Wess & Zumino found that the most general symmetry of the S-matrix involves charges which obey both commutation and anti-commutation relations. Such an algebra is called a graded Lie algebra. These algebras are generalizations of the Poincaré algebra. The simplest ($N=1$) supersymmetry (SUSY) algebra involves the generators of the Poincaré group P^μ, the generators of translations, $M^{\mu\nu}$, the generators of Lorentz transformations and two anti-commuting (Grassmann) spinor charges Q_α and $\bar{Q}_{\dot\alpha}$, the generators of supersymmetry transformations.

The $N=1$ SUSY graded Lie algebra consists of the Poincaré Algebra

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma})$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda})$$

$$\left(\begin{aligned} &= i\,[D_{\mu\nu}]_\lambda{}^\rho P_\rho \\ &= i(\delta_\mu{}^\rho g_{\nu\lambda} - \delta_\nu{}^\rho g_{\mu\lambda})P_\rho \end{aligned} \right)$$

Plus the anti-commutation relations

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = +2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

and the fact that the SUSY charges are spinors

$$[M^{\mu\nu}, Q_\alpha] = -\tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[M^{\mu\nu}, \bar{Q}_{\dot\alpha}] = +\tfrac{1}{2}(\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}$$

and finally the trivial zero commutators

$$[Q_\alpha, P^\mu] = 0 = [\bar{Q}_{\dot\alpha}, P^\mu] \quad .$$

The SUSY algebra is invariant under multiplication of Q_α by a phase that is

$$(Q'_\alpha =)\, e^{+i\alpha R} Q_\alpha e^{-i\alpha R} \equiv e^{+i\alpha} Q_\alpha$$

$$(\bar{Q}'_{\dot\alpha} =)\, e^{+i\alpha R} \bar{Q}_{\dot\alpha} e^{-i\alpha R} \equiv e^{-i\alpha} \bar{Q}_{\dot\alpha}$$

$$(P'_\mu =)\, e^{+i\alpha R} P_\mu e^{-i\alpha R} \equiv P_\mu$$

$$(M'_{\mu\nu} =)\, e^{-i\alpha R} M_{\mu\nu} e^{-i\alpha R} \equiv M_{\mu\nu}$$

This additional $U(1)$ automorphism group of the SUSY algebra is known as $U(1)_R$; the additional commutators are

$$[R, Q_\alpha] = +Q_\alpha$$
$$[R, \bar{Q}_{\dot\alpha}] = -\bar{Q}_{\dot\alpha}$$

$$[R, P^\mu] = 0 = [R, M^{\mu\nu}]$$

When studying representations of this algebra on single particle states (as we will do later) we note that $P^2 = P_\mu P^\mu$ still commutes with all the generators $P_\mu, M_{\mu\nu}, Q_\alpha, \bar{Q}_{\dot\alpha}$. Hence states (and fields) in a supermultiplet will have the same mass $P^2 = m^2$. However $W^2 = W_\mu W^\mu$ (where $W^\alpha = \frac{i}{4}\epsilon^{\alpha\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski covariant spin operator) does not commute with the SUSY generators. Thus the particles in the same supermultiplet will have different spins. Fermions & bosons will be combined in the same supermultiplet and will have the same mass.

Represent the SUSY algebra by means of linear differential operators, as we did for the Poincaré generators P_μ & $M_{\mu\nu}$, acting on spinor & tensor fields. Since we now have anti-commuting charges we must extend space-time, x^μ, to include anti-commuting spinor parameters, $\theta_\alpha, \bar\theta^{\dot\alpha}$, to form superspace. A point in superspace is defined by

$$z^M \equiv (x^\mu, \theta_\alpha, \bar\theta^{\dot\alpha}) \quad \text{where}$$

the $\theta_\alpha, \bar\theta^{\dot\alpha}$ are (two component, complex) Weyl spinors which anti-commute, that is, are elements of a Grassmann algebra:

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha \quad \text{and since}$$

$\alpha = 1, 2$ we find

$$\theta^\alpha \theta^\beta \theta^\gamma = 0 \quad \text{and}$$

similarly for

$$\bar\theta_{\dot\alpha} \bar\theta_{\dot\beta} = -\bar\theta_{\dot\beta} \bar\theta_{\dot\alpha} \quad \text{with}$$

$$\bar\theta_{\dot\alpha} \bar\theta_{\dot\beta} \bar\theta_{\dot\gamma} = 0 .$$

Differentiation with respect to the anti-commuting parameters can be defined by the Taylor expansion formulae

$$\phi(\theta+\delta\theta) \equiv \phi(\theta) + \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\phi(\theta)$$

$$\phi(\bar\theta+\delta\bar\theta) \equiv \phi(\bar\theta) - \delta\bar\theta_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\phi(\bar\theta) \quad .$$

Choosing $\phi(\theta) = \theta^\alpha$ or $\phi(\bar\theta) = \bar\theta^{\dot\alpha}$, we find

$$\Big(\text{i.e. } (\theta+\delta\theta)^\beta = \theta^\beta + \delta\theta^\beta \equiv \theta^\beta + \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\theta^\beta$$
$$\Rightarrow \delta\theta^\alpha \delta_\alpha{}^\beta = \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\theta^\beta \Rightarrow \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta\Big)$$

$$\boxed{\begin{array}{ll} \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta & \frac{\partial}{\partial\theta^\alpha}\theta_\beta = -\delta^\alpha{}_\beta \\ \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} & \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta_{\dot\beta} = -\delta^{\dot\alpha}{}_{\dot\beta} \end{array}}$$

with $\frac{\partial}{\partial\theta_\alpha} \equiv \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}$

$$\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \equiv \epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}}$$

Using these derivatives we can define linear differential operators that act on functions of $x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}$.

The SUSY algebra generators can then be represented as linear superspace differential operators acting on a superfield $\phi = \phi(x, \theta, \bar\theta)$. These are obtained by recalling the general transformation formula for operators now extended to include SUSY charges. Define the superfield $\phi(x, \theta, \bar\theta)$ so that

$$\phi(x,\theta,\bar\theta) = e^{i x^\mu P_\mu} e^{i(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})} \phi(0,0,0) \times e^{-i(\theta Q + \bar\theta \bar Q)} e^{-ix\cdot P}$$

where we have translated the field from the origin of superspace to the point $(x, \theta, \bar\theta)$.

Using the SUSY algebra we can transform the field by a further translation

$$e^{i a^\mu P_\mu} e^{i x^\nu P_\nu} = e^{i(x+a)^\mu P_\mu}$$

hence for an invariant field under translations

$$e^{i a^\mu P_\mu} \phi(x,\theta,\bar\theta) e^{-i a^\mu P_\mu} = \phi(x', \theta', \bar\theta')$$

$$= e^{i a\cdot P} e^{i x\cdot P} e^{i(\theta Q + \bar\theta \bar Q)} \phi(0,0,0) e^{-i(\theta Q + \bar\theta \bar Q)} e^{-i x\cdot P} \times e^{-i a\cdot P}$$

$$= e^{i(x+a)\cdot P} e^{i(\theta Q + \bar\theta \bar Q)} \phi(0,0,0) e^{-i(\theta Q + \bar\theta \bar Q)} e^{-i(x+a)\cdot P}$$

$$= \phi(x+a,\theta,\bar\theta) = e^{a^\mu\partial_\mu}\phi(x,\theta,\bar\theta)$$

$\Rightarrow$

$$\phi(x',\theta',\bar\theta') = \phi(x+a,\theta,\bar\theta).$$

So $$e^{ia^\mu P_\mu}\phi(x,\theta,\bar\theta)e^{-ia^\mu P_\mu} = \phi(x+a,\theta,\bar\theta)$$

$$= e^{a^\mu\partial_\mu}\phi(x,\theta,\bar\theta)$$

for infinitesimal a^μ

$\Rightarrow$ $$\phi(x,\theta,\bar\theta) + ia^\mu[P_\mu,\phi] = \phi(x,\theta,\bar\theta) + a^\mu\partial_\mu\phi$$

$\Rightarrow$

$$[P_\mu,\phi] = -i\partial_\mu\phi \quad \left(= -\text{"}P_\mu\text{"}\phi\right)$$

(Note an abuse of notation: The quantum symmetry generators should be written as $\mathcal{P}_\mu, \mathcal{M}_{\mu\nu}, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}^{\dot\alpha}$ with their representation as differential operators written as $P_\mu, M_{\mu\nu}, Q_\alpha, \bar Q^{\dot\alpha}$ as we did earlier for the Poincaré algebra. We will use the block letters for both, when the context is clear.)

