1.3. Consequences and Physical Interpretation

1.3.1. Continuity Equation and Conservation of Probability

Certainly if $\rho(\vec{r},t) \equiv |\psi(\vec{r},t)|^2$ is to be interpreted as the position probability density, then the probability of observing the particle somewhere in space is a constant in time. This follows directly from the Schrödinger equation, since, if $\psi(\vec{r},t)$ is a solution

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi ,$$

then the complex conjugate of the equation is

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V^*\psi^* .$$

Hence

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}|\psi|^2 = \psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi$$

$$= \frac{1}{i\hbar}\Big[\psi^*\Big(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\Big) - \Big(-\frac{\hbar^2}{2m}\nabla^2\psi^* + V^*\psi^*\Big)\psi\Big]$$

$$= -\frac{\hbar}{2im}\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right)$$

$$+ \frac{1}{i\hbar}\psi^*\psi\underbrace{(V^*-V)}_{=0}$$

since the potential energy V is real ($\mathrm{Im}\,V \neq 0$ for unstable particles, since they decay, probability will change in time)

Thus defining the probability current density

$$\vec{S}(\vec{r},t) \equiv \frac{\hbar}{2im}\left[\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right](\vec{r},t)$$

we find

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0 ,$$

the continuity equation.

Integrating the continuity equation over some volume V with boundary S we find

$$\int_V d^3r \frac{\partial}{\partial t}|\psi|^2 = \int_V d^3r \vec{\nabla}\cdot \frac{i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right)$$

$$= \oint_S d\vec{S}\cdot \frac{i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right)$$

by Gauss' theorem $\int_V d^3r\, \vec{\nabla}\cdot\vec{F} = \oint_S d\vec{S}\cdot\vec{F}$.

Since the LHS is the time rate of change of the probability of finding the particle in volume V, the RHS can be interpreted as the probability flux through the surface S bounding V. For localized particles we have that $\psi \to 0$ as $S \to \infty$ sufficiently fast so that the RHS vanishes. Thus the integral over all space of $\frac{\partial\rho}{\partial t}$ vanishes

$$\frac{d}{dt}\int_{\substack{\text{all}\\\text{space}}} d^3r\, |\psi(\vec{r},t)|^2 = 0 .$$

The probability of finding the particle somewhere in space is constant in time, as desired.

Note that the probability density ρ and current $\vec{S}$ are invariant under multiplication of the wavefunction by a constant phase, that is

$$\psi(\vec{r},t) \to e^{i\omega}\psi(\vec{r},t) \quad \text{with } \omega\in\mathbb{R}$$

leaves $\rho, \vec{S}$ unchanged: $\rho \to \rho$
$\vec{S} \to \vec{S}$

Such changes in the overall phase of the wavefunction does not change any physical observations. We are free to choose any value of ω we desire. Alternatively stated, the phase of the overall normalization constant N is irrelevant, that is $N \to e^{i\omega}N$ has no impact on any physics. Note though, that if $\psi = \psi_1 + \psi_2$, then letting

$$\psi_1 \to e^{i\omega_1}\psi_1,\ \psi_2 \to e^{i\omega_2}\psi_2 \quad \text{with } \omega_1 \neq \omega_2$$

changes the relative phase between ψ_1 & ψ_2 in ψ and does change the physics.

Finally the interpretation of $\vec{S}$ as a probability current density leads again to the interpretation of $-i\hbar\vec{\nabla}$ as the momentum operator ~~and~~ and hence $\frac{-i\hbar}{m}\vec{\nabla}$ the velocity operator. This implies that the probability current density is