Of course the wave-function is no longer normalizable \( \int_{-\infty}^{\infty} dx |\psi(x)\|^2 = \frac{1}{\sqrt{2\pi}a} = a \sqrt{\frac{\pi}{2}} \rightarrow \infty \) as expected. However, the observables will be finite in this limit since this normalization factor is cancelled from numerator and denominator, as in the calculation of energy, say, before the limit as \( a \rightarrow 0 \).

1.3.3. Expectation Values

Since \( |\Psi(\vec{r},t)|^2 \) is the position probability density (let \( \int d^3r |\Psi(\vec{r},t)|^2 = 1 \) for simplicity), the expectation value of any function, \( f(\vec{r}) \), of position is given by

\[
\langle f \rangle = \int d^3r f(\vec{r}) |\Psi(\vec{r},t)|^2.
\]

Further, any square-integrable wave function (even for \( V \neq 0 \)) can be Fourier expanded

\[
\Psi(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} g(\vec{k},t).
\]
Since \( \varphi(\vec{r},t) \) is normalized to one, we have

\[
1 = \int d^3r \left| \varphi(\vec{r},t) \right|^2 = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \varphi^*(\vec{k}',t) \varphi(\vec{k},t) \int d^3r e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} .
\]

But

\[
\int d^3k e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}') ,
\]

hence

\[
1 = \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \varphi^*(\vec{k}',t) \varphi(\vec{k},t) \delta^3(\vec{k}-\vec{k}')
\]

\[
\Rightarrow \quad 1 = \frac{1}{(2\pi)^3} \int d^3k \varphi^*(\vec{k},t) \varphi(\vec{k},t)
\]

\[
= \frac{1}{(2\pi)^3} \int d^3k \left| \varphi(\vec{k},t) \right|^2
\]

This is known as Parseval's Theorem.

\[
\int d^3r \left| \varphi(\vec{r},t) \right|^2 = \frac{1}{(2\pi)^3} \int d^3k \left| \varphi(\vec{k},t) \right|^2 = 1 .
\]

Since \( \left| \varphi(\vec{k},t) \right|^2 \) integrates to one like a probability density, we interpret \( \varphi(\vec{k},t) \) to be the momentum space wavefunction of the particle and that the momentum probability density is

\[
dP(\vec{k},t) = \left| \varphi(\vec{k},t) \right|^2 \frac{d^3k}{(2\pi)^3} .
\]
That is, \( dP(\vec{r}, t) \) is the probability that a particle at time \( t \) has momentum differentially close to \( \vec{p} \).

This interpretation also follows from Postulate 3: The principle of spectral decomposition applied to the momentum operator \( \vec{p} = -i\hbar \nabla \).

Recall the postulate states that we can expand any wavefunction in terms of the eigenfunctions of a Hermitian operator. Indeed, if \( \varphi_r \) is any normalizable wavefunction,

\[
\int d^3r \left[ \varphi_1 (\vec{r}, t+\Delta t) \right] \ast \varphi_2 (\vec{r}, t)
= \int d^3r \left[ -i\hbar \nabla \varphi_1 \right] \ast \varphi_2
= \int d^3r \left[ (i\hbar \nabla \varphi_1) \ast \varphi_2 \right] - \varphi_1 \ast \nabla (\varphi_2)
= i\hbar \int d^3r \nabla (\varphi_1 \ast \varphi_2)
+ \int d^3r \varphi_1 \ast \nabla \varphi_2
\]

but \( \varphi_1, \varphi_2 \rightarrow 0 \) as \( r \rightarrow \infty \) sufficiently fast.
so that \[ \int d^3r \text{ } \hat{\nabla} (\psi^* \psi) = \int d^3r \text{ } (\phi^* \phi) = 0 \]
hence \( \hat{\mathcal{P}} \) is Hermitian.

\[ \int d^3r \text{ } \left[ \hat{\mathcal{P}} \psi_1 (F,t) \right]^* \psi_2 (F,t) \]

\[ = \int d^3r \text{ } \psi_1^* (F,t) \left[ \hat{\mathcal{P}} \psi_2 (F,t) \right] \]

As we have shown, the eigenfunctions of momentum are plane waves,

\[ \psi_{\mathbf{k}} (F) = A e^{i \mathbf{k} \cdot \mathbf{F}} \]

with \( A \) a constant and \( \mathbf{k} \in \mathbb{R}^3 \), so that

\[ \hat{\mathcal{P}} \psi_{\mathbf{k}} (F) = -i \hbar \hat{\mathbf{k}} \psi_{\mathbf{k}} (F) \]

\[ = (i \hbar \mathbf{k}) \psi_{\mathbf{k}} (F) \]