So $$\text{"}P_\mu\text{"} = i\partial_\mu \text{ as previously.}$$

Likewise consider the scalar field under Rotations of superspace:

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\phi(0,0,0)\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \equiv \phi(0,0,0)$$

So using $e^A e^B e^{-A} = e^{B+[A,B]}$ for infinitesimal A

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{ix\cdot P}\,e^{i(\theta Q+\bar\theta\bar Q)}$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{ix\cdot P}\,e^{i(\theta Q+\bar\theta\bar Q)}\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$= e^{ix\cdot P+[\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},\,ix\cdot P]}\;e^{i(\theta Q+\bar\theta\bar Q)+[\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},\,i(\theta Q+\bar\theta\bar Q)]} \times e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$= e^{i[x^\rho-\frac{1}{2}\omega^{\mu\nu}x^\lambda[D_{\mu\nu}]_\lambda{}^\rho]P_\rho} \times e^{i[(\theta^\beta-\frac{i}{4}\omega^{\mu\nu}\theta^\alpha(\sigma_{\mu\nu})_\alpha{}^\beta)Q_\beta + (\bar\theta_{\dot\beta}-\frac{i}{4}\omega^{\mu\nu}\bar\theta_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta})\bar Q^{\dot\beta}]} \times e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, e^{ix\cdot P}\, e^{i(\theta Q+\bar{\theta}\bar{Q})}$$
$$= e^{i[x^\mu - \omega^{\mu\nu}x_\nu]P_\mu} \times$$
$$\times e^{i\left[\left(\theta^\beta - \frac{i}{4}\omega^{\mu\nu}(\theta\sigma_{\mu\nu})^\beta\right)Q_\beta + \left(\bar{\theta}_{\dot\beta} - \frac{i}{4}\omega^{\mu\nu}(\bar\theta\bar\sigma_{\mu\nu})_{\dot\beta}\right)\bar{Q}^{\dot\beta}\right]} \times$$
$$\times e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

So

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, \phi(x,\theta,\bar\theta)\, e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$
$$= \left(e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, e^{ix\cdot P}\, e^{i(\theta Q+\bar\theta\bar Q)}\, e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right) \times$$
$$\times \left(e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, \phi(0,0,0)\, e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right) \times$$
$$\times \left(e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, e^{-i(\theta Q+\bar\theta\bar Q)}\, e^{-ix\cdot P}\, e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right)$$

Scalar field $\quad e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\, \phi(0,0,0)\, e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$

$$= \text{"}D^{-1(\alpha)}{}_{\beta}\text{"}(s)\, \phi(0,0,0)$$
$$\underset{1}{\parallel}$$
$$= \phi(0,0,0)$$

So

$$\boxed{\begin{aligned} & e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,\phi(x,\theta,\bar\theta)\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \\ & = \phi\left(x^\mu - \omega^{\mu\nu}x_\nu,\ \theta^\alpha - \tfrac{i}{4}\omega^{\mu\nu}(\theta\sigma_{\mu\nu})^\alpha,\ \bar\theta_{\dot\alpha} - \tfrac{i}{4}\omega^{\mu\nu}(\bar\theta\bar\sigma_{\mu\nu})_{\dot\alpha}\right) \\ & = \phi(x,\theta,\bar\theta) \\ & + \left[-\omega^{\mu\nu}x_\nu\partial_\mu - \tfrac{i}{4}\omega^{\mu\nu}\left(\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta}\right) + \tfrac{i}{4}\omega^{\mu\nu}\left(\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\right)\right] \\ & \qquad \times \phi(x,\theta,\bar\theta) \end{aligned}}$$

$$\boxed{= \phi(x,\theta,\bar\theta) + \left[\tfrac{i}{2}\omega^{\mu\nu}M_{\mu\nu},\ \phi(x,\theta,\bar\theta)\right]}$$

$\Rightarrow$

$$\boxed{\begin{aligned} [M_{\mu\nu}, \phi(x,\theta,\bar\theta)] &= -i\Big[x_\mu\partial_\nu - x_\nu\partial_\mu \\ &\qquad - \tfrac{i}{2}\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta} + \tfrac{i}{2}\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\Big]\phi \\ &\equiv -\mathcal{M}_{\mu\nu}\,\phi(x,\theta,\bar\theta) \end{aligned}}$$

And finally consider a SUSY transformation

$$e^{i(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})}\phi(x,\theta,\bar\theta)\, e^{-i(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})}$$

$$= e^{ix^\mu P_\mu} e^{i(\xi\cdot Q + \bar\xi\bar Q)} e^{i(\theta Q + \bar\theta\bar Q)} \phi(0,0,0) \times$$
$$\times e^{-i(\theta Q + \bar\theta \bar Q)} e^{-i(\xi Q + \bar\xi\bar Q)} e^{-ix\cdot P}$$

But $e^{i(\xi Q + \bar\xi\bar Q)} e^{i(\theta Q + \bar\theta\bar Q)}$

$$= e^{i[(\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q] + \frac{1}{2}[i(\xi Q + \bar\xi\bar Q),\, i(\theta Q + \bar\theta\bar Q)]}$$

$$= e^{i[(\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q] + \frac{1}{2}i^2([\xi Q, \bar\theta\bar Q] + [\bar\xi\bar Q, \theta Q])}$$

Now

$$[\xi Q, \bar\theta\bar Q] = -\xi^\alpha \bar\theta_{\dot\alpha}\{Q_\alpha, \bar Q^{\dot\alpha}\}$$

$$= +\xi^\alpha \bar\theta^{\dot\alpha}\{Q_\alpha, \bar Q_{\dot\alpha}\} = \xi^\alpha\bar\theta^{\dot\alpha}\, 2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

$$= (2\xi\sigma^\mu\bar\theta)P_\mu$$

Similarly

$$[\bar\xi\bar Q, \theta Q] = +\bar\xi^{\dot\alpha}\theta^\alpha\{Q_\alpha, \bar Q_{\dot\alpha}\}$$

$$= 2\bar\xi^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha P_\mu = +(2\bar\xi\bar\sigma^\mu\theta)P_\mu$$

$$\boxed{e^{i(\xi Q+\bar{\xi}\bar{Q})}\, e^{i(\theta Q+\bar{\theta}\bar{Q})} = e^{+i[\,i(\xi\sigma^\mu\bar{\theta}+\bar{\xi}\bar{\sigma}^\mu\theta)P_\mu]} \times e^{i[(\theta+\xi)Q+(\bar{\theta}+\bar{\xi})\bar{Q}]}}$$

For $\xi, \bar{\xi}$ finite or infinitesimal

So

$$e^{i(\xi Q+\bar{\xi}\bar{Q})}\,\phi(x,\theta,\bar{\theta})\,e^{-i(\xi Q+\bar{\xi}\bar{Q})}$$

$$= e^{i[x^\mu+i(\xi\sigma^\mu\bar{\theta}+\bar{\xi}\bar{\sigma}^\mu\theta)]P_\mu} \times e^{i[(\theta+\xi)Q+(\bar{\theta}+\bar{\xi})\bar{Q}]}\,\phi(0,0,0)\,e^{-i[(\theta+\xi)Q+(\bar{\theta}+\bar{\xi})\bar{Q}]} \times e^{-ix\cdot P}$$

$$= \phi(x^\mu+i\xi\sigma^\mu\bar{\theta}+i\bar{\xi}\bar{\sigma}^\mu\theta,\ \theta+\xi,\ \bar{\theta}+\bar{\xi})$$

$$= \phi(x^\mu+i(\xi\sigma^\mu\bar{\theta}-\theta\sigma^\mu\bar{\xi}),\ \theta+\xi,\ \bar{\theta}+\bar{\xi})$$

$$= \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha}+i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\right)\phi(x,\theta,\bar{\theta}) + \bar{\xi}_{\dot{\alpha}}\left(-\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}+i(\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu\right)\phi(x,\theta,\bar{\theta}) + \phi(x,\theta,\bar{\theta})$$

$\Rightarrow$

$$i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}, \phi(x,\theta,\bar\theta)]$$
$$= \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right)\phi(x,\theta,\bar\theta)$$
$$+ \bar\xi_{\dot\alpha}\left(-\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\right)\phi(x,\theta,\bar\theta)$$

$\Rightarrow$

$$\boxed{\begin{aligned}[Q_\alpha, \phi(x,\theta,\bar\theta)] &= -i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not\partial\bar\theta)_\alpha\right]\phi(x,\theta,\bar\theta) \\ &\equiv -\mathcal{Q}_\alpha\phi(x,\theta,\bar\theta) \\ [\bar{Q}_{\dot\alpha}, \phi(x,\theta,\bar\theta)] &= -i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not\partial)_{\dot\alpha}\right]\phi(x,\theta,\bar\theta) \\ &\equiv -\bar{\mathcal{Q}}_{\dot\alpha}\phi(x,\theta,\bar\theta)\end{aligned}}$$

