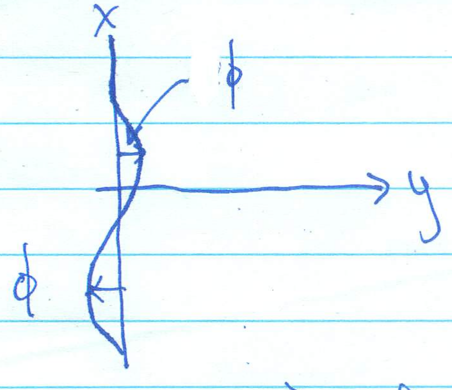
Brane Oscillations

Before considering gravity & supergravity suppose we consider the formation of a $p=3$ brane in flat 5 dimensional space-time. The equations of motion — or more precisely the action can be found by the coset method or the Green-Schwarz method. Suppose the brane is centered on the $y=0$ 4^{th} spatial dimension coordinate origin. Again the invariant interval for flat 5-D space-time is

$$ds^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2 - dy^2$$

$$= dx^\mu \eta_{\mu\nu} dx^\nu - dy^2 .$$

The oscillations of the brane into the "5th" dimension are denoted by $\phi(x)$, i.e.



That is on the brane

$$y = \phi(x)$$

hence $dy = \partial_\mu \phi \, dx^\mu$

$$\Rightarrow \quad ds^2 = dx^\mu (\eta_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi) dx^\nu$$

$$= dx^\mu g_{\mu\nu} dx^\nu$$

where we have induced a metric in 4-dimensional space-time. Since ds^2 is 5D invariant we have that the 5D Poincaré invariant action is given by

$$\Gamma = -\sigma \int d^4x \sqrt{-\det g_{\mu\nu}}$$

For example 5^{th} dimension translations are given by

$$\phi'(x) = \phi(x) + \epsilon \qquad (\text{i.e. } y' = y + \epsilon)$$

hence, simply, $\partial'_\mu \phi'(x) = \partial_\mu \phi(x)$ & $\Gamma' = \Gamma$, likewise for 5D rotations. Note ϕ transforms into a constant — hence it is a Nambu-Goldstone boson associated with the spontaneous breaking of translation invariance in the 5^{t} dimension (i.e. y) by the presence of the brane. The action is the Nambu-Goto action.

$$\text{Now } \det g_{\mu\nu} = \det(\eta_{\mu\nu} - \partial_\mu\phi\,\partial_\nu\phi)$$

$$= \det[\eta_{\mu\rho}(\delta^\rho{}_\nu - \partial^\rho\phi\,\partial_\nu\phi)]$$

$$= \underbrace{\det\eta_{\mu\rho}}_{=-1} \det(\delta^\rho{}_\nu - \partial^\rho\phi\,\partial_\nu\phi)$$

but
$$\det(\delta^\rho{}_\nu - \partial^\rho\phi\partial_\nu\phi) = e^{\operatorname{Tr}\ln[\delta^\rho{}_\nu - \partial^\rho\phi\partial_\nu\phi]}$$
$$= e^{\sum_n \frac{(-1)}{n} \underbrace{(\partial^\rho\phi\partial_\nu\phi)(\partial^\nu\phi\partial_\mu\phi)(\partial^\mu\phi\partial\phi)\cdots(\partial\phi\partial_\rho\phi)}_{n\text{-terms}}}$$
$$= e^{\sum_n \frac{(-1)}{n} (\partial_\nu\phi\partial^\nu\phi)(\partial_\mu\phi\partial^\mu\phi)\cdots(\partial_\rho\phi\partial^\rho\phi)}$$
$$= e^{\sum_n \frac{(-1)}{n} (\partial_\mu\phi\partial^\mu\phi)^n} = e^{\ln[1-\partial_\mu\phi\partial^\mu\phi]}$$
$$= [1-\partial_\mu\phi\partial^\mu\phi]$$

So
$$\sqrt{\det g_{\mu\nu}} = \sqrt{1-\partial_\mu\phi\partial^\mu\phi}$$ and the

Nambu-Goto action is
$$\boxed{\Gamma = -\sigma\int d^4x\,\sqrt{1-\partial_\mu\phi\partial^\mu\phi}}$$

σ = brane tension.

