General Relativity — Review

Background geometry (Rigid space) first:
Consider M_4 as the flat limit of a hyper dal space embedded in a pseudo-Euclidean space of one dimension higher, 5:

$$\pm R^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2 \pm X_4^2$$

The + signs denote a AdS_4 hyperboloidal hypersurface; the − signs a dS_4 surface. For $R \to \infty$ ($X_4 \to R$), the surface is our Minkowski space M_4. The metric for the space with these Cartesian coordinates is $\hat{\eta}_{MN} = (+1, -1, -1, -1, \pm 1)$. Let's focus on the AdS case.
Hence the AdS_4 hyperboloidal surface has the equation

$$R^2 = X^M \hat{\eta}_{MN} X^N$$ with isometry group $SO(2,3)$ and the invariant interval is

$$ds^2 = dX^M \hat{\eta}_{MN} dX^N .$$

The generators of the $SO(2,3)$ group obey the algebra

$$[M^{MN}, M^{RS}] = -i[\eta^{MR} M^{NS} - \eta^{MS} M^{NR} + \eta^{NS} M^{MR} - \eta^{NR} M^{MS}]$$

and can be realized by the Killing vectors:

$$K^{MN} \equiv i(X^M \partial^N - X^N \partial^M);$$

just the usual infinitesimal "Taylor" term transformation generator.

The Lorentz group involving transformations of the first 4 coordinates, i.e. $SO(1,3)$, is a subgroup and $AdS_4 \cong SO(2,3)/SO(1,3)$ correspond to (pseudo-) translations in the space. The Lorentz transformations are just generated by the first 4×4 M^{MN}, that is

$$M^{\mu\nu} \equiv M^{MN} \quad \text{for } M,N = 0,1,2,3.$$
$$\mu,\nu = 0,1,2,3.$$

The remaining generators are (pseudo-) translations in AdS_4

$$P^\mu = \frac{1}{R} M^{4, M=\mu} = m M^{4, M=\mu} \quad \text{where } m \equiv \frac{1}{R}$$

The SO(2,3) algebra can be written in terms of the P^μ, $M^{\mu\nu}$ as

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i[\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\sigma}]$$

$$[M^{\mu\nu}, P^\lambda] = +i[P^\mu \eta^{\nu\lambda} - P^\nu \eta^{\mu\lambda}]$$

$$[P^\mu, P^\nu] = -i m^2 M^{\mu\nu}$$

with our usual $D=4$ Minkowski metric $\eta^{\mu\nu} = (+1,-1,-1,-1)$

For a flat hyperboloidal surface $R \to \infty$ i.e. $m \to 0$ we obtain the usual M_4 and the Poincaré algebra.

Note if we go one more dimension up and repeat all, we have AdS_5 as the hyperboloidal hypersurface

$$\bar{X}^M \hat{\eta}_{MN} \bar{X}^N = R^2 = \frac{1}{m^2}$$

with invariant interval

$$ds^2 = d\bar{X}^M \hat{\eta}_{MN} \, d\bar{X}^N$$

and isometries $SO(2,4)$

We can choose coordinates on the hyperbola describing the embedding of a 4 dimensional brane in the 5 d space. For an AdS_4 brane embedded in AdS_5 use coordinates

$$\bar{X}^M = (\bar{X}^\mu, \bar{X}^4, \bar{X}^5) \quad \text{where the brane is located at } r = 0 = \bar{X}^4$$

$$\bar{X}^\mu = \left[\frac{4}{4 + m^2 x^2}\right] x^\mu \cosh\sqrt{m^2 r^2}$$

$$\bar{X}^4 = \frac{1}{\sqrt{m^2}} \sinh\sqrt{m^2 r^2}$$

$$\bar{X}^5 = \frac{1}{\sqrt{m^2}} \frac{(4 - m^2 x^2)}{(4 + m^2 x^2)} \cosh\sqrt{m^2 r^2}$$

The invariant interval becomes

$$ds^2 = e^{2A(r)} \overbrace{dx^\mu g_{\mu\nu} dx^\nu}^{d\bar{s}^2} - dr^2$$

where the warp factor $A(r) = \ln\cosh mr$

and

$d\bar{s}^2 = dx^\mu g_{\mu\nu} dx^\nu$ has the isometries of $SO(2,3)$ and $AdS_4 = SO(2,3)/SO(1,3)$; i.e. it is a hyperboloidal hypersurface in the hyperboloidal hypersurface in AdS_5 with $X^4 = 0$

$$R^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2 + X_5^2$$

This subspace has coordinates as previously with $r = 0$

$$X^\mu = \frac{4}{4 + m^2 x^2} x^\mu$$

$$X^4 = 0$$

$$X^5 = \frac{1}{\sqrt{m^2}} \frac{(4 - m^2 x^2)}{(4 + m^2 x^2)}$$

and as above

$$g_{\mu\nu} = \left[\frac{4}{4 + m^2 x^2}\right]^2 \eta_{\mu\nu}$$ a form of the AdS_4 metric

So for AdS_4 we have with this choice of coordinates

$$g_{\mu\nu} = a^2 \eta_{\mu\nu} \quad \text{with}$$

$$a(x) = \left(\frac{4}{4+m^2x^2}\right). \quad \text{Hence } g^{\mu\nu} = \frac{1}{a^2}\eta^{\mu\nu}$$

$$\partial_\rho g_{\mu\nu} = - m^2 a^3 \eta_{\mu\nu}\eta_{\rho\alpha} x^\alpha \qquad \left(\partial_\rho a = -\frac{m^2}{2} a^2 \eta_{\rho\alpha} x^\alpha\right)$$

and so $\Gamma_{\mu,\nu\rho} = \frac{1}{2}\left[\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}\right]$

$$\Gamma^\mu_{\nu\rho} = \frac{1}{a^2}\eta^{\mu\sigma}\Gamma_{\sigma,\nu\rho}$$

$$= -\frac{1}{2} m^2 a x^\alpha \left[\eta_{\alpha\rho}\delta^\mu{}_\nu + \eta_{\alpha\nu}\delta^\mu_\rho - \delta^\mu_\alpha \eta_{\nu\rho}\right]$$

Consequently (calculate away)

$$R^\mu{}_{\sigma\rho\tau} = \partial_\rho \Gamma^\mu_{\sigma\tau} - \partial_\tau \Gamma^\mu_{\sigma\rho} + \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\sigma\tau} - \Gamma^\mu_{\nu\tau}\Gamma^\nu_{\sigma\rho}$$

$$= - m^2 a^2 \left[\eta_{\sigma\rho}\delta^\mu_\tau - \eta_{\sigma\tau}\delta^\mu_\rho\right]$$

$$R_{\nu\rho} = R^\mu{}_{\nu\mu\rho} = (d-1)m^2 a^2 \eta_{\nu\rho} = (d-1) m^2 g_{\nu\rho}$$

(d=4 here)

$$R = d(d-1)m^2 = 12m^2$$

$$\parallel$$

$$g^{\nu\rho}R_{\nu\rho}$$

So the Einstein equation in empty space

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu} \Lambda = 0 \quad \left(= 8\pi G_N T_{\mu\nu} \right)$$

For AdS_4 we found

$$R_{\mu\nu} = (d-1) m^2 g_{\mu\nu}$$

$$R = d(d-1) m^2$$

So $\quad (d-1) m^2 g_{\mu\nu} - \frac{1}{2} d(d-1) m^2 g_{\mu\nu} = \Lambda g_{\mu\nu}$

The cosmological constant

$$\Lambda = m^2 \left[1 - \tfrac{1}{2} d \right] (d-1)$$

$$= -\tfrac{1}{2} m^2 (d-2)(d-1)$$

$$\Lambda = -3 m^2 \qquad AdS_4$$

Note for $\Lambda = +3m^2$ dS_4.

There is currently evidence for $\Lambda \sim (3\,\text{meV})^4$, a "dark energy" pervades the Universe.

$(\Lambda = 8\pi \rho_\Lambda \;;\; \rho_\Lambda \approx 10^{-29}\, g/cm^3 = -p_\Lambda)$

$\bar{X}^\mu \to x^\mu$
$\bar{X}^4 \to 0$
$\bar{X}^5 \to R$

We can continue along this general vein, but to avoid the extra notation lets consider the Minkowski case and let $R \to \infty$ ($m \to 0$). The $SO(2,3)$ group contracts to the usual Poincaré group with generator algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i[\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}]$$

$$[M^{\mu\nu}, P^\lambda] = +i[P^\mu \eta^{\nu\lambda} - P^\nu \eta^{\mu\lambda}]$$

$$[P^\mu, P^\nu] = 0 .$$

Suppose we consider the coset element $ISO(1,3)/SO(1,3) = \Omega(x)$

inhomogeneous $SO(1,3)$ $= P_4$ = Poincaré group. ($= T(1,3) \rtimes SO(1,3)$)

$$\Omega(x) = e^{i x^\mu P_\mu}$$

where x^μ are the usual Cartesian coordinates of M_4 and P_μ the translation generator in M_4.