So one can check explicitly that the group multiplication laws ($e^{ia\cdot P}e^{ix\cdot P} = e^{i(x+a)\cdot P}$ etc.) ~~has~~ given us a representation of the commutation relations by superspace linear differential operators acting on superfields:

$$\boxed{\begin{aligned} P_\mu \phi &= i\partial_\mu \phi \\ M_{\mu\nu}\phi &= i\left[x_\mu \partial_\nu - x_\nu \partial_\mu - \tfrac{i}{2}\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta} + \tfrac{i}{2}\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\right]\phi \\ Q_\alpha \phi &= i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right]\phi \\ \bar{Q}_{\dot\alpha}\phi &= i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right]\phi \end{aligned}}$$

For example the SUSY charge differential operators yield:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right][i]\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right] + \bar{Q}_{\dot\alpha}Q_\alpha$$

$$= +2i\not{\partial}_{\alpha\dot\alpha} = +2\sigma^\mu_{\alpha\dot\alpha}P_\mu \quad .$$

The other commutators can be similarly checked.

This is called the <u>real representation</u> of the SUSY algebra. There are 2 other representations of the algebra that are quite useful.

Real Representation:

Group Element $\Omega(x,\theta,\bar\theta) \equiv e^{ix^\mu P_\mu} e^{i(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})}$

$\Rightarrow$

$$e^{i(\xi Q + \bar\xi \bar Q)} \Omega(x,\theta,\bar\theta) = \Omega(x + i(\xi\sigma\bar\theta - \theta\sigma\bar\xi), \theta+\xi, \bar\theta+\bar\xi)$$

1) Chiral Representation:

Group Element $\Omega_1(x,\theta,\bar\theta) \equiv e^{ix\cdot P} e^{i\theta Q} e^{i\bar\theta\bar Q}$

$$e^{i(\xi Q + \bar\xi \bar Q)} \Omega_1(x,\theta,\bar\theta) = \Omega_1(x - 2i\theta\sigma\bar\xi, \theta+\xi, \bar\theta+\bar\xi)$$

2) Anti-Chiral Representation:

Group Element $\Omega_2(x,\theta,\bar\theta) \equiv e^{ix\cdot P} e^{i\bar\theta\bar Q} e^{i\theta Q}$

$$e^{i(\xi Q + \bar\xi \bar Q)} \Omega_2(x,\theta,\bar\theta) = \Omega_2(x + 2i\xi\sigma\bar\theta, \theta+\xi, \bar\theta+\bar\xi)$$

In each case we can define a superfield in that representation

Real representation:

$$\phi(x,\theta,\bar\theta) \equiv \Omega(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega^{-1}(x,\theta,\bar\theta)$$

Chiral representation:

$$\phi_1(x,\theta,\bar\theta) \equiv \Omega_1(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega_1^{-1}(x,\theta,\bar\theta)$$

Anti-Chiral representation:

$$\phi_2(x,\theta,\bar\theta) \equiv \Omega_2(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega_2^{-1}(x,\theta,\bar\theta)$$

The SUSY changes in each representation becomes

0) Real:
$$Q_\alpha \phi = i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right]\phi$$
$$\bar{Q}_{\dot\alpha} \phi = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right]\phi$$

1) Chiral:
$$Q_{1\alpha}\phi_1 = i\left[\frac{\partial}{\partial\theta^\alpha}\right]\phi_1$$
$$\bar{Q}_{1\dot\alpha}\phi_1 = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i(\theta\not{\partial})_{\dot\alpha}\right]\phi_1$$

2) Anti-Chiral:
$$Q_{2\alpha}\phi_2 = i\left[\frac{\partial}{\partial\theta^\alpha} + 2i(\not{\partial}\bar\theta)_\alpha\right]\phi_2$$
$$\bar{Q}_{2\dot\alpha}\phi_2 = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right]\phi_2$$

For each representation

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = +2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

Since the group elements are related according to

$$\Omega_1(x,\theta,\bar\theta) = \Omega(x+i\theta\sigma\bar\theta, \theta, \bar\theta)$$

$$\Omega_2(x,\theta,\bar\theta) = \Omega(x-i\theta\sigma\bar\theta, \theta, \bar\theta)$$

The fields are related as

$$\boxed{\begin{aligned}\phi(x,\theta,\bar\theta) &= \phi_1(x-i\theta\sigma\bar\theta, \theta, \bar\theta) \\ &= \phi_2(x+i\theta\sigma\bar\theta, \theta, \bar\theta)\end{aligned}}$$

Hence $\phi(x,\theta,\bar\theta) = e^{-i\theta\gamma\bar\theta}\,\phi_1(x,\theta,\bar\theta)$

$$= e^{+i\theta\gamma\bar\theta}\,\phi_2(x,\theta,\bar\theta)$$

Likewise $Q_\alpha = e^{-i\theta\gamma\bar\theta}\,Q_{1\alpha}\,e^{+i\theta\gamma\bar\theta}$

$$\bar{Q}_{\dot\alpha} = e^{-i\theta\gamma\bar\theta}\,\bar{Q}_{1\dot\alpha}\,e^{+i\theta\gamma\bar\theta}$$

and $Q_\alpha = e^{+i\theta\gamma\bar\theta}\,Q_{2\alpha}\,e^{-i\theta\gamma\bar\theta}$

$$\bar{Q}_{\dot\alpha} = e^{+i\theta\gamma\bar\theta}\,\bar{Q}_{2\dot\alpha}\,e^{-i\theta\gamma\bar\theta}$$

So $e^{-i\theta\gamma\bar\theta}\,Q_{1\alpha}\phi_1 = Q_\alpha\phi$, etc., as can be checked explicitly using the differential operators.

From the transformation property of the superfield,

$$U(\xi,\bar\xi)\,\phi(x,\theta,\bar\theta)\,U^{-1}(\xi,\bar\xi) = \phi(x',\theta',\bar\theta')$$

(p. 237)

$$= \phi(x+i(\xi\sigma\bar\theta-\theta\sigma\bar\xi),\ \theta+\xi,\ \bar\theta+\bar\xi),$$

we see that SUSY transformations correspond to <u>translations</u> in superspace

$$\boxed{\begin{aligned} x'^{\mu} &= x^{\mu} + i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi) \\ \theta'^{\alpha} &= \theta^{\alpha} + \xi^{\alpha} \\ \bar\theta'_{\dot\alpha} &= \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha} \end{aligned}}$$

Note that

$$[i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi)]^{*}$$

$$= [i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi)]$$

is <u>real</u>; thus $\phi(x,\theta,\bar\theta)$ can be taken as a real superfield $\phi = \phi^*$ and is called a vector superfield it forms a real representation of SUSY.

In the chiral representation x^μ is translated by a pure imaginary vector

$$\phi_1(x,\theta,\bar\theta) = \phi(x + i\theta\sigma\bar\theta, \theta, \bar\theta)$$

with $[i\theta\sigma\bar\theta]^* = -[i\theta\sigma\bar\theta]$

and likewise $\phi_2(x,\theta,\bar\theta) = \phi(x - i\theta\sigma\bar\theta, \theta, \bar\theta)$

Hence ϕ_1 & ϕ_2 transform as complex representations of SUSY

Chiral:
$$x'^\mu = x^\mu - 2i\theta\sigma\bar\xi$$
$$\theta'^\alpha = \theta^\alpha + \xi^\alpha$$
$$\bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

Anti-Chiral:
$$x'^\mu = x^\mu + 2i\xi\sigma^\mu\bar\theta$$
$$\theta'^\alpha = \theta^\alpha + \xi^\alpha$$
$$\bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

[Notation: Complex conjugation changes $\theta_\alpha \to \bar\theta_{\dot\alpha}$ and also interchanges the order of Grassmann spinors e.g.

$$[\theta^\alpha \chi_\alpha]^* = \bar\chi_{\dot\alpha}\bar\theta^{\dot\alpha} = \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}.$$
)

The real vector superfield can be Taylor expanded in powers of θ & $\bar{\theta}$. This expansion terminates after the $\theta^2\bar{\theta}^2$ power due to the anti-commutivity of θ & $\bar{\theta}$

$$\theta^\alpha \theta^\beta \theta^\gamma = 0 = \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}_{\dot{\gamma}} .$$

So

$$\begin{aligned}\phi(x,\theta,\bar{\theta}) = {} & C(x) + \theta^\alpha \chi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) \\ & + \tfrac{1}{2}\theta^2 M(x) + \tfrac{1}{2}\bar{\theta}^2 M^\dagger(x) \\ & + \theta\sigma^\mu\bar{\theta}\, V_\mu(x) + \tfrac{1}{2}\theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) \\ & + \tfrac{1}{2}\bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2 D(x),\end{aligned}$$

where $\phi = \phi^*$ and the Lorentz transformation properties of the ordinary field coefficients are determined by the fact that ϕ is a Lorentz scalar superfield and the Lorentz property of the corresponding power of θ & $\bar{\theta}$. Also the fields $C, M, M^\dagger, V_\mu, D$ are bosonic and

$\chi^\alpha, \bar{\chi}_{\dot{\alpha}}, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}$ are fermionic spinors.