Now $\Gamma = -\sigma\int d^4x\left(1-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \ldots\right)$

re-scaling the field so that $\pi = \sqrt{\sigma}\,\phi$ yields
$$\Gamma = \int d^4x\left[-\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{\sigma}\frac{1}{8}(\partial_\mu\pi\partial^\mu\pi)^2 + \cdots\right]$$

From the non-linear realization point of view we have broken down the 5D Poincaré group $ISO(1,4)$ to the 4D Poincaré group $ISO(1,3)$. The coset element will consist of the broken generators and the 4D space-time coordinates —

$$\Omega(x) \equiv e^{ix^\mu P_\mu} e^{i\phi(x) Z} e^{iv^\mu K_\mu} \in \frac{ISO(1,4)}{SO(1,3)}$$

where $\quad Z \equiv -P^4 \quad ; \quad K^\mu = 2M^{4\mu}$

where the 5D Poincaré algebra is as usual

$$[M^{MN}, M^{RS}] = -i(\eta^{MR} M^{NS} - \eta^{MS} M^{NR} + \eta^{NS} M^{MR} - \eta^{NR} M^{MS})$$

$$[M^{MN}, P^L] = i(P^M \eta^{NL} - P^N \eta^{ML})$$

$$[P^M, P^N] = 0 .$$

This algebra can be re-expressed in terms of the unbroken Lorentz group $SO(1,3)$ represent- ations they lie in, again choosing the brane to be oscillating into the 5^{th} dimension i.e. $M = 4$.

So we have the 4D Poincaré algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\sigma}\right)$$

$$[M^{\mu\nu}, P^{\lambda}] = i\left(P^{\mu}\eta^{\nu\lambda} - P^{\nu}\eta^{\mu\lambda}\right)$$

$$[P^{\mu}, P^{\nu}] = 0$$

and the commutators involving the broken generators:

$$[M^{\mu\nu}, Z] = 0 \quad ; \quad [M^{\mu\nu}, K^{\lambda}] = i\left(K^{\mu}\eta^{\nu\lambda} - K^{\nu}\eta^{\mu\lambda}\right)$$

$$[P^{\mu}, Z] = 0 \quad ; \quad [K^{\mu}, P^{\nu}] = -2i\eta^{\mu\nu} Z$$

$$[K^{\mu}, Z] = -2iP^{\mu} \; ; \; [K^{\mu}, K^{\nu}] = 4iM^{\mu\nu}$$

The transformed coordinates and fields — under a 5D Poincaré variation

$$g = e^{i\epsilon^{\mu}P_{\mu}} e^{izZ} e^{ib^{\mu}K_{\mu}} e^{-\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}}$$

yields

$$g\,\Omega(x) = \Omega(x')\,h(x) \quad \text{with}$$

$$\Omega'(x) = e^{i x'^\mu P_\mu}\, e^{i\phi'(x') Z}\, e^{i v'^\mu(x') K_\mu}$$

& $$h(x) = e^{-\frac{i}{2}\alpha^{\mu\nu}(x) M_{\mu\nu}} .$$

Applying the BCH formulae to the product $g\Omega$ we find

$$x'^\mu = (\eta^{\mu\nu} + \lambda^{\mu\nu}) x_\nu + \epsilon^\mu + 2 b^\mu \phi(x)$$

$$\phi'(x') = \phi(x) + z + 2 b^\mu x_\mu$$

$$v'^\mu(x') = (\eta^{\mu\nu} + \lambda^{\mu\nu}) v_\nu(x) - 4v^2 \coth\sqrt{4v^2}\, P_T^{\mu\nu}(v) b_\nu - P_L^{\mu\nu}(v)\, b_\nu$$

$$\alpha^{\mu\nu}(x) = \lambda^{\mu\nu} + 2\left(\frac{\tanh\sqrt{v^2}}{\sqrt{v^2}}\left(b^\mu v^\nu - b^\nu v^\mu\right)\right)$$

where $P_L^{\mu\nu}(x) = \frac{x^\mu x^\nu}{x^2}$, $P_T^{\mu\nu}(x) = \eta^{\mu\nu} - \frac{x^\mu x^\nu}{x^2}$

Note: general coordinate transformations have been induced

$$dx'^\mu = dx^\nu G_\nu{}^\mu(x) \text{ that is}$$

$$G_\nu{}^\mu(x) = \partial_\nu x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} .$$

and LLT : $\alpha^{\mu\nu}(x)$; $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \alpha^{\mu\nu}(x)$.

