Of course the wave-function is no longer normalizable $\int_{-\infty}^{+\infty} dx\, |\psi(x,t)|^2 = \frac{|A|^2}{\sqrt{8\pi}\, a} = a\sqrt{2\pi} \xrightarrow[a\to\infty]{} \infty$

as expected. However the relevant observables will be finite in this limit since this normalization factor is cancelled from numerator and denominator, as in the calculation of energy, say, before the limit $a\to\infty$ is taken.

1.3.3. Expectation Values

Since $|\psi(\vec{r},t)|^2$ is the position probability density (let $\int d^3r\, |\psi(\vec{r},t)|^2 = 1$ for simplicity) the expectation value of any function, $f(\vec{r})$, of position is given by

$$\langle f \rangle \equiv \int d^3r\, f(\vec{r})\, |\psi(\vec{r},t)|^2 .$$

Further, any square-integrable wave function (even for $V\neq 0$) can be Fourier expanded

$$\psi(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, g(\vec{k},t) .$$

Since $\psi(\vec{r},t)$ is normalized to one, we have

$$1 = \int d^3r\, |\psi(\vec{r},t)|^2 = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3}\, g^*(\vec{k}',t)\, g(\vec{k},t) \int d^3r\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} .$$

But
$$\int d^3k\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}') ,$$

hence
$$1 = \frac{1}{(2\pi)^3} \int d^3k \int d^3k'\, g^*(\vec{k}',t)\, g(\vec{k},t)\, \delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow \quad \boxed{\begin{aligned} 1 &= \int \frac{d^3k}{(2\pi)^3}\, g^*(\vec{k},t)\, g(\vec{k},t) \\ &= \int \frac{d^3k}{(2\pi)^3}\, |g(\vec{k},t)|^2 \end{aligned}}$$

This is known as Parseval's Theorem

$$\int d^3r\, |\psi(\vec{r},t)|^2 = \int \frac{d^3k}{(2\pi)^3}\, |g(\vec{k},t)|^2 = 1 .$$

Since $|g(\vec{k},t)|^2$ integrates to 1 like a probability density, we interpret $g(\vec{k},t)$ to be the momentum space wavefunction of the particle and that the momentum probability density is

$$dP(\vec{k},t) = |g(\vec{k},t)|^2 \frac{d^3k}{(2\pi)^3}$$

That is, $dP(\vec{k},t)$ is the probability that a particle at time t has momentum differentially close to $\hbar\vec{k}$.

This interpretation also follows from Postulate 3: the principle of spectral decomposition applied to the momentum operator

$$\vec{P} = -i\hbar\vec{\nabla} .$$

Recall the postulate states that we can expand any wavefunction in terms of the eigenfunctions of a Hermitian operator. Indeed $\vec{P}$ is Hermitian, for ψ_1, ψ_2 normalizable wavefunctions

$$\int d^3r \left[\vec{P}\psi_1(\vec{r},t)\right]^* \psi_2(\vec{r},t)$$

$$= \int d^3r \left[-i\hbar\vec{\nabla}\psi_1\right]^* \psi_2$$

$$= \int d^3r\,(+i\hbar)\left(\vec{\nabla}(\psi_1^*\psi_2) - \psi_1^*\vec{\nabla}\psi_2\right)$$

$$= i\hbar\int d^3r\,\vec{\nabla}(\psi_1^*\psi_2)$$

$$+ \int d^3r\,\psi_1^*\left(-i\hbar\vec{\nabla}\psi_2\right)$$

but $\psi_1, \psi_2 \to 0$ as $r \to \infty$ sufficiently fast

so that
$$\int d^3r \, \vec{\nabla}(\psi_1^* \psi_2) = \int_{S\to\infty} d\vec{S}\,(\psi_1^*\psi_2) = 0$$
hence $\vec{P}$ is Hermitian
$$\int d^3r \left[\vec{P}\psi_1(\vec{r},t)\right]^* \psi_2(\vec{r},t)$$
$$= \int d^3r \, \psi_1^*(\vec{r},t)\left[\vec{P}\psi_2(\vec{r},t)\right].$$
As we have shown, the eigenfunctions of momentum are plane waves
$$\varphi_{\vec{k}}(\vec{r}) = A e^{i\vec{k}\cdot\vec{r}}$$
with A a constant and $\vec{k} \in \mathbb{R}^3$, so that
$$\vec{P}\varphi_{\vec{k}}(\vec{r}) = -i\hbar\vec{\nabla}\varphi_{\vec{k}}(\vec{r})$$
$$= (\hbar\vec{k})\varphi_{\vec{k}}(\vec{r}).$$
According to Postulate 3, any wavefunction can be expanded in terms of $\varphi_{\vec{k}}(\vec{r})$
$$\psi(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k},t)\,\varphi_{\vec{k}}(\vec{r})$$
$$= \int \frac{d^3k}{(2\pi)^3} g(\vec{k},t)\, e^{i\vec{k}\cdot\vec{r}}. \quad (\text{let } Ag \to g)$$