ϕ is called a vector superfield because it contains the ordinary vector field $V_\mu(x)$.

Given the expansion of the vector superfield in terms of its component ordinary space-time fields we can determine how the space-time component fields transform under SUSY:

Recall, letting $Q \equiv \xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha} = Q(\xi, \bar{\xi})$

$$[Q, \phi(x,\theta,\bar\theta)] = -i\left[\xi \frac{\partial}{\partial\theta} + i\xi\gamma\bar\theta - \bar\xi\frac{\partial}{\partial\bar\theta} - i(\theta\gamma\bar\xi)\right]\phi$$

$$\|$$

$$[Q, C(x)] + \theta^\alpha[Q, \chi_\alpha] + \bar\theta_{\dot\alpha}[Q, \bar\chi^{\dot\alpha}] + \dots + \tfrac{1}{4}\theta^2\bar\theta^2[Q, D(x)]$$

$$= -i\xi\chi - i\xi\theta M - i\xi\sigma^\mu\bar\theta V_\mu + \dots$$

$$\dots - \tfrac{1}{2}\theta\gamma\bar\xi\,\bar\theta^2\theta\lambda$$

So equating powers of θ & $\bar\theta$ and performing tedious $\theta, \bar\theta$ and σ^μ algebra, the component field transformations are found

$$i[Q(\xi,\bar\xi),\phi] \equiv \underbrace{\delta^Q(\xi,\bar\xi)\phi}_{\text{definition of intrinsic variation}} = -i\,\mathcal{Q}(\xi,\bar\xi)\phi(x,\theta,\bar\theta)$$

Q: Quantum operator — $\mathcal{Q}$: diff. operator

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$$[Q,C(x)] = -i[\xi^\alpha\chi_\alpha(x)+\bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha}(x)] \equiv -\text{"}\mathcal{Q}\text{"}C(x)$$

$$\equiv -i\,\delta^Q_{(\xi,\bar\xi)}C(x)$$

$$[Q,\chi_\alpha] = -i\xi_\alpha M - i(\sigma^\mu\bar\xi)_\alpha(V_\mu - i\partial_\mu C)$$

$$\equiv -i\delta^Q\chi_\alpha$$

$$[Q,\bar\chi^{\dot\alpha}] = -i\bar\xi^{\dot\alpha}M^\dagger - i(\xi\sigma^\mu)^{\dot\alpha}(V_\mu + i\partial_\mu C)$$

$$\equiv -i\delta^Q\bar\chi^{\dot\alpha}$$

$$[Q,M] = -i\bar\xi\bar\lambda + \chi\overleftarrow{\partial\!\!\!/}\,\bar\xi \equiv -i\delta^Q M$$

$$[Q,M^\dagger] = -i\xi\lambda - \xi\overrightarrow{\partial\!\!\!/}\,\bar\chi \equiv -i\delta^Q M^\dagger$$

$$[Q,V^\mu] = -\frac{i}{2}\lambda\sigma^\mu\bar\xi + \frac{i}{2}\xi\sigma^\mu\bar\lambda$$

$$-\frac{1}{2}\partial_\nu\chi\sigma^\mu\bar\sigma^\nu\xi - \frac{1}{2}\partial_\nu\bar\chi\bar\sigma^\mu\sigma^\nu\bar\xi$$

$$\equiv -i\delta^Q V^\mu$$

$$[Q,\bar\lambda^{\dot\alpha}] = -i\bar\xi^{\dot\alpha}D + (\xi\partial\!\!\!/)^{\dot\alpha}M - \bar\xi^{\dot\alpha}\partial_\mu V^\mu$$
$$-\frac{i}{2}(\bar\sigma^{\mu\nu}\bar\xi)^{\dot\alpha}V_{\mu\nu}$$

$$\equiv -i\delta^Q\bar\lambda^{\dot\alpha}$$

$$[Q, \lambda_\alpha] = -i\xi_\alpha D - (\not{\partial}\bar{\xi})_\alpha M^\dagger + \xi_\alpha \partial_\mu V^\mu + \frac{i}{2}(\sigma^{\mu\nu}\xi)_\alpha V_{\mu\nu}$$

$$\equiv -i\delta^Q \lambda_\alpha$$

$$[Q, D] = -\xi\overrightarrow{\not{\partial}}\bar{\lambda} - \lambda\overleftarrow{\not{\partial}}\bar{\xi} \equiv -i\delta^Q D$$

where $V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$.

Note that the highest weight $\theta^2\bar{\theta}^2$ field D transforms as a total divergence under SUSY. Since the product of superfields ϕ^n is also a superfield, i.e. transforms by the chain rule with the same linear differential operator as did ϕ, its $\theta^2\bar{\theta}^2$ term will <u>also</u> transform as a total divergence. Thus if we integrate $\int d^4x$ the last term (<u>D-term</u>) of a superfield it will be SUSY invariant (ignoring surface terms involving fermions). We will use this later to build SUSY invariant actions.

Now the vector superfield is the most general superfield. We can define constrained superfields with less degrees of freedom — that is component fields. Consider the SUSY charges in the Chiral Representation

$$Q_{1\alpha} = i\frac{\partial}{\partial\theta^\alpha} \quad ; \quad \bar{Q}_{1\dot\alpha} = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i(\theta\sigma)_{\dot\alpha}\right]$$

Since Q & $\bar{Q}$ have no explicit $\bar\theta$ dependence we see that $\frac{\partial}{\partial\bar\theta_{\dot\beta}}$ will anti-commute with the charges $\left\{\frac{\partial}{\partial\bar\theta_{\dot\beta}}, Q_{1\alpha}\right\} = 0 = \left\{\frac{\partial}{\partial\bar\theta_{\dot\beta}}, \bar{Q}_{1\dot\alpha}\right\}$.

Hence the condition $\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1 = 0$ is a SUSY covariant constraint since $Q_{1\alpha}\left(\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1\right) = \left(i\frac{\partial}{\partial\theta^\alpha}\right)\left(\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1\right)$ is the same SUSY representation (i.e. $[Q_\alpha, \phi_1] = -i\frac{\partial}{\partial\theta^\alpha}\phi_1$

& $\left[Q_{\alpha}, \left(\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1\right)\right] = -i\frac{\partial}{\partial\theta^\alpha}\left(\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1\right)$)

Likewise for $\bar{Q}_{1\dot\alpha}$. So the condition

$$\frac{\partial}{\partial\bar\theta_{\dot\beta}}\phi_1 = 0 \text{ is a covariant constraint.}$$

It implies that $\phi_1 = \phi_1(x,\theta)$ only, indep. of $\bar\theta$.

So we define the SUSY covariant spinor derivative in the chiral representation as

$$\boxed{\bar{D}_{1\dot{\alpha}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}}$$

and a chiral superfield in the chiral representation as a superfield satisfying the constraint

$$\boxed{\bar{D}_{1\dot{\alpha}} \phi_1 = 0}$$

which implies $\phi_1 = \phi_1(x, \theta)$ which we can expand in powers of θ

$$\boxed{\phi_1 = \phi_1(x, \theta) = A(x) + \theta^\alpha \psi_\alpha + \theta^2 F(x)}$$

The chiral superfield consists of complex scalar fields $A(x)$ & $F(x)$ and a Weyl spinor $\psi_\alpha(x)$.