The Poincaré group element of infinitesimal transformations is given by

$$g = e^{i\epsilon^\mu P_\mu} e^{-i\frac{\lambda^{\mu\nu}}{2} M_{\mu\nu}} .$$

The action of a transformation on the coordinates of M_4 is

$$g\Omega(x) = e^{i\epsilon^\mu P_\mu} e^{-i\frac{\lambda^{\rho\sigma}}{2} M_{\rho\sigma}} e^{ix^\nu P_\nu}$$

$$\equiv e^{ix'^\mu P_\mu} h = \Omega(x') h$$

where $h = e^{-i\frac{\lambda^{\mu\nu}}{2} M_{\mu\nu}}$

and using BCH $e^A e^B e^{-A} = e^{B + [A,B]}$
for A infinitesimal, B arbitrary

(later use: $e^B e^A = e^{B + \mathcal{L}_{B/2}\cdot[A + \coth(\mathcal{L}_{B/2})\cdot A]}$

$$e^A e^B = e^{B - \mathcal{L}_{B/2}\cdot[A - \coth(\mathcal{L}_{B/2})\cdot A]}$$

with Lie derivative

$$\mathcal{L}_A \cdot B \equiv [A, B]$$

Arbitrary A, B: $e^A B e^{-A} = e^{\mathcal{L}_A}\cdot B$

Feynman Formula

$$e^{-iA}\,\delta\, e^{+iA} = \int_0^1 dt\, e^{-itA}\,(i\delta A)\, e^{+itA}$$

$$= \frac{e^{\mathcal{L}_{-iA}} - 1}{\mathcal{L}_{-iA}} \cdot (i\delta A)$$

$$= i\delta A - \frac{(-i)^2}{2!}[A, \delta A] - \ldots$$

$$- \frac{(-i)^{n+1}}{(n+1)!}\underbrace{[A,[A,\ldots[A,\delta A]\ldots]]}_{n\text{-commutators}}$$

So

$$g\Omega(x) = e^{i\epsilon^\nu P_\nu}\,\underbrace{e^{-i\frac{\lambda^{\rho\sigma}}{2}M_{\rho\sigma}}\, e^{ix^\mu P_\mu}\, e^{+i\frac{\lambda^{\alpha\beta}}{2}M_{\alpha\beta}}}_{\text{use BCH}}\, e^{-i\frac{c^{\gamma\delta}}{2}M_{\gamma\delta}}$$

$$= e^{i\epsilon^\nu P_\nu}\, e^{ix^\mu P_\mu + \left[-i\frac{\lambda^{\rho\sigma}}{2}M_{\rho\sigma},\, ix^\mu P_\mu\right]}\, e^{-i\frac{c^{\gamma\delta}}{2}M_{\gamma\delta}}$$

Now

$$-i\frac{\lambda^{\rho\sigma}}{2}\, ix^\mu\,[M_{\rho\sigma}, P_\mu] = i[P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu}]\left(+\frac{\lambda^{\rho\sigma}}{2}x^\mu\right)$$

$$= +\frac{i}{2}\left[\lambda^{\rho}{}_{\cdot\mu} P_\rho x^\mu - \lambda_\mu{}^{\cdot\sigma} P_\sigma x^\mu\right] = +i\lambda^\mu{}_{\cdot\nu} x^\nu P_\mu$$

$$g\Omega(x) = e^{i\epsilon^\rho P_\rho} e^{i x^\mu P_\mu + i\lambda^\mu{}_\nu x^\nu P_\mu} e^{-i\frac{\lambda^{\gamma\delta}}{2}M_{\gamma\delta}}$$

$$= e^{i[x^\mu + \epsilon^\mu + \lambda^\mu{}_\nu x^\nu]P_\mu} e^{-i\frac{\lambda^{\rho\sigma}}{2}M_{\rho\sigma}}$$

$$\equiv e^{i x'^\mu P_\mu} h$$

$\Rightarrow$

$$\boxed{\begin{aligned} x'^\mu &= x^\mu + \epsilon^\mu + \lambda^\mu{}_\nu x^\nu \\ h &= e^{-i\frac{\lambda^{\rho\sigma}}{2}M_{\rho\sigma}} \end{aligned}}$$

So the Poincaré transformations are given by the usual expression

$$x'^\mu = x^\mu + \epsilon^\mu + \lambda^\mu{}_\nu x^\nu$$

i.e.

$$\delta x^\mu = \epsilon^\mu + \lambda^\mu{}_\nu x^\nu \quad .$$

with Killing vectors $K^\mu = i\partial^\mu$

$K^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$

as usual they generate infinitesimal transformations.

Now recall a field belonging to the finite dimensional rep. of Lorentz group $(D_{\mu\nu})^\alpha{}_\beta$ is has the transformation

$$\phi'^{\alpha}(x') = S^{\alpha}_{\cdot\beta}\,\phi^{\beta}(x) \equiv \left[e^{-\frac{1}{2}\lambda^{\rho\sigma}D_{\rho\sigma}}\right]^{\alpha}_{\cdot\beta}\phi^{\beta}(x).$$

Recall for quantum field operators

$$g\,\phi^{\alpha}(x)\,g^{-1} = S^{-1\alpha}{}_{\beta}\,\phi^{\beta}(x')$$

Now the field operator at x^{μ} is related to the operator at the origin by the translation operator

$$\phi^{\alpha}(x) = e^{i x^{\mu}P_{\mu}}\phi^{\alpha}(0)\,e^{-i x^{\nu}P_{\nu}}$$

$$= \Omega(x)\,\phi^{\alpha}(0)\,\Omega^{-1}(x)$$

So a Poincaré transformation of the field operator at x^{μ} is

$$g\,\phi^{\alpha}(x)\,g^{-1} = g\,\Omega(x)\,\phi^{\alpha}(0)\,(g\,\Omega(x))^{-1}$$

$$= \Omega(x')\,h\,\phi^{\alpha}(0)\,h^{-1}\,\Omega^{-1}(x')$$

But $h\,\phi^{\alpha}(0)\,h^{-1} = e^{-i\frac{\lambda^{\rho\sigma}}{2}M_{\rho\sigma}}\phi^{\alpha}(0)\,e^{+i\frac{\lambda^{\mu\nu}}{2}M_{\mu\nu}}$

$$= S^{-1\alpha}{}_{\beta}\,\phi^{\beta}(0) = \left[e^{+\frac{1}{2}\lambda^{\rho\sigma}D_{\rho\sigma}}\right]^{\alpha}_{\cdot\beta}\phi^{\beta}(0)$$

where the $(D^{\mu\nu})^{\alpha}{}_{\cdot\beta}$ = finite dim. rep. of $SO(1,3)$

ex.

Weyl Spinors $(\frac{1}{2},0)$ $(D^{\mu\nu})_{\alpha}{}^{\beta} = \frac{i}{2}(\sigma^{\mu\nu})_{\alpha}{}^{\beta}$

$(0,\frac{1}{2})$ $(D^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = -\frac{i}{2}(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$

Vector $(D^{\mu\nu})_{\alpha\beta} = -\left[\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right]$

etc. (see 658 notes IIA)

So

$$g\phi^{\alpha}(x)g^{-1} = \left[e^{+\frac{1}{2}\lambda^{\rho\sigma}D_{\rho\sigma}}\right]^{\alpha}{}_{\cdot\beta}\,\Omega(x')\phi^{\beta}(0)\Omega^{-1}(x')$$

$$= \left[e^{+\frac{1}{2}\lambda^{\rho\sigma}D_{\rho\sigma}}\right]^{\alpha}{}_{\cdot\beta}\,\phi^{\beta}(x^{\mu}+\epsilon^{\mu}+\lambda^{\mu\nu}x_{\nu})$$

$$\left(= S^{-1\alpha}{}_{\beta}\phi^{\beta}(x')\right)$$

$$= \left[\delta^{\alpha}_{\beta} + \frac{1}{2}\lambda^{\rho\sigma}(D_{\rho\sigma})^{\alpha}{}_{\cdot\beta}\right]\left[\phi^{\beta}(x) + \epsilon^{\mu}\partial_{\mu}\phi^{\beta}(x) + \lambda^{\mu\nu}x_{\nu}\partial_{\mu}\phi^{\beta}(x)\right]$$

$$= \phi^{\alpha}(x) + \epsilon^{\mu}\partial_{\mu}\phi(x) + \frac{i}{2}\lambda^{\mu\nu}\left[i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\delta^{\alpha}{}_{\cdot\beta} - i(D_{\mu\nu})^{\alpha}{}_{\cdot\beta}\right]\phi^{\beta}(x)$$

As usual the intrinsic variations of the field is

$$\phi'(x) - \phi(x) \equiv \delta\phi(x)$$

It is related to the total variation

$$\phi'(x') - \phi(x) \equiv \Delta\phi(x)$$

by

$$\Delta\phi(x) = \phi'(x') - \phi(x') + \phi(x') - \phi(x)$$

$$= \delta\phi(x) + \delta x^\mu \partial_\mu \phi$$

$$\boxed{\delta\phi(x) = \Delta\phi(x) - \delta x^\mu \partial_\mu \phi(x)}$$

$$\delta\phi^\beta(x) = (S-1)^\alpha_{\ \beta}\,\phi^\beta(x) - \delta x^\mu \partial_\mu \phi^\alpha(x)$$

$$= -\epsilon^\mu \partial_\mu \phi + \frac{\lambda^{\mu\nu}}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi^\alpha(x)$$

$$- \frac{\lambda^{\mu\nu}}{2}(D_{\mu\nu})^\alpha_{\ \beta}\,\phi^\beta(x)$$

$$\Big(= g(-\epsilon,-\lambda)\,\phi^\alpha(x)\,g^{-1}(-\epsilon,-\lambda) - \phi^\alpha(x)$$

$$= g^{-1}(\epsilon,\lambda)\,\phi^\alpha(x)\,g(\epsilon,\lambda) - \phi^\alpha(x) \Big)$$