According to Postulate 3, any wavefunction can be expanded in terms of \( \psi_{\mathbf{k}} (F) \)

\[ \psi (F,t) = \int \frac{d^3k}{(2\pi)^3} \mathcal{G}(\mathbf{k},t) \psi_{\mathbf{k}} (F) \]

\[ = \int \frac{d^3k}{(2\pi)^3} \mathcal{G}(\mathbf{k},t) e^{i \mathbf{k} \cdot \mathbf{F}} \left( \text{let } A \to 2 \hbar \phi \right) \]
Further, the postulate interprets the coefficients in the expansion of probability amplitudes. That is, the probability for observing the momentum of the particle at time \( t \) to be in the interval \( \hbar^3 d^3k \) about \( \hbar \tilde{t} \tilde{k} \) is

\[ dP(\hbar \tilde{t}, t) = |g(\hbar \tilde{t}, t)|^3 \frac{d^3k}{(2\pi)^3} \]

Hence the expectation value of any function of momentum \( \tilde{p} = \hbar \tilde{t} \tilde{k} \), \( f(\tilde{p}) \), is

\[ \langle f \rangle = \int \frac{d^3k}{(2\pi)^3} f(\hbar \tilde{t} \tilde{k}) |g(\hbar \tilde{t}, t)|^2 \]

For example, \( \langle \tilde{p} \rangle = \int \frac{d^3k}{(2\pi)^3} \tilde{t} \tilde{k} |g(\hbar \tilde{t}, t)|^2 \).

The Fourier transform of \( \tilde{t} \) can be inverted to obtain

\[ g(\hbar \tilde{t}, t) = \int d^3v e^{-i\tilde{v}\cdot\tilde{t}} \tilde{t}(\tilde{F}, t) \]

This can be used to re-express expectation values of the momentum,
\begin{align*}
\langle f \rangle &= \int \frac{d^3k}{(2\pi)^3} f(k) \left| g_t(k, t) \right|^2 = g_0^*(k, t) \\
&= \int \frac{d^3k}{(2\pi)^3} f(k) \int d^3r e^{i k \cdot \mathbf{r}} \mathcal{A}(\mathbf{r}, t) \times \\
&\quad \times \int d^3r' e^{-i k' \cdot \mathbf{r}'} \mathcal{A}(\mathbf{r}', t) \\
&= f(k) \mathcal{A}(\mathbf{r}, t) \mathcal{A}^*(\mathbf{r}', t) \\
&= \int d^3r d^3r' \mathcal{A}(\mathbf{r}, t) f(-i \hbar \nabla) \mathcal{A}^*(\mathbf{r}', t) \\
\langle f \rangle &= \int d^3r \mathcal{A}(\mathbf{r}, t) f(-i \hbar \nabla) \mathcal{A}^*(\mathbf{r}, t) \\
&= \int \frac{d^3k}{(2\pi)^3} f(k) \left| g_t(k, t) \right|^2 \\
\end{align*}

As expected in coordinate space the momentum is represented by the gradient operator \( \hat{\mathbf{p}} = -i \hbar \nabla \).
Combining this with the previous result for expectation values of \( \hat{F} \), we have the expectation value for a general function of \( \hat{F} \) and \( \hat{P} \), \( f(\hat{F}, \hat{P}) \), as

\[
\langle f(\hat{F}, \hat{P}) \rangle = \int d^3x \, \hat{F}^\dagger (F,t) f(F, -i\hbar \hat{P}) \hat{F}(F,t).
\]

1.3.4. Canonical Commutation Relations

As seen above, care must be taken in the ordering of the \( \hat{F} \) and \( \hat{P} \) arguments in the function \( f \). Since in the expected value, the momentum is replaced with the gradient \( \frac{\partial}{\partial F} \) \( \hat{P} \), clearly factors of \( \hat{F} \) and \( \hat{P} \) cannot be simply interchanged since their commutator does not vanish

\[
[X_i, P_j] = [X_i, -i\hbar \frac{\partial}{\partial X_j}] = X_i (-i\hbar \frac{\partial}{\partial X_j}) - (-i\hbar \frac{\partial}{\partial X_j}) X_i = -i\hbar X_i \frac{\partial}{\partial X_j} + i\hbar \delta_{ij} + i\hbar X_i \frac{\partial}{\partial X_j} = i\hbar \delta_{ij},
\]

\[
[Y_i, P_j] = [Y_i, -i\hbar \frac{\partial}{\partial Y_j}] = Y_i (-i\hbar \frac{\partial}{\partial Y_j}) - (-i\hbar \frac{\partial}{\partial Y_j}) Y_i = -i\hbar Y_i \frac{\partial}{\partial Y_j} + i\hbar \delta_{ij} + i\hbar Y_i \frac{\partial}{\partial Y_j} = i\hbar \delta_{ij},
\]