We can convert $\bar{D}_{1\dot{\alpha}}$ & the chiral field to the real representation

recall $\quad \phi(x, \theta, \bar{\theta}) = \phi_1(x - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})$

So $\quad \phi(x, \theta, \bar{\theta}) = e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu} \phi_1(x, \theta, \bar{\theta})$

The chiral field in the real representation is given by

$$\boxed{\begin{aligned}\phi(x,\theta,\bar\theta) &= e^{-i\theta\not\partial\bar\theta}\,\phi_1(x,\theta) \\ &= e^{-i\theta\not\partial\bar\theta}\left[A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x)\right]\end{aligned}}$$

The chiral constraint gives the definition of the covariant spinor derivative in the real representation. Since $\bar D_{1\dot\alpha}\phi_1$ is a superfield in the chiral representation it is taken to the real representation according to

$$e^{-i\theta\not\partial\bar\theta}\left(\bar D_{1\dot\alpha}\phi_1\right) = e^{-i\theta\not\partial\bar\theta}\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} e^{+i\theta\not\partial\bar\theta}\phi(x,\theta,\bar\theta)\right]$$

$$\parallel$$

Real Rep. $$\bar D_{\dot\alpha}\,\phi(x,\theta,\bar\theta) = \left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\not\partial)_{\dot\alpha}\right]\phi(x,\theta,\bar\theta)$$

$\Rightarrow$

$$\boxed{\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\not\partial)_{\dot\alpha}}$$

The susy covariant spinor derivative in the real representation.

Hence $\bar{D}_{\dot\alpha}\phi(x,\theta,\bar\theta) = 0$ defines

a chiral superfield in the real representation
the solution (algebraic not differential) to
this is

$$\phi(x,\theta,\bar\theta) = e^{-i\theta\not{\sigma}\bar\theta}\left[A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x)\right]$$

We can also convert this to the anti-chiral
representation - although not as useful

$$\phi(x,\theta,\bar\theta) = \phi_2(x+i\theta\sigma\bar\theta,\theta,\bar\theta) = e^{i\theta\not{\sigma}\bar\theta}\phi_2(x,\theta,\bar\theta)$$

So

$$e^{-i\theta\not{\sigma}\bar\theta}\bar{D}_{\dot\alpha}\phi(x,\theta,\bar\theta) \equiv \bar{D}_{2\dot\alpha}\phi_2(x,\theta,\bar\theta)$$

$$\|$$

$$e^{-i\theta\not{\sigma}\bar\theta}\left[\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\not{\sigma})_{\dot\alpha}\right)e^{+i\theta\not{\sigma}\bar\theta}\phi_2(x,\theta,\bar\theta)\right]$$

$$= \left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i(\theta\not{\sigma})_{\dot\alpha}\right]\phi_2(x,\theta,\bar\theta)$$

$$\Rightarrow \quad \boxed{\bar{D}_{2\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i(\theta\not{\sigma})_{\dot\alpha}}$$

The anti-chiral representation is useful however to define the complex conjugate field to the chiral field — the anti-chiral superfield. Recall

$$Q_{2\alpha} = i\left[\frac{\partial}{\partial\theta^\alpha} + 2i(\not{\partial}\bar{\theta})_\alpha\right]$$

$$\bar{Q}_{2\dot{\alpha}} = i\left[-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right]$$

Since the SUSY charges are indep. of θ we see that the SUSY covariant spinor derivative

$$\boxed{D_{2\alpha} \equiv \frac{\partial}{\partial\theta^\alpha}}$$

anti-commutes with $Q_2, \bar{Q}_2$.

$$\{D_{2\alpha}, Q_{2\beta}\} = 0 = \{D_{2\alpha}, \bar{Q}_{2\dot{\alpha}}\}$$

Hence we have the covariant constraint

$$\boxed{\begin{aligned} &D_{2\alpha}\bar{\Phi}_2 = 0 \\ \Rightarrow\ &\bar{\Phi}_2 = \bar{\Phi}_2(x,\bar{\theta}) \\ &\quad = \bar{A}(x) + \bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x) + \bar{\theta}^2\bar{F}(x) \end{aligned}}$$

and Φ_2 is an anti-chiral superfield in the anti-chiral representation. As previously we can convert this to the real representation.

$$\boxed{\begin{aligned}\Phi(x,\theta,\bar\theta) &= e^{i\theta\not{\partial}\bar\theta}\,\phi_2(x,\bar\theta)\\ &= e^{i\theta\not{\partial}\bar\theta}\left[\bar A+\bar\theta\bar\psi+\bar\theta^2\bar F\right]\end{aligned}}$$

Next

$$\begin{aligned}D_\alpha\Phi(x,\theta,\bar\theta) &= e^{i\theta\not{\partial}\bar\theta}\left[D_{2\alpha}\Phi_2\right]\\ &= e^{i\theta\not{\partial}\bar\theta}\frac{\partial}{\partial\theta^\alpha}e^{-i\theta\not{\partial}\bar\theta}\,\Phi(x,\theta,\bar\theta)\\ &= \left[\frac{\partial}{\partial\theta^\alpha}-i(\not{\partial}\bar\theta)_\alpha\right]\Phi(x,\theta,\bar\theta)\end{aligned}$$

$$\Rightarrow\quad \boxed{D_\alpha=\frac{\partial}{\partial\theta^\alpha}-i(\not{\partial}\bar\theta)_\alpha}$$

And the anti-chiral superfield in the real representation satisfies the constraint

$$\boxed{\begin{aligned}&D_\alpha\Phi(x,\theta,\bar\theta)=0\\ \Rightarrow\ &\Phi(x,\theta,\bar\theta)=e^{i\theta\not{\partial}\bar\theta}\left[\bar A+\bar\theta\bar\psi+\bar\theta^2\bar F\right]\end{aligned}}$$

Likewise we can convert the spinor derivative to the chiral representation also

$$\phi\,(x,\theta,\bar\theta) = e^{-i\theta\not{x}\bar\theta}\,\phi_1(x,\theta,\bar\theta)$$

So

$$D_{1\alpha}\,\phi_1\,(x,\theta,\bar\theta) = e^{+i\theta\not{x}\bar\theta}\,D_\alpha\phi$$

$$= e^{+i\theta\not{x}\bar\theta}\,D_\alpha\,e^{-i\theta\not{x}\bar\theta}\,\phi_1$$

$$= \left[\frac{\partial}{\partial\theta^\alpha} - 2i(\not{\partial}\bar\theta)_\alpha\right]\phi_1(x,\theta,\bar\theta)$$

$\Rightarrow$

$$\boxed{D_{1\alpha} = \frac{\partial}{\partial\theta^\alpha} - 2i(\not{\partial}\bar\theta)_\alpha}$$

So we can summarize the susy spinor covariant derivatives in the different representations — in all reps they anti-commute with the susy charges

$$\{D_\alpha, Q_\beta\} = 0 = \{D_\alpha, \bar Q_{\dot\beta}\}$$

$$\{\bar D_{\dot\alpha}, Q_\beta\} = 0 = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\}$$

and anti-commute amongst themselves to yield

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = +2i\not{\partial}_{\alpha\dot\alpha}$$

$$\{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\}$$

Real Representation:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\not{\partial}\bar\theta)$$

$$\bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\not{\partial})_{\dot\alpha}$$

Chiral Representation:

$$D_{1\alpha} = \frac{\partial}{\partial\theta^\alpha} - 2i(\not{\partial}\bar\theta)_\alpha$$

$$\bar{D}_{1\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$

Anti-Chiral Representation:

$$D_{2\alpha} = \frac{\partial}{\partial\theta^\alpha}$$

$$\bar{D}_{2\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i(\theta\not{\partial})_{\dot\alpha}$$

If ϕ is a superfield so are $D_\alpha\phi$ & $\bar{D}_{\dot\alpha}\phi$.

Chiral Superfields obey the supersymmetric constraint

$$\boxed{\bar{D}_{\dot{\alpha}}\,\phi(x,\theta,\bar{\theta}) = 0}$$

Its (algebraic) solution is

$$\boxed{\phi(x,\theta,\bar{\theta}) = e^{-i\theta\not{\partial}\bar{\theta}}\left[A + \theta\psi + \theta^2 F\right]}$$

Anti-chiral Superfields obey the conjugate constraint

$$\boxed{D_\alpha\,\bar{\phi}(x,\theta,\bar{\theta}) = 0}$$

with solution

$$\boxed{\bar{\phi}(x,\theta,\bar{\theta}) = e^{+i\theta\not{\partial}\bar{\theta}}\left[\bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F}\right]}$$

(Notation: $\bar{A} = A^\dagger$, $\bar{F} = F^\dagger$,

$\bar{\psi} = \psi^\dagger$; so $\bar{\phi} = \phi^\dagger$.

use bar notation.)

