frac{1}{4a^2} - \frac{im}{2\hbar t}}}\; e^{\frac{\left(\frac{imx}{\hbar t} - ik_0\right)^2}{4\left(\frac{1}{4a^2} - \frac{im}{2\hbar t}\right)}}$$

$$= \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}} \sqrt{\frac{4\pi a^2 \frac{i\hbar t}{2m}}{a^2 + \frac{i\hbar t}{2m}}} \times$$

$$\times e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}}\; e^{\frac{-\left(\frac{m^2x^2}{\hbar^2t^2} + k_0^2 - \frac{2mk_0x}{\hbar t}\right)}{-i4\left(\frac{i2\hbar t + m4a^2}{4a^2 2\hbar t}\right)}}$$

$$= \frac{A}{2\sqrt{\pi}} \sqrt{\frac{1}{a^2 + \frac{i\hbar t}{2m}}}\; \exp\left[\frac{-1}{4\left(a^2 + i\frac{\hbar t}{2m}\right)} \times \right.$$

$$\left. \times \left(\frac{2i\hbar k_0^2 a^2 t}{m} - 4ia^2k_0x + \cancel{\frac{2imx^2a^2}{\hbar t}} - i\cancel{\frac{2mx^2a^2}{\hbar t}} + x^2\right)\right]$$

$$= \frac{A}{2\sqrt{\pi}} \sqrt{\frac{1}{a^2 + \frac{i\hbar t}{2m}}}\; e^{\frac{4ik_0x}{4\left(a^2 + \frac{i\hbar t}{2m}\right)}\left(\frac{i\hbar t}{2m} + a^2\right)} \times$$

$$\times e^{\frac{-1}{4\left(a^2 + \frac{i\hbar t}{2m}\right)}\left(x^2 + \frac{2i\hbar t a^2 k_0^2}{m} - \frac{2xk_0\hbar t}{m}\right)}$$

Thus we finally obtain

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}} \frac{1}{\sqrt{a^2 + \frac{i\hbar t}{2m}}} e^{i\left(k_0 x - \frac{\hbar k_0^2 t}{2m}\right)} \times$$

$$\times e^{-\frac{\left(x - \frac{\hbar k_0 t}{m}\right)^2}{4\left(a^2 + \frac{i\hbar t}{2m}\right)}}$$

in agreement with our previous result obtained from Schrödinger's equation directly (i.e. as an expansion in terms of plane wave solutions to Schrödinger's equation).

Similarly one could calculate the propagation kernel K for other Hamiltonia, for instance the one-dimensional harmonic oscillator Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 .$$

The Lagrangian is simply

$$L = p\dot{x} - H = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2} x^2 .$$

The Green function is then

$$K(x,t;x_0 t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx]\, e^{\frac{i}{\hbar}\int_{t_0}^{t} dt \left(\frac{1}{2}m\dot{x}^2(t) - \frac{m\omega^2}{2}x^2(t)\right)}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega(t-t_0)}} \times \theta(t-t_0) \times e^{\frac{im\omega}{\hbar\, 2\sin\omega(t-t_0)}\left[(x^2+x_0^2)\cos\omega(t-t_0) - 2xx_0\right]}.$$

Note for $\omega \to 0$ we just recover the free particle kernel of page -100-.