So this as usual gives the representati– of the variations in terms of the Killing vectors & finite dim. Rep. of the Lorentz group

$$P_\mu = i\partial_\mu = K_\mu$$

$$(M_{\mu\nu})^{\alpha}_{\cdot\beta} = (K_{\mu\nu})^{\alpha}_{\cdot\beta} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^\alpha_\beta - i(D_{\mu\nu})^{\alpha}_{\cdot\beta}$$
$$= (K_{\mu\nu})^{\alpha}_{\cdot\beta}$$

Since these are all global symmetry transformations the covariant derivative is simply the derivative ∂_μ

$$\delta\partial_\mu\phi^\alpha = \partial_\mu\delta\phi^\alpha$$
$$= \left[-\epsilon^\rho\partial_\rho\delta^\alpha_\beta - \frac{i}{2}\lambda^{\rho\sigma}\left[i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\delta^\alpha_{\cdot\beta} - i(D_{\rho\sigma})^\alpha_{\cdot\beta}\right]\right]\times$$
$$\times\,\partial_\mu\phi^\beta$$

just as for $\delta\phi^\alpha$.

Thus the Maurer-Cartan 1-form is trivial:

$$\Omega^{-1}(x)\,d\Omega(x) = e^{-ix^\nu P_\nu}\,dx^\mu\partial_\mu\,e^{+ix^\rho P_\rho} = i\,dx^\mu P_\mu \equiv i\omega^m P_m$$

That is $\omega^m = dx^\mu e_\mu{}^m$ with $e_\mu{}^m = \delta_\mu{}^m$.

The invariant interval can be written as

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = \omega^m \eta_{mn} \omega^n$$
$$= dx^\mu e_\mu{}^m \eta_{mn} e_\nu{}^n dx^\nu$$

So $g_{\mu\nu} = e_\mu{}^m \eta_{mn} e_\nu{}^n = \delta_\mu{}^m \eta_{mn} \delta_\nu{}^n = \eta_{\mu\nu}$.

In general we have that

$$x'^\mu = x^\mu + \epsilon^\mu + \lambda^{\mu\nu} x_\nu$$

$$\Rightarrow \quad dx'^\mu = dx^\mu + \lambda^{\mu\nu} dx_\nu$$
$$\equiv dx^\nu G_\nu{}^\mu$$

which in the global case is the Lorentz transformation $G_\nu{}^\mu = \delta_\nu{}^\mu - \lambda_\nu{}^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu}$

On the other have $\partial'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} \partial_\nu = G^{-1}{}_\mu{}^\nu \partial_\nu$
$$= (\delta_\mu{}^\nu + \lambda_\mu{}^\nu) \partial_\nu$$

So $d = dx^\mu \partial_\mu = d' = dx'^\mu \partial'_\mu = dx^\nu G_\nu{}^\mu G^{-1}{}_\mu{}^\rho \partial_\rho$
$$= dx^\nu \partial_\nu \checkmark !$$

So we see that directly $i\omega(x') = \Omega^{-1}(x')d'\Omega(x')$

$$= i\,dx'^{\mu} P_{\mu}$$

$$i\omega(x') = \underbrace{i\,dx^{\nu}}_{= i\omega^{\nu}} G_{\nu}{}^{\mu} P_{\mu} = i\omega'^{m}(x') P_{m}$$

$$\Rightarrow \omega'^{m}(x') = \omega^{n}(x) G_{n}{}^{m}$$

$$= \omega^{n}(x) \underbrace{(\delta_{n}{}^{m} - \lambda_{n}{}^{m})}_{= \Lambda_{n}{}^{m}(\lambda)}$$

This can be arrived at in a general way

$$g\Omega(x) = \Omega(x')h \Rightarrow \Omega(x') = g\Omega(x)h^{-1}$$

So

$$i\omega(x') = \Omega^{-1}(x')d'\Omega(x')$$

$$= (h\Omega^{-1}(x)g^{-1})\,d\,(g\Omega(x)h^{-1})$$

$$= h(\Omega^{-1}(x)d\Omega(x))h^{-1}$$

$$= h\,i\omega(x)h^{-1} = i\omega^{m}(x)\,hP_{m}h^{-1}$$

$$hP_{m}h^{-1} = P_{m} - \frac{i\lambda^{rs}}{2}[M_{rs}, P_{m}]$$

$$= P_{m} - \frac{i\lambda^{rs}}{2}\,i[P_{r}\eta_{sm} - P_{s}\eta_{rm}]$$

$$= P_{m} - \lambda_{m}{}^{r}P_{r} = \Lambda_{m}{}^{n}(\lambda)P_{n}$$

$\Rightarrow$

$$\omega'^m(x')P_m = \omega^n(x)\Lambda_n{}^m P_m$$

$$\Rightarrow \quad \boxed{\omega'^m(x') = \omega^n(x)\Lambda_n{}^m}$$

The point of all this being the Maurer-Cartan 1-form transforms only according to Lorentz transformation $\Lambda_n{}^m$ not the general coordinate translation ϵ^μ in the above sense. Hence

$$ds^2 = \omega^m \eta_{mn}\omega^n = ds'^2$$

Now, this is a review of general relativity.

So let's view GR as a gauge theory. Instead of just global transformations consider space-time dependent transformations

Let $\epsilon^\mu = \epsilon^\mu(x)$ & $\lambda^{\mu\nu} = \lambda^{\mu\nu}(x)$.

Then

$$g(x)\Omega(x) = \Omega(x')\,h(x) \quad \text{with the}$$

same form $x'^\mu = x^\mu + \epsilon^\mu(x) + \lambda^\mu{}_\nu(x)x^\nu$

and $$h(x) = e^{-i\frac{\lambda^{\rho\sigma}(x)}{2}M_{\rho\sigma}}$$

only $\epsilon^\mu = \epsilon^\mu(x)$ & $\lambda^{\mu\nu} = \lambda^{\mu\nu}(x)$.

Likewise $\delta\phi^\alpha = -\epsilon^\mu(x)\partial_\mu\phi^\alpha(x)$ for instance now and ∂_μ will no longer be a covariant derivative

$$\delta(\partial_\mu\phi^\alpha(x)) = \partial_\mu\left(-\epsilon^\rho(x)\partial_\rho\phi^\alpha(x)\right)$$

$$= -\epsilon^\rho(x)\partial_\rho\left(\partial_\mu\phi^\alpha(x)\right) - \left(\partial_\mu\epsilon^\rho(x)\right)\left(\partial_\rho\phi^\alpha(x)\right)$$

$$\neq -\epsilon^\rho(x)\partial_\rho\left(\partial_\mu\phi^\alpha(x)\right)$$

This seems redundant to have $\epsilon^\mu(x)$ & $\lambda^\mu_{\ \nu}(x) x^\nu$ so let's call the general <u>local</u> transformation parameters

$$\xi^\mu(x) \equiv \epsilon^\mu(x) - \lambda^\mu_{\ \cdot\nu}(x) x^\nu$$

and

$$\lambda^{\mu\nu}(x)$$

<u>ξ^μ, $\lambda^{\mu\nu}$ are independent.</u> So the coordinate transformations are just the usual general coordinate transformations

$$x'^\mu = x^\mu + \xi^\mu(x) .$$

Recall the intrinsic transformations of the fields were given by

$$g(x)\phi^\alpha(x) g^{-1}(x)$$

$$= \phi^\alpha(x) + \epsilon^\mu(x)\partial_\mu\phi^\alpha(x)$$

$$+ \frac{i}{2}\lambda^{\rho\sigma}(x)\left[i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\delta^\alpha_\beta\right.$$

$$\left. - i\,(D_{\rho\sigma})^\alpha_{\ \cdot\beta}\right]\phi^\beta(x)$$

That is

$$\boxed{g(x)\,\phi^{\alpha}(x)\,g^{-1}(x) = \phi^{\alpha}(x) + \xi^{\mu}(x)\,\partial_{\mu}\phi(x) + \frac{\lambda^{\rho\sigma}(x)}{2}(D_{\rho\sigma})^{\alpha}_{\cdot\,\beta}\,\phi^{\beta}(x)}$$

Thus we see that

$$\delta x^{\mu} = \xi^{\mu}(x) \quad ; \quad \Delta\phi^{\alpha}(x) = -\frac{1}{2}\lambda^{mn}(x)(D_{mn})^{\alpha}_{\cdot\,\beta}\,\phi^{\beta}(x)$$

where now we will use Greek indices to denote general coordinate transformations (GCT) and Latin indices for local Lorentz transformations (LLT)

From the intrinsic variation point of view

$$\delta\phi^{\alpha}(x) = -\xi^{\mu}(x)\,\partial_{\mu}\phi^{\alpha}(x) - \frac{1}{2}\lambda^{mn}(x)(D_{mn})^{\alpha}_{\cdot\,\beta}\,\phi^{\beta}(x)$$

The GCT can be set to zero ($\xi^{\mu}=0$) & have just LLT $\Delta\phi^{\alpha} = \delta\phi^{\alpha} = -\frac{1}{2}\lambda^{mn}(x)(D_{mn})^{\alpha}_{\cdot\,\beta}\,\phi^{\beta}(x)$

or visaversa $\lambda^{mn}=0$; $\delta x^{\mu} = \xi^{\mu}(x)$ $\Delta\phi^{\alpha}=0$

$$\delta\phi^{\alpha} = -\xi^{\mu}(x)\,\partial_{\mu}\phi^{\alpha}(x).$$

The LLT are transformations of the tangent space basis while the GCT are world vector transformations.