The SUSY transformations of the component fields for the chiral super-fields can be found most easily using the chiral & anti-chiral representations:

Consider the chiral superfield ϕ $(\bar{D}_{\dot\alpha}\phi = 0)$ in the chiral representation

$$\phi_1(x,\theta) = A + \theta\psi + \theta^2 F$$

The SUSY charges are given by

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$$Q_1(\xi,\bar\xi) = \xi^\alpha Q_{1\alpha} + \bar\xi_{\dot\alpha}\bar Q_1^{\dot\alpha}$$

$$= i\left[\xi\frac{\partial}{\partial\theta} - \bar\xi\frac{\partial}{\partial\bar\theta} - 2i(\theta\partial\!\!\!/\,\bar\xi)\right]$$

So

$$[Q(\xi,\bar\xi),\phi_1] = -Q_1(\xi,\bar\xi)\,\phi_1(x,\theta)$$

$$= -i\left[\xi\psi + \theta^\alpha\left(2\xi_\alpha F - 2i(\partial\!\!\!/\,\bar\xi)_\alpha A\right) + i\theta^2\,\psi\overleftarrow{\partial\!\!\!/}\,\bar\xi\right]$$

Hence we find

$$[Q, A(x)] = -i\xi\psi$$

$$\equiv -i\,\delta^Q(\xi,\bar\xi)\,A(x)$$

$$[Q, \psi_\alpha(x)] = -i\left[2\xi_\alpha F(x) - 2i(\partial\!\!\!/\,\bar\xi)_\alpha A(x)\right]$$

$$\equiv -i\,\delta^Q(\xi,\bar\xi)\,\psi_\alpha(x)$$

$$[Q, F(x)] = +\psi\overleftarrow{\partial\!\!\!/}\,\bar\xi$$

$$\equiv -i\,\delta^Q(\xi,\bar\xi)\,F(x)$$

Or in detail

$$\boxed{\begin{array}{ll} [Q_\alpha, A] = -i\psi_\alpha & ; \quad [\bar Q_{\dot\alpha}, A] = 0 \\ \{Q_\alpha, \psi_\beta\} = +2i\epsilon_{\alpha\beta}F & ; \quad \{\bar Q_{\dot\alpha}, \psi_\beta\} = +2\partial\!\!\!/_{\beta\dot\alpha}A \\ [Q_\alpha, F] = 0 & ; \quad [\bar Q_{\dot\alpha}, F] = +(\partial_\mu\psi\sigma^\mu)_{\dot\alpha} \end{array}}$$

Again the highest weight field F transforms as a total space-time derivative under SUSY. The product of chiral fields is again chiral (by chain rule for $\bar D$)

So we can make SUSY invariant terms in the action by integrating the θ^2 component of a product of chiral fields over $\int d^4x$ (F-term).

Similarly we can find the SUSY transformations of the hermitean conjugate anti-chiral superfield component fields by either taking the conjugate of the above or explicitly in the anti-chiral representation: The anti-chiral superfield ($D_\alpha \bar{\Phi} = 0$) in the anti-chiral representation is

$$\bar{\Phi}_2(x,\bar{\theta}) = \bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2 \bar{F}.$$

The SUSY changes are given by

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$$Q_2(\xi,\bar{\xi}) = \xi^\alpha Q_{2\alpha} + \bar{\xi}_{\dot{\alpha}} \bar{Q}_2{}^{\dot{\alpha}}$$

$$= i\left[\xi \frac{\partial}{\partial\theta} + 2i\,\xi\gamma\bar{\theta} - \bar{\xi}\frac{\partial}{\partial\bar{\theta}}\right]$$

So $[Q(\xi,\bar{\xi}), \bar{\Phi}_2] = -Q_2(\xi,\bar{\xi})\bar{\Phi}_2(x,\bar{\theta})$

$$= -i\left[\bar{\xi}\bar{\psi} + \bar{\theta}_{\dot{\alpha}}(2\bar{\xi}^{\dot{\alpha}}\bar{F} + i2(\xi\gamma)^{\dot{\alpha}}\bar{A}) - i\bar{\theta}^2\xi\overrightarrow{\gamma}\bar{\psi}\right]$$

Hence we find

$$[Q, \bar{A}(x)] = -i\,\bar{\xi}\bar{\psi} \equiv -i\,\delta^Q(\xi,\bar{\xi})\,\bar{A}(x)$$

$$[Q, \bar{\psi}^{\dot{\alpha}}(x)] = -i\left[2\bar{\xi}^{\dot{\alpha}}\bar{F} + i2(\xi \not{\partial})^{\dot{\alpha}}\bar{A}\right]$$

$$\equiv -i\,\delta^Q(\xi,\bar{\xi})\,\bar{\psi}^{\dot{\alpha}}(x)$$

$$[Q, \bar{F}(x)] = -\xi\not{\partial}\bar{\psi} \equiv -i\,\delta^Q(\xi,\bar{\xi})\,\bar{F}(x)$$

Or individually

$$\boxed{\begin{aligned} &[Q_\alpha, \bar{A}] = 0 \quad ; \quad [\bar{Q}_{\dot{\alpha}}, \bar{A}] = -i\bar{\psi}_{\dot{\alpha}} \\ &\{Q_\alpha, \bar{\psi}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \bar{A} \; ; \; \{\bar{Q}_{\dot{\alpha}}, \bar{\psi}_{\dot{\beta}}\} = -2i\epsilon_{\dot{\alpha}\dot{\beta}}\bar{F} \\ &[Q_\alpha, \bar{F}] = -(\not{\partial}\bar{\psi})_\alpha \; ; \; [\bar{Q}_{\dot{\alpha}}, \bar{F}] = 0 \end{aligned}}$$

Again SUSY invariants can be made by integrating the $\bar{\theta}^2$ component of products of anti-chiral superfields over d^4x, (F-terms).

Remark: The space-time derivative ∂_μ is also a SUSY covariant derivative as $D_\mu \equiv \partial_\mu$ commutes with Q & $\bar{Q}$

$$[D_\mu, Q_\alpha] = 0 = [D_\mu, \bar{Q}_{\dot{\alpha}}].$$

The covariant derivatives may also be defined through the multiplication of the group elements on the right rather than the left.

$$\Omega(x,\theta,\bar\theta)\, e^{i(\xi Q+\bar\xi\bar Q)} = \Omega(x+i(\theta\sigma\bar\xi-\xi\sigma\bar\theta), \theta+\xi, \bar\theta+\bar\xi)$$

$$= \Omega(x,\theta,\bar\theta)$$

$$+\xi^\alpha\left[\frac{\partial}{\partial\theta^\alpha}-i(\not{\partial}\bar\theta)_\alpha\right]\Omega(x,\theta,\bar\theta)$$

$$-\bar\xi^{\dot\alpha}\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+i(\theta\not{\partial})_{\dot\alpha}\right]\Omega(x,\theta,\bar\theta)$$

These are just the SUSY covariant spinor derivatives

$$\boxed{\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha}-i(\not{\partial}\bar\theta)_\alpha \\ \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+i(\theta\not{\partial})_{\dot\alpha} \end{aligned}}$$

From this we see that the covariant derivatives and the SUSY transformations anti-commute since left & right multiplication lead to the same result independent of the order

Let $g(\xi,\bar\xi) = e^{i(\xi Q + \bar\xi \bar Q)}$; $h(\zeta,\bar\zeta) = e^{i(\zeta Q + \bar\zeta \bar Q)}$

First multiply by g from the left

$$g(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta) = \Omega(x + i(\xi\sigma\bar\theta - \theta\sigma\bar\xi), \theta+\xi, \bar\theta+\bar\xi)$$

$$= -iQ(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta) + \Omega(x,\theta,\bar\theta)$$

Second multiply by h from the right

$$g(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)\,h(\zeta,\bar\zeta)$$

$$= \Omega(x,\theta,\bar\theta)\,h(\zeta,\bar\zeta) - iQ(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)\,h(\zeta,\bar\zeta)$$

$$= \Omega(x + i(\theta\sigma\bar\zeta - \zeta\sigma\bar\theta), \theta+\zeta, \bar\theta+\bar\zeta)$$

$$- iQ(\xi,\bar\xi)\,\Omega(x + i(\theta\sigma\bar\zeta - \zeta\sigma\bar\theta), \theta+\xi, \bar\theta+\bar\xi)$$

$$= \Omega(x,\theta,\bar\theta) + D(\zeta,\bar\zeta)\,\Omega(x,\theta,\bar\theta) - iQ(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)$$

$$- iQ(\xi,\bar\xi)\,D(\zeta,\bar\zeta)\,\Omega(x,\theta,\bar\theta)$$

---

with

$$D(\zeta,\bar\zeta) \equiv \zeta^\alpha D_\alpha + \bar\zeta_{\dot\alpha}\bar D^{\dot\alpha}$$