Further the postulate interprets the coefficients in the expansion as probability amplitudes. That is the probability for observing the momentum of the particle at time t to be in the interval $\hbar^3 d^3k$ about $\hbar\vec{k}$ is

$$d\mathcal{P}(\vec{k},t) = |g(\vec{k},t)|^3 \frac{d^3k}{(2\pi)^3} .$$

Hence the expectation value of any function of momentum $\vec{p} = \hbar\vec{k}$, $f(\vec{p})$, is

$$\langle f \rangle = \int \frac{d^3k}{(2\pi)^3} f(\hbar\vec{k}) |g(\vec{k},t)|^2 .$$

For example $\langle \vec{p} \rangle = \int \frac{d^3k}{(2\pi)^3} \hbar\vec{k} \, |g(\vec{k},t)|^2 .$

The Fourier transform of ψ can be inverted to obtain

$$g(\vec{k},t) = \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \psi(\vec{r},t) .$$

This can be used to re-express expectation values of the momentum,

$$\langle f \rangle = \int \frac{d^3k}{(2\pi)^3} f(\hbar\vec{k}) |g(\vec{k},t)|^2$$

$$= \int \frac{d^3k}{(2\pi)^3} f(\hbar\vec{k}) \overbrace{\int d^3r\, e^{+i\vec{k}\cdot\vec{r}}\, \psi^*(\vec{r},t)}^{=g^*(\vec{k},t)} \times$$

$$\times \underbrace{\int d^3r'\, e^{-i\vec{k}\cdot\vec{r}'}\, \psi(\vec{r}',t)}_{=g(\vec{k},t)}$$

$$= \int d^3r\, d^3r'\, \psi^*(\vec{r},t) \times$$

$$\times \int \frac{d^3k}{(2\pi)^3} \underbrace{f(-i\hbar\vec{\nabla})}_{=f(\hbar\vec{k})} e^{+i\vec{k}\cdot(\vec{r}-\vec{r}')} \psi(\vec{r}',t)$$

$$= \int d^3r\, d^3r'\, \psi^*(\vec{r},t) f(-i\hbar\vec{\nabla}) \delta^3(\vec{r}-\vec{r}') \psi(\vec{r}',t)$$

$$\boxed{\begin{aligned}\langle f \rangle &= \int d^3r\, \psi^*(\vec{r},t) f(-i\hbar\vec{\nabla}) \psi(\vec{r},t) \\ &= \int \frac{d^3k}{(2\pi)^3} f(\hbar\vec{k}) |g(\vec{k},t)|^2\end{aligned}}$$

As expected in coordinate space the momentum is represented by the gradient operator $\vec{p} = -i\hbar\vec{\nabla}$.

Combining this with the previous result for expectation values of $\vec{r}$, we have the expectation value for a general function of $\vec{r}$ and $\vec{p}$, $f(\vec{r},\vec{p})$, as

$$\langle f(\vec{r},\vec{p})\rangle = \int d^3r\, \psi^*(\vec{r},t)\, f(\vec{r},-i\hbar\vec{\nabla})\,\psi(\vec{r},t).$$

1.3.4. Canonical Commutation Relations

As seen above, care must be taken in the ordering of the $\vec{r}$ and $\vec{p}$ arguments in the function f since in the coordinate space expectation value the momentum is replaced with the gradient $\vec{p} = -i\hbar\vec{\nabla}$. Clearly factors of $\vec{r}$ and $\vec{p}$ cannot be simply interchanged since their commutator does not vanish

$$\begin{aligned}[x_i, p_j] &= [x_i, -i\hbar\tfrac{\partial}{\partial x_j}] \\ &= x_i(-i\hbar)\tfrac{\partial}{\partial x_j} - (-i\hbar)\tfrac{\partial}{\partial x_j} x_i \\ &= -i\hbar x_i\tfrac{\partial}{\partial x_j} + i\hbar\delta_{ij} + i\hbar x_i\tfrac{\partial}{\partial x_j} \\ &= i\hbar\delta_{ij},\end{aligned}$$