As expected with local transformations ∂_μ is no longer the covariant derivative. We must introduce additional gauge fields to compensate for the differentiation of the space-time dependent parameters

The locally covariant Maurer-Cartan one-form is then given by

$$i\omega \equiv \Omega^{-1}(d + iE)\Omega = i[\omega^m P_m - \tfrac{1}{2}\omega^{mn} M_{mn}]$$

where E is a Poincaré algebra valued one-form

$$E \equiv E^m P_m - \tfrac{1}{2}\gamma^{mn} M_{mn}$$

$$= dx^\mu [E_\mu{}^m P_m - \tfrac{1}{2}\gamma_\mu{}^{mn} M_{mn}]$$

So we have introduced 40 new fields

$E_\mu^m(x)$ and $\gamma_\mu^{mn}(x)$.

Now let $$\hat{E} \equiv e^{+ix^\mu P_\mu} E \, e^{-ix^\nu P_\nu}$$

$$= \hat{E}^m P_m - \tfrac{1}{2}\hat{\gamma}^{mn} M_{mn}$$

and using BCH $\Rightarrow$

$$\hat{E}^m = E^m - \gamma^{mn}\gamma_n$$

$$\hat{\gamma}^{mn} = \gamma^{mn}$$

Now recall that $g(x)\Omega(x) = \Omega(x')h(x) \Rightarrow$

$\Omega(x') = g(x)\Omega(x)h^{-1}(x)$ So

$$i\omega'(x') = \left[\Omega^{-1}(x)(d + iE(x))\Omega(x)\right]'$$

$$= \Omega^{-1}(x')(d' + iE'(x'))\Omega(x')$$

Now to evaluate this we have

$$x'^\mu = x^\mu + \xi^\mu \Rightarrow dx'^\mu = dx^\mu + \frac{\partial \xi^\mu}{\partial x^\nu}dx^\nu$$

$$\equiv dx^\nu G_\nu{}^\mu(x),$$

a GCT $G_\nu{}^\mu(x) = \dfrac{\partial x'^\mu}{\partial x^\nu}$.

As in the global case $\partial'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu}\dfrac{\partial}{\partial x^\nu} = G^{-1}{}_\mu{}^\nu(x)\partial_\nu$

hence $d' = dx'^\mu \partial'_\mu = dx^\nu G_\nu{}^\mu(x) G^{-1}{}_\mu{}^\rho(x)\partial_\rho$

$$= dx^\mu \partial_\mu = d$$

So

$$i\omega'(x') = [g(x)\Omega(x)h^{-1}(x)]^{-1}[d + iE'(x')][g(x)\Omega(x)h^{-1}(x)]$$

$$= h\Omega^{-1}g^{-1}d(g\Omega h^{-1})$$

$$+ h\Omega^{-1}g^{-1}(iE'(x'))g\Omega h^{-1}$$

$$= h\Omega^{-1}(g^{-1}dg)\Omega h^{-1} + h(\Omega^{-1}d\Omega)h^{-1}$$

$$+ (hdh^{-1}) + h\Omega^{-1}g^{-1}(iE'(x'))g\Omega h^{-1}$$

Add & Subtract $h\Omega^{-1}iE(x)\Omega h^{-1}$

$$i\omega'(x') = h[\Omega^{-1}(x)(d + iE(x))\Omega(x)]h^{-1} + hdh^{-1}$$

$$+ ih\Omega^{-1}[g^{-1}E'(x')g - E(x) - ig^{-1}dg]$$

So define the transformation property of $E(x)$ to be

$$g^{-1}(x)E'(x')g(x) = E(x) + ig^{-1}dg$$

That is

$$\boxed{\begin{aligned} E'(x') &= g(x)E(x)g^{-1}(x) + idg(x)g^{-1}(x) \\ &= g(x)E(x)g^{-1}(x) - ig(x)dg^{-1}(x) \end{aligned}}$$

Thus $E(x)$ transforms as a gauge field under local Poincaré transformations while

$$\boxed{i\omega'(x') = h(x)\, i\omega(x)\, h^{-1}(x) + h(x)\, d h^{-1}(x)}$$

Expanding ω in terms of the Poincaré generators

$$i\omega(x) = i\left[\omega^m(x) P_m - \tfrac{1}{2}\omega^{mn}(x) M_{mn}\right]$$

$$i\omega'(x') = i\left[\omega'^m(x') P_m - \tfrac{1}{2}\omega'^{mn}(x') M_{mn}\right]$$

we find

$$i\omega'(x') = i\left[\omega^m(x)\, h(x) P_m h^{-1}(x) - \tfrac{1}{2}\omega^{mn}(x)\, h(x) M_{mn} h^{-1}(x)\right] + h\,dh^{-1}$$

$$h(x) P_m h^{-1}(x) = e^{-\frac{i}{2}\lambda^{rs}(x) M_{rs}}\, P_m\, e^{+\frac{i}{2}\lambda^{rs}(x) M_{rs}}$$

$$= P_m - \tfrac{i}{2}\lambda^{rs}(x)\left[M_{rs}, P_m\right]$$

$$= P_m - \tfrac{i}{2}\lambda^{rs}(x)\, i\left[P_r \eta_{sm} - P_s \eta_{rm}\right]$$

$$= P_m - \lambda_m{}^n P_n = (\delta_m{}^n - \lambda_m{}^n) P_n$$

$$= \Lambda_m{}^n(\lambda(x)) P_n$$ as in the global (but now local) case

and likewise

$$h(x) M_{mn} h^{-1}(x) = e^{-\frac{i}{2}\lambda^{rs}_{(x)} M_{rs}} M_{mn}\, e^{+\frac{i}{2}\lambda^{rs}_{(x)} M_{rs}}$$

$$= M_{mn} - \frac{i}{2}\lambda^{rs}_{(x)} [M_{rs}, M_{mn}]$$

$$= M_{mn} + \frac{i}{2}\lambda^{rs}_{(x)} [-i][\eta_{mr} M_{ns} - \eta_{ms} M_{nr} + \eta_{ns} M_{mr} - \eta_{nr} M_{ms}]$$

$$= M_{mn} - \lambda_m{}^r M_{rn} - \lambda_n{}^r M_{mr}$$

$$= (\delta_m{}^r \delta_n{}^s - \lambda_m{}^r \delta_n{}^s - \delta_m{}^r \lambda_n{}^s) M_{rs}$$

$$= (\delta_m{}^r - \lambda_m{}^r)(\delta_n{}^s - \lambda_n{}^s) M_{rs}$$

$$= \Lambda_m{}^r(\lambda(x)) \Lambda_n{}^s(\lambda(x)) M_{rs}$$

---

and finally

$$h\, dh^{-1} = \frac{i}{2} d\lambda^{rs}_{(x)} M_{rs} .$$

Putting this together we find the component Maurer-Cartan one-forms transform under LLT as

$$i\omega'(x') = i\Big[\omega^m(x)\,\Lambda_m{}^n(\lambda(x))\,P_n - \tfrac{1}{2}\omega^{mn}(x)\,\Lambda_m{}^r(\lambda(x))\,\Lambda_n{}^s(\lambda(x))\,M_{rs} + \tfrac{1}{2}d\lambda^{rs}(x)\,M_{rs}\Big]$$

$$= i\Big[\omega'^n(x')\,P_n - \tfrac{1}{2}\omega'^{rs}(x')\,M_{rs}\Big]$$

$\Longrightarrow$

$$\boxed{\begin{aligned} \omega'^n(x') &= \omega^m(x)\,\Lambda_m{}^n(\lambda(x)) \\ \omega'^{rs}(x') &= \omega^{mn}(x)\,\Lambda_m{}^r(\lambda(x))\,\Lambda_n{}^s(\lambda(x)) - d\lambda^{rs}(x) \end{aligned}}$$

where $\Lambda_m{}^n = \delta_m{}^n - \lambda_m{}^n(x)$ an infinitesimal LLT

So the M-C one-forms transform under LLT.