So reversing the order of multiplication

1) Multiply by h from the right first

$$\Omega(x,\theta,\bar\theta)\,h(\zeta,\bar\zeta) = \Omega(x,\theta,\bar\theta) + D(\zeta,\bar\zeta)\,\Omega(x,\theta,\bar\theta)$$

2) Second multiply by g from the left

$$g(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)\,h(\zeta,\bar\zeta)$$

$$= g(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta) + D(\zeta,\bar\zeta)\,g(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)$$

$$= \Omega(x,\theta,\bar\theta) - iQ(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta) + D(\zeta,\bar\zeta)\,\Omega(x,\theta,\bar\theta)$$

$$- i\,D(\zeta,\bar\zeta)\,Q(\xi,\bar\xi)\,\Omega(x,\theta,\bar\theta)$$

So we end with $g\Omega h$ in both cases

$$\Rightarrow \quad Q(\xi,\bar\xi)\,D(\zeta,\bar\zeta) = D(\zeta,\bar\zeta)\,Q(\xi,\bar\xi)$$

$$\Rightarrow \{Q_\alpha, D_\beta\} = 0 = \{Q_\alpha, \bar D_{\dot\beta}\}$$

$$\{\bar Q_{\dot\alpha}, D_\beta\} = 0 = \{\bar Q_{\dot\alpha}, \bar D_{\dot\beta}\}$$

as required of a Susy covariant derivative.

Let's return to the view of superspace as the coset coordinates for $SP_4/SO(1,3)$.

The real representation coset element was parameterized as

$$\Omega(x,\theta,\bar\theta) = e^{ix^\mu P_\mu} e^{i[\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}]}$$

Viewing the points in superspace as the triplet of parameters $z^M = (x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha})$ we defined the differential operator by means of the Taylor expansion

Check Wess & Bagger for up/down & sign conventions!

$$d \equiv dz^M \partial_M \equiv dx^\mu \frac{\partial}{\partial x^\mu} + d\theta^\alpha \frac{\partial}{\partial\theta^\alpha} - d\bar\theta_{\dot\alpha} \frac{\partial}{\partial\bar\theta_{\dot\alpha}}$$

$$= dx^\mu \partial_\mu + d\theta^\alpha \frac{\partial}{\partial\theta^\alpha} + d\bar\theta^{\dot\alpha} \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$

So let $dz^M = (dx^\mu, d\theta^\alpha, d\bar\theta^{\dot\alpha})$, $\partial_M = (\partial_\mu, \frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar\theta^{\dot\alpha}})$.
These derivatives ∂_M are not SUSY covariant (except for ∂_μ) i.e. $\{Q_\alpha, \frac{\partial}{\partial\theta^\alpha}\} \neq 0$ since the superspace coordinate differentials dz^M are not covariant. We can find the SUSY covariant coordinate differentials

ω^A and SUSY covariant derivatives D_A by means the Maurer-Cartan 1-form

Then the invariant differential

$$d = dz^M \partial_M = \omega^A D_A \quad \text{where}$$

$$\omega^A = dz^M E_M{}^A \quad \& \quad dz^M = \omega^A E^{-1}{}_A{}^M$$

with $E_M{}^A$ the supervielbein and $E^{-1}{}_A{}^M$ its inverse.

Hence $d = \omega^A D_A = dz^M \partial_M = \omega^A E^{-1}{}_A{}^M \partial_M$

$$\Rightarrow \boxed{D_A = E^{-1}{}_A{}^M \partial_M}$$

To find the vielbein consider the Maurer-Cartan 1-form

$$i\omega \equiv \Omega^{-1} d\Omega = i\omega^A \mathcal{Q}_A = i dz^M E_M{}^A \mathcal{Q}_A$$

where we have combined all the charges into $\mathcal{Q}_A = (P_m, Q_\alpha, \bar{Q}^{\dot\alpha})$ then

$$\Omega = e^{i z^M \mathcal{Q}_M} = e^{i[x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot\alpha} \bar{Q}^{\dot\alpha}]}$$

So
$$\Omega^{-1} d\Omega = e^{-iz^M g_M} d\, e^{+iz^M g_M}$$

$$= i dz^M g_M - \tfrac{1}{2}[dz^M g_M, z^N g_N]$$

$$= i dz^M g_M - \tfrac{1}{2}[dx^\mu P_\mu + d\theta^\alpha Q_\alpha + d\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha},$$
$$x^\nu P_\nu + \theta^\beta Q_\beta + \bar\theta_{\dot\beta}\bar Q^{\dot\beta}]$$

$$= i[dx^\mu P_\mu + d\theta^\alpha Q_\alpha + d\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}]$$
$$-\tfrac{1}{2}[d\theta^\alpha Q_\alpha, \bar\theta_{\dot\beta}\bar Q^{\dot\beta}] - \tfrac{1}{2}[d\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}, \theta^\beta Q_\beta]$$

$$= i[(dx^\mu - i\theta\sigma^\mu d\bar\theta + i d\theta\sigma^\mu\bar\theta)P_\mu$$
$$+ d\theta^\alpha Q_\alpha + d\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}]$$

$\Rightarrow$

$$\omega^A = (dx^\mu - i\theta\sigma^\mu d\bar\theta + i d\theta\sigma^\mu\bar\theta,\ d\theta^\alpha,\ d\bar\theta_{\dot\alpha})$$

$$= dx^\mu \delta_\mu{}^A + d\theta^\alpha[\delta_\alpha{}^A + i(\sigma^\mu\bar\theta)_\alpha]$$
$$+ d\bar\theta^{\dot\alpha}[-\delta_{\dot\alpha}{}^A + i(\theta\sigma^\mu)_{\dot\alpha}]$$

So we can read off the matrix els. of the super Vielbein $E_M{}^A$

$$E_M{}^A = \begin{array}{c|ccc} {}_M\backslash^A & m & a & \dot{a} \\ \hline \mu & & & \\ \alpha & & & \\ \dot{\alpha} & & & \end{array} \quad \begin{array}{c} \\ \mu \\ \alpha \\ \dot\alpha \end{array}\begin{array}{c} m \quad\quad a \quad\quad \dot a \\ \begin{bmatrix} \delta_\mu{}^m & 0 & 0 \\ i(\sigma^m\bar\theta)_\alpha & \delta_\alpha{}^a & 0 \\ +i(\theta\sigma^m)_{\dot\alpha} & 0 & -\delta_{\dot\alpha}{}^{\dot a} \end{bmatrix} \end{array}$$

$\Rightarrow$

$$E^{-1}{}_A{}^M = \begin{array}{c} \\ m \\ a \\ \dot a \end{array}\begin{array}{c} \mu \quad\quad \alpha \quad\quad \dot\alpha \\ \begin{bmatrix} \delta_m{}^\mu & 0 & 0 \\ -i(\sigma^\mu\bar\theta)_a & \delta_a{}^\alpha & 0 \\ +i(\theta\sigma^\mu)_{\dot a} & 0 & -\delta^{\dot\alpha}{}_{\dot a} \end{bmatrix} \end{array}$$

So we find the SUSY covariant derivatives

$$D_A = E^{-1}{}_A{}^M \partial_M$$

$$= \begin{bmatrix} \delta_m{}^\mu & 0 & 0 \\ -i(\sigma^\mu\bar\theta)_a & \delta_a{}^\alpha & 0 \\ +i(\theta\sigma^\mu)_{\dot a} & 0 & -\delta^{\dot\alpha}{}_{\dot a} \end{bmatrix} \begin{bmatrix} \partial_\mu \\ \dfrac{\partial}{\partial\theta^\alpha} \\ \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{bmatrix}$$

$$= \begin{bmatrix} \partial_m \\ \dfrac{\partial}{\partial\theta^a} - i(\not\partial\bar\theta)_a \\ -\dfrac{\partial}{\partial\bar\theta^{\dot a}} + i(\theta\not\partial)_{\dot a} \end{bmatrix} = \begin{bmatrix} D_m \\ D_a \\ \bar D_{\dot a} \end{bmatrix}$$

Thus we find the SUSY covariant derivatives in the real representation:

$$D_\mu = \partial_\mu$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\not{\partial}\bar{\theta})_\alpha$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\not{\partial})_{\dot{\alpha}}$$

Finally SUSY invariant integration measures and delta functions can be introduced. Integration over Grassmann variables is defined so that it is translationally invariant mimicing that property of infinite limit ordinary integrals

$$\int_{-\infty}^{+\infty} dx\, f(x) = \int_{-\infty}^{+\infty} d(x+a)\, f(x+a) = \int_{-\infty}^{+\infty} dx\, f(x+a) .$$