As usual the Maurer-Cartan one-form will yield the associated vierbein & induced metric.

$$i\omega(x) = \Omega^{-1} d\Omega = i[\omega^m P_m + \omega_Z Z + \omega_K^m K_m - \tfrac{1}{2}\omega^{mn} M_{mn}]$$

The M-C one-form transforms as

$$\omega'(x') = h(x)\,\omega(x)\,h^{-1}(x) - i\,h(x)\,dh^{-1}(x)$$

$\Rightarrow$ the component forms transform according to their local Lorentz character

$$\omega'^m(x') = \omega^n(x)\Lambda_n{}^m(\alpha(x))$$

$$\omega'_Z(x') = \omega_Z(x)$$

$$\omega'^m_K(x') = \omega_K^n(x)\Lambda_n{}^m(\alpha(x))$$

$$\omega'^{mn}(x') = \omega^{rs}(x)\Lambda_r{}^m(\alpha(x))\Lambda_s{}^n(\alpha(x)) - d\alpha^{mn}(x)$$

Again using the Feynman formula & BCH formula ⇒

$$\omega^m = dx^\mu e_\mu{}^m = dx^\mu \Big[\delta_\mu{}^m + (\cosh\sqrt{4v^2} - 1) P_{L\mu}{}^m(v) - \partial_\mu \phi \frac{\sinh\sqrt{4v^2}}{\sqrt{v^2}} v^m \Big]$$

$$\omega_Z = dx^\mu \cosh\sqrt{4v^2} \Big[\partial_\mu \phi - \frac{\tanh\sqrt{4v^2}}{\sqrt{v^2}} v_\mu \Big]$$

$$\omega_K^m = dx^\mu \partial_\mu v^n \frac{\sinh\sqrt{4v^2}}{\sqrt{4v^2}} P_T(v)_n{}^m$$

$$\omega^{mn} = -4 dx^\mu [\partial_\mu v^r] \Big[\frac{(\cosh\sqrt{4v^2} - 1)}{4v^2} P_T(v)_r{}^m v^n - \frac{(\cosh\sqrt{4v^2} - 1)}{4v^2} P_T(v)_r{}^n v^m \Big]$$

So we find

$$e_\mu{}^m = \Big[\delta_\mu{}^m + (\cosh\sqrt{4v^2} - 1) P_{L\mu}{}^m(v) - \partial_\mu \phi \frac{\sinh\sqrt{4v^2}}{\sqrt{v^2}} v^m \Big]$$

Since $d' = d \Rightarrow e'_\mu{}^m(x') = G^{-1}{}_\mu{}^\nu(x) e_\nu{}^n(x) \Lambda_n{}^m(\alpha(x))$

So

$$\Gamma = -\sigma \int d^4x \det e(x)$$

Since

$$\Gamma' = -\sigma \int d^4x' \det e'(x')$$

$$= -\sigma \int d^4x \det G \left(\det G^{-1} \det e \det \Lambda \right)$$

$$= -\sigma \int d^4x \det e = \Gamma$$

since $\det \Lambda = 1$.

But what about $v^m(x)$ another field?

Note $e_\mu{}^m$ contains no derivatives of v^m → it's field equation is algebraic — so we can eliminate v^m in terms of $\partial_\mu \phi$. Equivalently since w_Z is invariant we can set $w_Z = 0$ in all frames as a covariant constraint

$$w_Z = 0 \Rightarrow \boxed{\partial_\mu \phi = v_\mu \frac{\tanh\sqrt{v^2}}{\sqrt{v^2}}} = v_\mu + \cdots$$

we have eliminated v_μ in terms of $\partial_\mu \phi$.

This is called the "inverse Higgs mechanism".