The $\omega^m(x)$ are tangent space coordinate differentials while the $\omega^{mn}(x)$ is the spin connection. Indeed the vierbein is given by the transformation matrix between the tangent space differentials and the coordinate differentials

$$\boxed{\omega^m(x) \equiv dx^\mu e_\mu{}^m(x)}$$

Hence we find the transformation property of the vierbein

$$\omega'^m(x') = dx'^\mu e'_\mu{}^m(x')$$

$$= dx^\nu G_\nu{}^\mu(x) e'_\mu{}^m(x')$$

but

$$\omega'^m(x') = \omega^n(x) \Lambda_n{}^m(x)$$

$$= dx^\nu e_\nu{}^n(x) \Lambda_n{}^m(x)$$

So

$$G_\nu{}^\mu(x) e'_\mu{}^m(x') = e_\nu{}^n(x) \Lambda_n{}^m(x)$$

$$\Rightarrow \quad \boxed{e'_\mu{}^m(x') = G^{-1}{}_\mu{}^\nu(x) e_\nu{}^n(x) \Lambda_n{}^m(x)}$$

The spin connection one-form can be expanded in terms of the coordinate differentials as well

$$\omega^{mn}(x) = dx^\mu\, \omega_\mu^{mn}(x)$$

and

$$\omega'^{mn}_\mu(x) = G^{-1}{}_{\mu\cdot}{}^{\nu}(x)\, \omega_\nu^{rs}(x)\, \Lambda_{r\cdot}{}^{m}(x)\, \Lambda_{s\cdot}{}^{n}(x) - \partial_\mu \lambda^{mn}(x)$$

where recall

$$G_{\nu\cdot}{}^{\mu}(x) = \frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu{}^\mu + \partial_\nu \xi^\mu(x)$$

and $\Lambda_{m\cdot}{}^{n}(x) = \delta_{m\cdot}{}^{n} - \lambda_{m\cdot}{}^{n}(x)$

Finally the components of the vierbein and spin connection are given by the original Maurer-Cartan one-form

$$i\omega \equiv \Omega^{-1}(d+iE)\Omega$$

$$= i\left[\omega^m(x) P_m - \tfrac{1}{2}\omega^{mn}(x) M_{mn}\right]$$

$$= i\,dx^\mu\left[\omega_\mu^m(x) P_m - \tfrac{1}{2}\omega_\mu^{mn}(x) M_{mn}\right]$$

Now

$$\Omega^{-1} d\Omega = i\, dx^{\mu} \delta_{\mu}^{\ m} P_m$$

and $i\Omega^{-1} E \Omega = i e^{-ix^{\mu}P_{\mu}} \left[E^m P_m - \frac{1}{2} \gamma^{mn} M_{mn} \right] \times$

$$\times e^{+ix^{\nu}P_{\nu}}$$

$$= iE^m P_m - \frac{i}{2} \gamma^{mn} e^{-ix^{\mu}P_{\mu}} M_{mn} e^{+ix^{\lambda}P_{\lambda}}$$

$$= iE^m P_m - \frac{i}{2} \gamma^{mn} \left(M_{mn} + i x^{\lambda} [M_{mn}, P_{\lambda}] \right)$$

(higher order terms vanish since $[P_{\mu}, P_{\nu}] = 0$)

$$= iE^m P_m + \frac{1}{2} \gamma^{mn} x^{\lambda}\, i [P_m \eta_{n\lambda} - P_n \eta_{m\lambda}] - \frac{i}{2} \gamma^{mn} M_{mn}$$

$$= iE^m P_m + \frac{i}{2} \gamma^{mn} [x_n P_m - x_m P_n] - \frac{i}{2} \gamma^{mn} M_{mn}$$

$$= iE^m P_m + i \gamma^{mn} x_n P_m - \frac{i}{2} \gamma^{mn} M_{mn}$$

$$= i(E^m + \gamma^{mn} x_n) P_m - \frac{i}{2} \gamma^{mn} M_{mn}$$

And finally $\Omega^{-1} i\, dx^{\mu} E_{\mu} \Omega$

$$= i\, dx^{\mu} \left[(E_{\mu}^{\ m} + \gamma_{\mu}^{\ mn} x_n) P_m - \frac{1}{2} \gamma_{\mu}^{\ mn} M_{mn} \right]$$

Now $\omega^m = dx^\mu e_\mu{}^m$ that is $\omega_\mu{}^m = e_\mu{}^m$

and $\omega^{mn} = dx^\mu \omega_\mu{}^{mn}$, hence

$$i\omega = \Omega^{-1}(d + iE)\Omega$$

$$= i\,dx^\mu\left[\left(\delta_\mu{}^m + E_\mu{}^m + \gamma_\mu{}^{mn}x_n\right)P_m - \tfrac{1}{2}\gamma_\mu{}^{mn}M_{mn}\right]$$

so that

$$\boxed{\begin{aligned} e_\mu{}^m(x) &= \delta_\mu{}^m + E_\mu{}^m(x) + \gamma_\mu{}^{mn}x_n \\ \omega_\mu{}^{mn}(x) &= \gamma_\mu{}^{mn}(x) \end{aligned}}$$

$e_\mu{}^m$ is the vierbein, while $\gamma_\mu{}^{mn}$ is the spin connection.

They transform as

$$\boxed{\begin{aligned} e'_\mu{}^m(x') &= G^{-1}{}_\mu{}^\nu(x)\, e_\nu{}^n(x)\, \Lambda_n{}^m(x) \\ \text{and} \quad \gamma'_\mu{}^{mn}(x') &= G^{-1}{}_\mu{}^\nu(x)\, \gamma_\nu{}^{rs}(x)\, \Lambda_r{}^m(x)\, \Lambda_s{}^n(x) \\ &\quad + \underbrace{\eta^{rs} \Lambda^{-1}{}_r{}^m(x)\, \partial_\mu \Lambda_s{}^n(x)}_{= -\partial_\mu \lambda^{mn}(x)} \end{aligned}}$$

Recall we introduced the fields

$\hat{E} = \Omega^{-1} E \Omega$ which from above

$$= dx^\mu \left[(E_\mu{}^m + \gamma_\mu{}^{mn} x_n) P_m - \tfrac{1}{2} \gamma_\mu{}^{mn} M_{mn} \right]$$

As we found

$$\hat{E}_\mu{}^m = E_\mu{}^m + \gamma_\mu{}^{mn} x_n$$

$$\hat{\gamma}_\mu{}^{mn} = \gamma_\mu{}^{mn}$$

So

$$\boxed{e_\mu{}^m(x) = \delta_\mu{}^m + \hat{E}_\mu{}^m(x)}$$

$\hat{E}_\mu{}^m$ will be the gravitational fluctuations about the Minkowski flat metric.

Also $e_\mu{}^m$ transforms homogeneously so that

$$ds^2 = \omega^m \eta_{mn} \omega^n \text{ is invariant}$$
$$= \omega'^m \eta_{mn} \omega'^n = ds'^2$$

since $\omega'^m = \omega^n \Lambda_n{}^m$ so

$$ds'^2 = \omega^r \Lambda_r{}^m \eta_{mn} \omega^s \Lambda_s{}^n$$
$$= \omega^r \underbrace{[\Lambda \eta \Lambda^T]_{rs}}_{= \eta_{rs}} \omega^s$$
$$= \omega^r \eta_{rs} \omega^s = ds^2$$

That yields the metric

$$ds^2 = \omega^m \eta_{mn} \omega^n = dx^\mu e_\mu{}^m \eta_{mn} e_\nu{}^n dx^\nu$$
$$\equiv dx^\mu g_{\mu\nu} dx^\nu$$

$$\Rightarrow \quad \boxed{g_{\mu\nu}(x) = e_\mu{}^m(x)\,\eta_{mn}\,e_\nu{}^n(x)}$$

and

$$g'_{\mu\nu}(x') = G^{-1}{}_\mu{}^\rho(x)\,G^{-1}{}_\nu{}^\sigma(x)\,g_{\rho\sigma}(x)$$
$$= (\delta_\mu{}^\rho - \partial_\mu\xi^\rho)(\delta_\nu{}^\sigma - \partial_\nu\xi^\sigma)\,g_{\rho\sigma}$$
$$= g_{\mu\nu}(x) - (\partial_\mu\xi^\rho\,g_{\rho\nu} + \partial_\nu\xi^\rho\,g_{\mu\rho})$$

as required of the metric tensor.

Also we see in general that an invariant action will be made from a Lagrangian density $\mathcal{L}$ det e

$$\Gamma \equiv \int d^4x\,(\det e)\,\mathcal{L}$$

Since

$$\Gamma' = \int d^4x'\,\det e'\,\mathcal{L}'(x')$$

but $d^4x' = \det G\, d^4x$ and

$$\det e' = \det(G^{-1}e\Lambda) = \det G^{-1}\det e\det\Lambda$$

The $\det \Lambda = 1$ for a Lorentz transformation

i.e.

$$\det \Lambda = \det(1-\lambda) = 1 - \mathrm{Tr}\,\lambda = 1 - \underbrace{\lambda^{\mu}{}_{\mu}}_{=0}$$

$$= 1$$

since $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$

So

$$\Gamma' = \int d^4x \det G \det G^{-1} \det e\, \mathcal{L}'(x')$$

$$= \int d^4x \det e\, \mathcal{L}'(x')$$

$$\equiv \Gamma = \int d^4x \det e\, \mathcal{L}(x)$$

Thus $\boxed{\mathcal{L}'(x') = \mathcal{L}(x) \text{ must be a scalar for the action to be invariant.}}$

Certainly we have, at this point,

$$\Gamma = \int d^4x\, \Lambda \det e$$

↑ cosmological constant

but there are more invariants to be made.