Thus for a single Grassmann variable θ (i.e. $\theta^2=0$) we define

$$\int d\theta\, f(\theta) \equiv \int d\theta\, f(\theta+\xi) .$$

Hence for $f=\theta$ we have

$$\int d\theta\, \theta = \int d\theta\, (\theta+\xi) = \int d\theta\, \theta + \xi\int d\theta$$

$$\Rightarrow \boxed{\int d\theta = 0 .}$$

Since $\theta^2=0$ all we are left with is $\int d\theta\,\theta$ which we choose as 1 as a normalization

$$\boxed{\int d\theta\, \theta \equiv 1} .$$

So we see that integration over Grassmann variables is equivalent to differentiation with respect to the variable since

$$\frac{d}{d\theta}\theta = 1 = \int d\theta\,\theta$$

$$\frac{d}{d\theta}1 = 0 = \int d\theta$$

Hence

$$\int d\theta\, f(\theta) = \frac{d}{d\theta} f(\theta).$$

For our 2-component, complex Grassmann parameters of superspace we define integration as

$$\int d\theta_\alpha\, \theta^\beta \equiv \delta_\alpha{}^\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta$$

$$\int d\bar\theta_{\dot\alpha}\, \bar\theta^{\dot\beta} \equiv \delta_{\dot\alpha}{}^{\dot\beta} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta}$$

Thus integration is the same as differentiation for Grassmann variables

$$\int d\theta_\alpha = \frac{\partial}{\partial\theta^\alpha}$$

$$\int d\bar\theta_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$

SUSY invariant measures are made by integrating over all space-time $\int d^4x$ and the highest $\theta, \bar\theta$ weight of the integrand

1) Vector measure (or integration)

$$\int dV \equiv \int d^4x\, d^2\theta\, d^2\bar\theta \equiv \int d^4x\, d\theta^\alpha d\theta_\alpha\, d\bar\theta_{\dot\alpha} d\bar\theta^{\dot\alpha}$$

$$= \int d^4x \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}$$

So for an integrand $\phi(x,\theta,\bar\theta)$ that is a vector superfield that decreases sufficiently fast at space-time infinity so that we can ignore surface terms

$$\int dV\, \phi = \int d^4x\, D^\alpha D_\alpha \bar D_{\dot\alpha} \bar D^{\dot\alpha} \phi$$

$$= \int d^4x\, DD\bar D\bar D\, \phi \quad \left(= \int d^4x\, \bar D\bar D DD\phi \text{, etc.} \right)$$

2) Chiral measure (or integration)

$$\int dS \equiv \int d^4x\, d^2\theta = \int d^4x \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}$$

again ignoring surface terms for a chiral superfield $S(x,\theta,\bar\theta)$ we have

$$\int dS\, S = \int d^4x\, D^\alpha D_\alpha S = \int d^4x\, DD\, S$$

3) Anti-Chiral measure (or integration)

$$\int d\bar{S} \equiv \int d^4x\, d^2\bar{\theta} = \int d^4x \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}$$

For a anti-chiral superfield of sufficient decrease at space-time infinity $\bar{S}(x,\theta,\bar{\theta})$

$$\int d\bar{S}\, \bar{S} = \int d^4x\, \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{S} = \int d^4x\, \bar{D}\bar{D}\, \bar{S} .$$

Note that $\boxed{\int d^2\theta\, \theta^2 = -4}$

i.e. $$\int d^2\theta\, \theta^2 = \frac{\partial}{\partial\theta_\alpha} \frac{\partial}{\partial\theta^\alpha}\, \theta^\beta \theta_\beta$$

$$= \frac{\partial}{\partial\theta_\alpha} \left[\delta_\alpha{}^\beta \theta_\beta - \theta^\beta \epsilon_{\beta\gamma} \frac{\partial}{\partial\theta^\alpha} \theta^\gamma \right]$$

$$= \frac{\partial}{\partial\theta_\alpha} \left[\theta_\alpha - \theta^\beta \epsilon_{\beta\gamma} \delta_\alpha{}^\gamma \right]$$

$$= 2 \frac{\partial}{\partial\theta_\alpha} \delta_\alpha{}^\beta \theta_\beta = -2\, \delta_\alpha{}^\beta \delta_\beta{}^\alpha$$

$$= -2\, \delta_\alpha{}^\alpha = -4$$

Likewise $\boxed{\int d^2\bar{\theta}\, \bar{\theta}^2 = -4}$

1) So if ϕ is a vector superfield

$$\phi = C + \theta\chi + \bar\theta\bar\chi + \dots + \tfrac{1}{4}\theta^2\bar\theta^2 D$$

we have

$$\boxed{\int dV\,\phi = 16\int d^4x\,\tfrac{1}{4}D(x)}$$

Since the SUSY variation of $D(x)$,

$$\delta^Q(\xi,\bar\xi)\,D(x) = -i\left[\xi\overrightarrow{\not\partial}\bar\lambda - \lambda\overleftarrow{\not\partial}\bar\xi\right],$$

is a total space-time derivative we have

$$\boxed{\delta^Q(\xi,\bar\xi)\int dV\,\phi = 0}$$ a SUSY invariant

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2) If S is a chiral superfield, $\bar D_{\dot\alpha} S = 0$,

$$S = e^{-i\theta\not\partial\bar\theta}\left[A + \theta\psi + \theta^2 F\right]$$

then

$$\boxed{\int dS\,S = -4\int d^4x\,F(x)}$$

and since $\delta^Q(\xi,\bar\xi)F = i\psi\overleftarrow{\not\partial}\bar\xi$ a total derivative we have

$$\boxed{\delta^Q(\xi,\bar\xi)\int dS\,S = 0}$$ a SUSY invariant

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2) If $\bar{S}$ is an anti-chiral superfield, $D_\alpha \bar{S} = 0$, $\overline{(-2.2.1-)}$

$$\bar{S} = e^{+i\theta\not{\partial}\bar{\theta}}\left\{\bar{A} + \bar{\theta}\bar{\psi} + \bar{\theta}^2\bar{F}\right\}$$

then

$$\boxed{\int d\bar{s}\;\bar{S} = -4\int d^4x\;\bar{F}(x)}$$

and since $\delta^Q(\xi,\bar{\xi})\bar{F} = -i\xi\not{\partial}\bar{\psi}$ a total derivative we have

$$\boxed{\delta^Q(\xi,\bar{\xi})\int d\bar{s}\;\bar{S} = 0}$$ a SUSY invariant.

In general SUSY invariant terms can be made since

1) vector times vector superfields

= vector superfield

2) (anti-) chiral times vector superfields

= vector superfield

3) chiral x anti-chiral superfield

= vector superfield

4) chiral × chiral superfield

= chiral superfield

5) anti-chiral × anti-chiral superfield

= anti-chiral superfield.

Hence SUSY invariants can be made by integrating these products over the appropriate measure e.g.

$S\bar{S}$ = vector superfield - it depends on θ & $\bar{\theta}$ in a non-trivial way

$\int dV S\bar{S}$ = susy invariant.

Note: The fields must be in the same representation when they are multiplied together if the product is to be a superfield — so either all the superfields in a product are in the real, chiral or anti-chiral representations.

ex. SS = chiral superfield

$$= e^{-i\theta\gamma\bar{\theta}}\left[S_1(x,\theta)S_1(x,\theta)\right]$$

and

$$\int dS\, S^2 = \int dS\, S_1^2$$

Similarly

$$\int dV\, S\bar{S} = \int dV\, e^{-i\theta\gamma\bar{\theta}} S_1\, e^{+i\theta\gamma\bar{\theta}} \bar{S}_2$$

$$= \int dV\, S_1\, e^{+2i\theta\gamma\bar{\theta}} \bar{S}_2$$

$$= \int dV\, S_1 \bar{S}_1 = \int dV\, S_2 \bar{S}_2$$

Lastly, it is convenient to define functional differentiation wrt superfields and hence superspace Dirac delta functions.

The Grassmann variable delta function is defined so that

$$\int d\theta'\, \delta(\theta'-\theta) f(\theta') = f(\theta)$$

For a single θ we have $f(\theta) = f_0 + \theta f_1$ and

$$\delta(\theta'-\theta) = (\theta'-\theta)$$

So

$$\int d\theta'\, \delta(\theta'-\theta) f(\theta') = \int d\theta' \left[\theta'-\theta\right]\left[f_0 + \theta' f_1\right]$$

$$= \int d\theta' \left[\theta' f_0 - \theta f_0 - \theta\theta' f_1\right] = f_0 + \theta f_1 = f(\theta)$$