So

$$\frac{v_\mu \tanh\sqrt{4v^2}}{\sqrt{v^2}} \frac{v_\nu \tanh\sqrt{4v^2}}{\sqrt{v^2}} = \partial_\mu\phi\,\partial_\nu\phi$$

$\Rightarrow$

$$\tanh^2\sqrt{4v^2} = \partial_\mu\phi\,\partial^\mu\phi$$

$$\Rightarrow P_{L\mu\nu}(v) = \frac{\partial_\mu\phi\,\partial_\nu\phi}{\tanh^2\sqrt{4v^2}} = \frac{\partial_\mu\phi\,\partial_\nu\phi}{\partial_\lambda\phi\,\partial^\lambda\phi}$$

$$= P_{L\mu\nu}(\partial\phi)$$

Hence the Nambu-Goto vierbein becomes

$$e_\mu{}^m = \delta_\mu{}^m + (\cosh\sqrt{4v^2} - 1)P_{L\mu}{}^m(v) - \partial_\mu\phi \frac{\sinh\sqrt{4v^2}}{\sqrt{v^2}} v^m$$

$$= \delta_\mu{}^m + (\cosh\sqrt{4v^2} - 1)P_{L\mu}{}^m(v) - v_\mu \frac{\tanh\sqrt{4v^2}}{\sqrt{v^2}} \frac{\sinh\sqrt{4v^2}}{\sqrt{v^2}} v^m$$

$$= \delta_\mu{}^m + P_{L\mu}{}^m(v)\left[\cosh\sqrt{4v^2} - 1 - \frac{\sinh^2\sqrt{4v^2}}{\cosh\sqrt{4v^2}}\right]$$

$$e_\mu{}^m = \delta_\mu{}^m + \left[\frac{\cosh^2\sqrt{4v^2} - \cosh\sqrt{4v^2} - \sinh\sqrt{4v^2}}{\cosh\sqrt{4v^2}}\right] P_{L\mu}{}^m(v)$$

$$e_\mu{}^m = \delta_\mu{}^m + \left[\frac{1-\cosh\sqrt{4v^2}}{\cosh\sqrt{4v^2}}\right] P_{L\mu}{}^m(v)$$

now $\underbrace{\tanh^2\sqrt{4v^2}}_{=\,\partial_\mu\phi\,\partial^\mu\phi} = 1 - \frac{1}{\cosh^2\sqrt{4v^2}}$

$$\Rightarrow \quad \frac{1}{\cosh^2\sqrt{4v^2}} = 1 - \partial_\mu\phi\,\partial^\mu\phi$$

$$\Rightarrow \quad \boxed{\frac{1}{\cosh\sqrt{4v^2}} = \sqrt{1-\partial_\mu\phi\,\partial^\mu\phi}}$$

So

$$\frac{1-\cosh\sqrt{4v^2}}{\cosh\sqrt{4v^2}} = \sqrt{1-\partial_\mu\phi\,\partial^\mu\phi}\left[1 - \frac{1}{\sqrt{1-\partial_\mu\phi\,\partial^\mu\phi}}\right]$$

$$= \sqrt{1-\partial_\mu\phi\,\partial^\mu\phi} - 1$$

$$\Rightarrow \quad \boxed{e_\mu{}^m = \delta_\mu{}^m + \left[\sqrt{1-\partial_\mu\phi\,\partial^\mu\phi} - 1\right]\frac{\partial_\mu\phi\,\partial^m\phi}{\partial_\lambda\phi\,\partial^\lambda\phi}}$$

As earlier

$$\det e = e^{\mathrm{Tr}\ln e_\mu{}^m} = e^{\ln\left[1+\left[\sqrt{1-\partial\phi\partial\phi}-1\right]\frac{\partial_\mu\phi\partial^\mu\phi}{\partial_\lambda\phi\partial^\lambda\phi}\right]}$$

$$= \sqrt{1-\partial_\mu\phi\partial^\mu\phi}$$

$$\Rightarrow \boxed{\Gamma = -\sigma\int d^4x \det e = -\sigma\int d^4x\sqrt{1-\partial_\mu\phi\partial^\mu\phi}}$$

Now we can couple say the standard model to the brane oscillation quanta — "branons". Since the branon oscillations induce a metric, vierbein & spin connection on the brane. The coupling is given by replacing derivatives with covariant ~~derivatives~~ derivatives etc, just like General Relativity.

$$\mathcal{L}_{SM}(\eta,\partial_\mu) \to \mathcal{L}_{SM}(g_{\mu\nu},\nabla_\mu)\det e$$

To lowest order in an expansion in terms of the brane tension we find

$$\mathcal{L}_{int} = +\frac{1}{\sigma}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\, T^{\mu\nu}_{SM}$$

energy-momentum tensor of SM fields.