Finally just a consistency check. The defining transformation of $E(x)$ was

$$E'(x') = g(x)E(x)g^{-1}(x) - ig(x)dg^{-1}(x)$$

which should agree with the transformations of the vierbein & spin connection we obtained as a result of the above.

Recall $g(x) = e^{i\epsilon^\mu(x)P_\mu} e^{-\frac{i}{2}\lambda^{mn}(x)M_{mn}}$

and $$E = \left[E^m P_m - \tfrac{1}{2}\gamma^{mn}M_{mn}\right]$$

$$E' = \left[E'^m_{(x')}P_m - \tfrac{1}{2}\gamma'^{mn}_{(x')}M_{mn}\right]$$

So applying BCH $\Rightarrow$

$$E'^m(x') = E^n(x)\Lambda_n{}^m{}_{(x)} - (d\delta^m_{\ n} + \gamma^m{}_{n}(x))\epsilon^n_{(x)}$$

$$\gamma'^{mn}(x') = \gamma^{rs}_{(x)}\Lambda_r{}^m(x)\Lambda_s{}^n(x) - d\lambda^{mn}_{(x)}$$

On the other hand we found that

$$e_\mu{}^m = \delta_\mu{}^m + E_\mu{}^m + \gamma_\mu{}^{mn}x_n$$

Now the variation of γ^{mn} is the same

$$\gamma'^{mn}_{\mu}(x') = G^{-1}{}_{\mu}{}^{\nu}\,\gamma_{\nu}^{rs}(x)\Lambda_r{}^m(x)\Lambda_s{}^n(x) - \partial_\mu \lambda^{mn}(x)$$

as we found for $\omega'^{mn}_{\mu}(x')$ on page -404-

From above we have

$$e'^{m}_{\mu}(x') = \delta_\mu{}^m + E'^{m}_{\mu}(x') + \gamma'^{mn}_{\mu}(x')\,x'_n$$

So

$$E'^{m}_{\mu}(x') = e'^{m}_{\mu}(x') - \delta_\mu{}^m - \gamma'^{mn}_{\mu}(x')\,x'_n$$

From previously

$$= G^{-1}{}_{\mu}{}^{\nu}(x)\,e_\nu{}^n(x)\Lambda_n{}^m(x) - \delta_\mu{}^m$$

$$- G^{-1}{}_{\mu}{}^{\nu}\,\gamma_\nu^{rs}(x)\Lambda_r{}^m\Lambda_s{}^n\,x'_n$$

$$+ \partial_\mu \lambda^{mn} x'_n$$

recall $x'^m = x^m + \underbrace{\epsilon^m(x) + \lambda^{mn}(x)\,x_n}_{=\xi^m(x)}$

So

$$E'_\mu{}^m(x') = G^{-1}{}_\mu{}^\nu(x)\left[\delta_\nu{}^n + E_\nu{}^n(x) + \gamma_\nu{}^{ns} x_s\right]\Lambda_n{}^m(x)$$

$$- \delta_\mu{}^m - G^{-1}{}_\mu{}^\nu \gamma_\nu{}^{rs} \Lambda_r{}^m \Lambda_s{}^n x_n$$

$$- \gamma_\mu{}^{mn}(\epsilon_n + \lambda_{nr} x^r) + \partial_\mu \lambda^{mn} x_n$$

$$= G^{-1}{}_\mu{}^\nu E_\nu{}^n \Lambda_n{}^m + G^{-1}{}_\mu{}^n \Lambda_n{}^m - \delta_\mu{}^m$$

$$\cancel{+ G^{-1}{}_\mu{}^\nu \gamma_\nu{}^{ns} x_s \Lambda_n{}^m}$$

$$\cancel{- G^{-1}{}_\mu{}^\nu \gamma_\nu{}^{rs} \Lambda_r{}^m \Lambda_s{}^n (\Lambda^{-1}{}_n{}^p x_p)}$$

$$- \gamma_\mu{}^{mn} \epsilon_n + \partial_\mu \lambda^{mn} x_n$$

$$= G^{-1}{}_\mu{}^\nu E_\nu{}^n \Lambda_n{}^m + \cancel{\delta_\mu{}^m} - \lambda_\mu{}^m$$

$$- \partial_\mu \xi^m - \cancel{\delta_\mu{}^m} - \gamma_\mu{}^{mn} \epsilon_n + \partial_\mu \lambda^{mn} x_n$$

$$= G^{-1}{}_\mu{}^\nu E_\nu{}^n \Lambda_n{}^m - \cancel{\lambda_\mu{}^m} - \partial_\mu \epsilon^m - \cancel{\partial_\mu(\lambda^{mn} x_n)}$$

$$- \gamma_\mu{}^{mn} \epsilon_n + \cancel{(\partial_\mu \lambda^{mn})} x_n$$

$$\boxed{E'_\mu{}^m(x') = G^{-1}{}_\mu{}^\nu E_\nu{}^n \Lambda_n{}^m - \partial_\mu \epsilon^m - \gamma_\mu{}^{mn} \epsilon_n} \checkmark$$

which indeed agrees with p.-411-.

Now the Maurer-Cartan one-forms are the building blocks of invariant actions

The one-form has the nature of a gauge field under LLT

$$i\omega \equiv \Omega^{-1}(d+iE)\Omega = i\Omega^{-1}E\Omega + \Omega^{-1}d\Omega$$

that is $\boxed{\omega = \Omega^{-1}E\Omega - i\Omega^{-1}d\Omega}$

To find homogeneously transforming quantities we can make the field strength 2 form

$$F \equiv d\omega + i\omega\wedge\omega$$

$$= \Omega^{-1}(dE + iE\wedge E)\Omega$$

and recall $E'(x') = gEg^{-1} - igdg^{-1}$

So likewise $\mathcal{F} \equiv dE + iE\wedge E$ transforms homogeneously $\mathcal{F}'(x') = g\mathcal{F}g^{-1}$ and so

$$F'(x') = \Omega^{-1}(x')g(x)\mathcal{F}(x)g^{-1}(x)\Omega(x')$$

Recall $g\,\Omega(x) = \Omega(x')\,h(x)$ so

$$g^{-1}(x)\,\Omega(x') = \Omega(x)\,h^{-1}(x)$$

& $$\Omega^{-1}(x')\,g(x) = h(x)\,\Omega^{-1}(x) \Rightarrow$$

$$F'(x') = h(x)\,\Omega^{-1}(x)\,\mathcal{F}(x)\,\Omega(x)\,h^{-1}(x)$$

$$\boxed{F'(x') = h(x)\,F(x)\,h^{-1}(x)}$$

where we used $F(x) = \Omega^{-1}(x)\,\mathcal{F}(x)\,\Omega(x)$

Now the components of the Poincaré algebra valued 2-form F are

$$F(x) = \left[F^m(x)\,P_m - \tfrac{1}{2} F^{mn}(x)\,M_{mn} \right]$$

$$= \tfrac{1}{2} dx^\mu \wedge dx^\nu \left[F_{\mu\nu}{}^m(x)\,P_m - \tfrac{1}{2} F_{\mu\nu}{}^{mn}(x)\,M_{mn} \right]$$

and

$$\omega = \omega^m P_m - \tfrac{1}{2}\omega^{mn} M_{mn} = dx^\mu\,\omega_\mu$$

$$= dx^\mu \left[e_\mu{}^m P_m - \tfrac{1}{2}\gamma_\mu{}^{mn} M_{mn} \right] \quad \text{(p. -406-)}$$

So $$d\omega = dx^\mu \wedge dx^\nu\,\tfrac{1}{2}\left[\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right]$$

$$\omega\wedge\omega = dx^\mu \wedge dx^\nu\,\tfrac{1}{2}\left[\omega_\mu, \omega_\nu\right]$$

So

$$F \equiv \tfrac{1}{2} dx^\mu \wedge x^\nu F_{\mu\nu} = d\omega + i\omega\wedge\omega$$

$$= \tfrac{1}{2} dx^\mu \wedge dx^\nu \left[\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i[\omega_\mu, \omega_\nu]\right]$$

$\Rightarrow$

$$\boxed{F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i[\omega_\mu, \omega_\nu]}$$

Now

$$\omega_\mu = e_\mu{}^m P_m - \tfrac{1}{2}\gamma_\mu{}^{mn} M_{mn}$$

So

$$[\omega_\mu, \omega_\nu] = [e_\mu{}^m P_m - \tfrac{1}{2}\gamma_\mu{}^{mn} M_{mn},$$

$$e_\nu{}^r P_r - \tfrac{1}{2}\gamma_\nu{}^{rs} M_{rs}]$$

$$= e_\mu{}^m e_\nu{}^r \cancelto{0}{[P_m, P_r]} + \tfrac{1}{4}\gamma_\mu{}^{mn}\gamma_\nu{}^{rs}[M_{mn}, M_{rs}]$$

$$-\tfrac{1}{2} e_\mu{}^m \gamma_\nu{}^{rs}[P_m, M_{rs}]$$

$$-\tfrac{1}{2} e_\nu{}^r \gamma_\mu{}^{mn}[M_{mn}, P_r]$$

$$= +\tfrac{i}{2} e_\mu^m \gamma_\nu^{rs}\left[P_r \eta_{sm} - P_s \eta_{rm}\right]$$

$$-\tfrac{i}{2} e_\nu^r \gamma_\mu^{mn}\left[P_m \eta_{nr} - P_n \eta_{mr}\right]$$

$$-\tfrac{i}{4}\gamma_\mu^{mn}\gamma_\nu^{rs}\left[\eta_{mr} M_{ns} - \eta_{ms} M_{nr} + \eta_{ns} M_{mr} - \eta_{nr} M_{ms}\right]$$

$$[\omega_\mu, \omega_\nu] = -\frac{i}{2} e^m_\mu \gamma_{\nu m}{}^{\cdot s} P_s + \frac{i}{2} e^m_\mu \gamma_{\nu \cdot m}^{\;r} P_r$$
$$+ \frac{i}{2} e^m_\nu \gamma_{\mu m}{}^{\cdot s} P_s - \frac{i}{2} e^m_\nu \gamma_{\mu \cdot m}^{\;r} P_r$$
$$-\frac{i}{4}\Big[\gamma_{\mu r \cdot}{}^{n} \gamma_{\nu \cdot}^{\;rm} M_{nm} - \gamma_{\mu s}{}^{\cdot n} \gamma_\nu{}^{ms} M_{nm}$$
$$+ \gamma_{\mu \cdot s}^{\;m} \gamma_\nu{}^{ns} M_{mn} - \gamma_{\mu \cdot r}^{\;m} \gamma_\nu{}^{rn} M_{mn}\Big]$$

$$[\omega_\mu, \omega_\nu] = -i\left[e^n_\mu \gamma_{\nu n}{}^{\cdot m} - e^n_\nu \gamma_{\mu n}{}^{\cdot m}\right] P_m$$
$$-\frac{i}{2}\left[\gamma_\nu{}^{mr} \gamma_{\mu r}{}^{\cdot n} - \gamma_\mu{}^{mr} \gamma_{\nu r}{}^{\cdot n}\right] M_{mn}$$

So we find

$$\boxed{\begin{aligned} F^m_{\mu\nu} &= \partial_\mu e^m_\nu - \partial_\nu e^m_\mu + e^n_\mu \gamma_{\nu n}{}^{\cdot m} - e^n_\nu \gamma_{\mu n}{}^{\cdot m} \\ F^{mn}_{\mu\nu} &= \partial_\mu \gamma_\nu{}^{mn} - \partial_\nu \gamma_\mu{}^{mn} + \gamma_\mu{}^{mr} \gamma_{\nu r}{}^{\cdot n} - \gamma_\nu{}^{mr} \gamma_{\mu r}{}^{\cdot n} \end{aligned}}$$

And recall

$$F'^m_{(x')} = F^n_{(x)} \Lambda_n{}^m$$

$$F'^{mn}_{(x')} = F^{rs}_{(x)} \Lambda_r{}^m \Lambda_s{}^n$$

In general the spin connection γ^{mn}_{μ} can be used to define local Lorentz covariant derivatives — and along with the vierbein (Einstein) world tensors as well. A m^{th} rank contravariant local Lorentz & n^{th} rank Einstein tensor is defined to transform as

$$T'^{m'_1 \cdots m'_m}_{\mu'_1 \cdots \mu'_n}(x') = G^{-1\,\mu_1}_{\mu'_1}(x) \cdots G^{-1\,\mu_n}_{\mu'_n}(x) \times$$
$$\times T^{m_1 \cdots m_m}_{\mu_1 \cdots \mu_n}(x) \times$$
$$\times \Lambda_{m_1}{}^{m'_1}(x) \cdots \Lambda_{m_m}{}^{m'_m}(x)$$

for example
$$e'^{\,m}_{\mu}(x') = G^{-1\,\nu}_{\mu}(x)\, e_{\nu}{}^{n}(x)\, \Lambda_{n}{}^{m}(x)$$
$$F'^{\,m}_{\mu\nu}(x') = G^{-1\,\mu'}_{\mu} G^{-1\,\nu'}_{\nu} F_{\mu'\nu'}{}^{m'} \Lambda_{m'}{}^{m}$$

and so on. Since the vierbein has mixed indices, it can be used to convert local Lorentz indices into world indices & vice versa.

(note: $e'^{-1\,\mu}_{m}(x') = \Lambda^{-1}{}_{m}{}^{n}\, e^{-1\,\nu}_{n}(x)\, G_{\nu}{}^{\mu}$)

Since η_{mn} is invariant under LLT

The metric tensor $g_{\mu\nu} = e_\mu{}^m \eta_{mn} e_\nu{}^n$

is a rank 2 Einstein tensor and it can be used to convert covariant Einstein tensors to contravariant and vice versa. Similarly η_{mn}, η^{mn} can convert local Lorentz tensors from contravariant to covariant and vice versa.

Now we can define the covariant derivative of a tensor using the inhomogeneously transforming spin connection

(p.-407-)

$$\gamma'^{mn}_\mu(x') = G_\mu{}^\nu \gamma_\nu{}^{rs}(x) \Lambda_r{}^m \Lambda_s{}^n + \underbrace{\Lambda^{-1}{}_r{}^m \partial_\mu \Lambda_s{}^n \eta^{rs}}_{= -\partial_\mu \lambda^{mn}}.$$

For example

$$\nabla_\rho T^{mn} \equiv \partial_\rho T^{mn} - \gamma_{\rho.r}^m T^{rn} - \gamma_{\rho.r}^n T^{mr}$$

Hence $\nabla_\rho T^{mn}$ transforms homogeneously

$$(\nabla_\rho T^{mn})' = G^{-1\sigma}_{\rho}(\nabla_\sigma T^{rs})\Lambda_r^{\ m}\Lambda_s^{\ n}$$

Converting the Lorentz index n to a world index ν using the vierbein, the covariant derivative for mixed tensors is obtained

$$\boxed{\nabla_\rho T^{m\nu} \equiv e^{-1\nu}_{n}\nabla_\rho T^{mn}}$$

$$= e^{-1\nu}_{n}\left[\partial_\rho T^{mn} - \gamma_{\rho\cdot r}^{m} T^{rn} - \gamma_{\rho\cdot r}^{n} T^{mr}\right]$$

$$= e^{-1\nu}_{n}\Big[\partial_\rho(e_\sigma^{\ n} T^{m\sigma}) - \gamma_{\rho\cdot r}^{m}(e_\sigma^{\ n} T^{r\sigma})$$

$$- \gamma_{\rho\cdot r}^{n}(e_\sigma^{\ r} T^{m\sigma})\Big]$$

$$= \partial_\rho T^{m\nu} - (e^{-1\nu}_{n}\partial_\rho e_\sigma^{\ n})T^{m\sigma}$$

$$- \gamma_{\rho\cdot r}^{m} T^{r\nu} - \gamma_{\rho\cdot r}^{n} e_\sigma^{\ r} T^{m\sigma} e^{-1\nu}_{n}$$

$$\boxed{\nabla_\rho T^{m\nu} = \partial_\rho T^{m\nu} - \gamma_{\rho\cdot r}^{m} T^{r\nu} - \Gamma^{\nu}_{\sigma\rho} T^{m\sigma}}$$

where

Affine Connection $\boxed{\Gamma^{\nu}_{\sigma\rho} \equiv e^{-1\nu}_{n}(\partial_\rho e_\sigma^{\ n} + \gamma_{\rho\cdot r}^{n} e_\sigma^{\ r})}$

Now we can apply this to η_{mn} & $g_{\mu\nu} = e_\mu{}^m e_\nu{}^n \eta_{mn}$

$$\nabla_\rho \eta^{mn} = \cancelto{0}{\partial_\rho \eta^{mn}} - \gamma^m_{\rho\cdot r}\eta^{rn} - \gamma^{n\cdot}_{\rho r}\eta^{mr}$$

$$= -\gamma_\rho{}^{mn} - \gamma_\rho{}^{nm} = 0$$

$$= e_\mu{}^m e_\nu{}^n \nabla_\rho g^{\mu\nu}$$

$$= e_\mu{}^m e_\nu{}^n \left[\partial_\rho g^{\mu\nu} + \Gamma^\mu_{\sigma\rho} g^{\sigma\nu} + \Gamma^\nu_{\sigma\rho} g^{\mu\sigma}\right]$$

$\Rightarrow$

$$\boxed{\partial_\rho g^{\mu\nu} + \Gamma^\mu_{\sigma\rho} g^{\sigma\nu} + \Gamma^\nu_{\sigma\rho} g^{\mu\sigma} = 0}$$

Assuming $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ (torsionless) the solution to this equation is

$$\boxed{\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right]}$$

Note:

$$e^{-1}{}_m{}^\rho F^m_{\mu\nu} = e^{-1}{}_m{}^\rho \partial_\mu e^m_\nu + e^{-1}{}_m{}^\rho \gamma^m_{\mu\cdot n} e_\nu{}^n$$

$$- e^{-1}{}_m{}^\rho \partial_\nu e^m_\mu - e^{-1}{}_m{}^\rho \gamma^m_{\nu\cdot n} e_\mu{}^n$$

$$= \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}$$

So $$\boxed{e^{-1\rho}_{\ m} F^{m}_{\mu\nu} \equiv F^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\mu\nu}}$$

Since $F^{m}_{\mu\nu}$ is a tensor we can covariantly constrain it to be zero $F^{m}_{\mu\nu} \equiv 0$.

$$\Rightarrow \quad \Gamma^{\rho}_{\nu\mu} = \Gamma^{\rho}_{\mu\nu} \quad \text{torsionless constraint}$$

The Riemann curvature tensor is given by

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\nu \Gamma^{\rho}_{\sigma\mu} - \partial_\mu \Gamma^{\rho}_{\sigma\nu} + \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\rho}_{\lambda\mu}$$

The Ricci tensor $R_{\mu\nu} \equiv R^{\rho}{}_{\mu\nu\rho}$

and hence the Ricci scalar curvature $(R'(x') = R(x))$

$$R = g^{\mu\nu} R_{\mu\nu}.$$

Since $$\omega_{\rho\ \cdot n}^{\ m} = e^{-1\mu}_{\ n} e_\nu^{\ m} \Gamma^{\nu}_{\mu\rho} - e^{-1\mu}_{\ n} \partial_\rho e^{m}_{\mu}$$

(i.e. definition of $\Gamma^{\nu}_{\sigma\rho}$ (p. -420-))

After some algebra it can be shown that

$$F^{mn}_{\mu\nu} = e^{-1n\sigma} e_\rho{}^m R^\rho{}_{\sigma\mu\nu} .$$

$$= e^{-1n}_{\tau} g^{\tau\sigma} e_\rho{}^m R^\rho{}_{\sigma\mu\nu}$$

Hence

$$R = g^{\mu\nu} R_{\mu\nu} = -e^{-1\mu}_{m} e^{-1\nu}_{n} F^{mn}_{\mu\nu}$$

is the scalar curvature $R'(x') = R(x)$

So the lowest order action we can make is the Einstein-Hilbert action

$$\boxed{\Gamma_E = \frac{1}{\kappa^2} \int d^4x \, \det e \left[\Lambda - \frac{1}{2} R \right]}$$

where $\kappa^2 = \frac{8\pi G_N}{c^4}$;

$$= \frac{8\pi\hbar}{M_{pl}^2 c^3} \quad ; \quad M_{pl} c^2 = 1.2 \times 10^{19} \text{ GeV}$$

For $\hbar = 1 = c$ $\quad \kappa^2 = \frac{8\pi}{M_{pl}^2}$

and recall $\det g = (\det e)^2 \det \eta = -(\det e)^2$

So $\det e = \sqrt{-\det g}$.

Now matter & gauge fields transform as (spinors) tensors, hence we can make covariant derivatives and invariant scalar combinations of those fields. For example

$$\phi'^{\alpha}(x') = \phi^{\alpha}(x) - \frac{1}{2}\lambda^{mn}(x)(D_{mn})^{\alpha}_{\cdot\beta}\phi^{\beta}(x)$$

So

$$\nabla_\rho \phi^{\alpha}(x) \equiv \partial_\rho \phi^{\alpha}(x) - \frac{1}{2}\gamma_\rho^{\ mn}(D_{mn})^{\alpha}_{\cdot\beta}\phi^{\beta}(x)$$

Then

$$(\nabla_\rho \phi^{\alpha})'(x') = G^{-1\,\sigma}_{\ \rho}\,\partial_\sigma\left[\phi^{\alpha} - \frac{1}{2}(\lambda\cdot D)^{\alpha}_{\ \beta}\phi^{\beta}\right]$$

$$-\frac{1}{2}\left[(G^{-1}\gamma\Lambda\Lambda)_\rho^{\ mn} - \partial_\rho\lambda^{mn}\right](D_{mn})^{\alpha}_{\ \beta}\times$$

$$\times\left(\phi^{\beta} - \frac{1}{2}(\lambda\cdot D)^{\beta}_{\cdot\gamma}\phi^{\gamma}\right)$$

$$= G^{-1\,\sigma}_{\ \rho}\Big[\partial_\sigma\phi^{\alpha} - \frac{1}{2}(\lambda\cdot D)^{\alpha}_{\ \beta}\partial_\sigma\phi^{\beta}$$

$$-\frac{1}{2}\gamma_\sigma^{\ rs}\Lambda_r^{\ m}\Lambda_s^{\ n}(D_{mn})^{\alpha}_{\ \beta}\phi^{\beta}$$

$$-\frac{1}{2}\gamma^{mn}_{\ \ \sigma}(D_{mn})^{\alpha}_{\ \beta}\left(-\frac{1}{2}(\lambda\cdot D)^{\beta}_{\cdot\gamma}\phi^{\gamma}\right)\Big]$$

$$\cancel{-\frac{1}{2}(\partial_\rho\lambda^{mn})(D_{mn})^{\alpha}_{\ \beta}\phi^{\beta}}$$

$$\cancel{+\frac{1}{2}\partial_\rho\lambda^{mn}(D_{mn})^{\alpha}_{\ \beta}\phi^{\beta}}$$

$$(D_\rho \phi^\alpha)'(x') = G^{-1}{}_\rho{}^\sigma \Big[\partial_\sigma \phi^\alpha - \tfrac{1}{2}(\lambda\cdot D)^\alpha{}_\beta \partial_\sigma \phi^\beta$$

$$-\tfrac{1}{2}\gamma_\sigma{}^{mn}(D_{mn})^\alpha{}_\beta \phi^\beta$$

$$-\tfrac{1}{2}\gamma_\sigma{}^{rs}(-\delta_r{}^m \lambda_s{}^n - \lambda_r{}^m \delta_s{}^n)(D_{mn})^\alpha{}_\beta \phi^\beta$$

$$+\tfrac{1}{4}\gamma^{mn}{}_\sigma \lambda^{rs}(D_{mn}D_{rs})^\alpha{}_\beta \phi^\beta \Big]$$

$$= G^{-1}{}_\rho{}^\sigma \Big[(\delta^\alpha{}_\beta - \tfrac{1}{2}(\lambda\cdot D)^\alpha{}_\beta)\partial_\sigma \phi^\beta$$

$$+(\delta^\alpha{}_\beta - \tfrac{1}{2}(\lambda\cdot D)^\alpha{}_\beta)\Big[-\tfrac{1}{2}\gamma_\sigma{}^{mn}(D_{mn})^\beta{}_\gamma \phi^\gamma\Big]$$

$$-\tfrac{1}{4}\big[(\lambda\cdot D)(\gamma_\sigma\cdot D)\phi\big]^\alpha + \tfrac{1}{4}\big[(\gamma_\sigma\cdot D\,\lambda\cdot D)\phi\big]^\alpha$$

$$+\tfrac{1}{2}(\gamma_\sigma{}^{ms}\lambda_s{}^n - \gamma_\sigma{}^{ns}\lambda_s{}^m)(D_{mn})^\alpha{}_\beta \phi^\beta \Big]$$

Now recall

$$[D_{mn}, D_{rs}] = +[\eta_{mr}D_{ns} - \eta_{ms}D_{nr}$$

$$+\eta_{ns}D_{mr} - \eta_{nr}D_{ms}]$$

So the last 3 terms cancel $\Rightarrow$

$$\boxed{(\nabla_\rho \phi^\alpha)'(x') = G^{-1}{}_\rho{}^\sigma \left(\delta^\alpha_\beta - \tfrac{1}{2}(\lambda \cdot D)^\alpha{}_\beta\right)(\nabla_\sigma \phi^\beta)(x)}$$

Hence if we make global Lorentz invariants we know how to make local Lorentz & Einstein invariants.

0) Scalar field: $D^{mn} = 0$; global invariant

$$\partial_m \phi \, \eta^{mn} \partial_n \phi \rightarrow \nabla_\mu \phi \, g^{\mu\nu} \nabla_\nu \phi$$

where $\nabla_\mu \phi = \partial_\mu \phi$

1) Spinor field (Dirac): $D^{mn} = \frac{i}{2}\sigma^{mn}$

global invariant $\bar{\psi} \gamma^m \partial_m \psi$

$$\rightarrow \bar{\psi} \gamma^m e^{-1}_m{}^\mu \nabla_\mu \psi$$

where $\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{2}\gamma_\mu{}^{mn} \frac{i}{2} \sigma_{mn} \psi$

Note: $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ where $\gamma^\mu \equiv e^{-1}_m{}^\mu \gamma^m$

2) Vector (gauge) fields: $(D^{mn})_{\alpha\beta} = -\delta^m_\alpha \delta^n_\beta + \delta^m_\beta \delta^n_\alpha$

$$\nabla_\mu A^\nu = \partial_\mu A^\nu - \Gamma^\nu_{\sigma\mu} A^\sigma$$

$$F_{A\mu}{}^\nu = \nabla_\mu A^\nu - \nabla^\nu A_\mu + [A_\mu, A^\nu]$$

The invariant Y-M term has the form

$$F_{A\mu\nu}\, g^{\mu\rho} g^{\nu\sigma} F_{A\rho\sigma}\,.$$

The details are left to the reader.

Remark: If ψ also belongs to a representation of a local internal symmetry group

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{i}{4}\, \gamma_\mu{}^{mn} \sigma_{mn} \psi - ig A_\mu^i T \psi$$

and so on.

On to supergravity.