Proton lifetime

$$\tau_p = \frac{1}{\alpha_5^2}\,\frac{M_{hc}^2}{m_p^5}\,M_{susy}^2\left[\frac{m_W}{m_s}\,\frac{m_W}{m_u}\right]^2$$

So

$$M_{hc} = \alpha_5\left[\frac{m_u m_s}{m_W m_W}\right]\frac{\sqrt{\tau_p m_p^5}}{M_{susy}}$$

Once again the latest SuperKamiokande result $\tau_p > 1.9\times10^{33}$ yrs. $\sim 6\times10^{40}$ sec.

$m_p \approx 1$ GeV ; $m_u \approx 2$ MeV ; $m_s \approx 100$ MeV

$m_W \approx 80$ GeV . From the RGE running with $M_{susy} \approx 100$ GeV we found $\alpha_5 \approx \frac{1}{24}$

So again using our conversion factor

$(1=)\ \hbar = 6.58\times10^{-25}$ GeV·sec.

$$\Rightarrow \quad 1\,\text{sec} = \frac{1}{6.58\times10^{-25}\,\text{GeV}}$$

$\Rightarrow$

$$M_{hc} \gtrsim 4\times10^{21}\,\text{GeV}$$ ! Recall

$M_{Pl} = \quad \approx 1.2\times10^{19}$ GeV — Trouble

We can still "save" SUSY SU(5) by appealing to higher dimension effective operator corrections to the action.
See for example arXiv: hep-ph/0407173.

The SUSY puts fermions & bosons on equal footing so we will apply the general 1-loop formula for $\beta$ that includes fermion & scalar loops

$$\beta = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}T_F - \frac{1}{3}T_S\right]$$

Now $C_2(SU(N)) = N$. $C_2(U(1)) = 0$ and for each super Yang-Mills field we have an ordinary Y-M field as well as a corresponding partner gaugino [this is a Weyl fermion in the adjoint representation]. So the Y-M field + gaugino contribute to $\beta$ according to

$$\frac{11}{3}C_2(G) - \frac{4}{3}\frac{1}{2}C_2(G) = 3C_2(G)$$

since $T_F = \frac{1}{2}C_2(G)$ for the adjoint representation Weyl fermions. i.e. $\delta^{ij}T_F^{adj} = \frac{1}{2}\mathrm{Tr}[T^i_{adj}T^j_{adj}]$

$$= -\frac{1}{2}f_{kil}f_{ljk}$$

$$= \frac{1}{2}C_2(G)\delta^{ij}$$

(recall $(T^i_R)_{jk}(T^i_R)_{kl} = \delta_{jl}C_2(R)$

$(T^i_R)_{kl}(T^j_R)_{lk} = T(R)\delta^{ij}$)

The each chiral superfield we have a Weyl fermion and a complex scalar in the same group representation. Therefore for each super matter and super Higgs multiplet we find a contribution to $\beta$ of

$$\left[-\frac{4}{3}\frac{1}{2}T_S - \frac{1}{3}T_S\right] = -T_S = -\frac{1}{d}\mathrm{Tr}[T^i T^i]$$

where $d$ = the dimension of the group and the trace is summed over the chiral superfields i.e. for quarks and leptons this is twice the number of flavors ($=4F$)

So for supersymmetric theories we find

$$\boxed{\begin{aligned}\beta &= -\frac{g^3}{16\pi^2}\left[3C_2(G) - \sum_{\substack{\text{chiral}\\ \text{super-}\\ \text{fields}}}' \frac{1}{d}\mathrm{Tr}[T^i T^i]\right] \\ &= -\frac{g^3}{16\pi^2}\left[3C_2(G) - \sum_{\substack{\text{chiral}\\ \text{super-}\\ \text{fields}}}' T_S\right]\end{aligned}}$$

Now let's apply this to the $SU(3)\times SU(2)\times U(1)$ MSSM

Recall we have

$$\mu \frac{d}{d\mu} \bar{g}_i(\mu) = b_i \bar{g}_i^3(\mu)$$

with $\boxed{\beta_i \equiv g_i^3 b_i .}$ So for the fine structure constant for each group

$$\boxed{\alpha_i(\mu) \equiv \frac{\bar{g}_i^2(\mu)}{4\pi}}$$

we have

$$\mu \frac{d}{d\mu} \alpha_i(\mu) = 8\pi b_i \alpha_i^2(\mu)$$

So integrating between the scales $\mu_1 \leq \mu_2$ we have as usual

$$\boxed{\frac{1}{\alpha_i(\mu_2)} = \frac{1}{\alpha_i(\mu_1)} - 8\pi b_i \ln\left(\frac{\mu_2}{\mu_1}\right)}$$

or

$$\boxed{\alpha_i(\mu_2) = \frac{\alpha_i(\mu_1)}{1 - 8\pi b_i \alpha_i(\mu_1) \ln\left(\frac{\mu_2}{\mu_1}\right)}}$$

for each $i = 1, 2, 3$.

From our SUSY $\beta$ formula we have

$$b_3 = -\frac{1}{16\pi^2}\left[3C_2(SU(3)) - \sum_{\substack{\text{quark}\\ \text{chiral}\\ \text{super-}\\ \text{fields}}} T_S\right]$$

$$\boxed{b_3 = -\frac{1}{16\pi^2}\left[9 - 2F\right]}$$

($F$ = # of families = # of generations)

i.e. For quark superfields

$$\delta^{ij} T_S = \mathrm{Tr}\left[\frac{\lambda^i}{2}\frac{\lambda^j}{2}\right] = \frac{1}{4}\mathrm{Tr}\{\lambda^i\lambda^j\}$$

$$= \frac{1}{4}\frac{1}{2}\mathrm{Tr}\left[\{\lambda^i, \lambda^j\}\right]$$

$$= \frac{1}{8}\mathrm{Tr}\left[\frac{4}{3}\delta^{ij}\mathbb{1}_{3\times3} + 2d_{ijk}\lambda^k\right]$$

$$= \delta^{ij}\frac{1}{2}$$ for each quark chiral superfield $u_F, u^c_F, d_F, d^c_F$, $F = 1,2,3$

$\Rightarrow 4 \cdot 3 = 4F$ = # of such contributions

For $g_2$:

$$b_2 = -\frac{1}{16\pi^2}\left[3C_2(SU(2)) - \sum_{\substack{\text{Doublets}\\ \text{chiral}\\ \text{superfields}}} T_S\right]$$

$$\boxed{b_2 = -\frac{1}{16\pi^2}\left[6 - 2F - \frac{1}{2}H\right]}$$

H = # of Higgs chiral superfield SU(2) Doublets

For ~~SU(2)~~ SU(2) doublets $\delta^{ij} T_S = \mathrm{Tr}\left[\frac{\sigma^i}{2}\frac{\sigma^j}{2}\right] = \frac{1}{2}\delta^{ij}$

$\Rightarrow T_S = \frac{1}{2}$ for each chiral superfield doublet

$\Rightarrow$

$$\left.\begin{array}{l} \frac{1}{2} \text{ for } Q^{a}_{F} \quad a \leftarrow 3 \text{ colours},\ F \leftarrow 3 \text{ families} \\ + \frac{1}{2} \text{ for } L_F \quad F \leftarrow 3 \text{ families} \end{array}\right\} = \frac{1}{2}\cdot(\underset{\substack{\uparrow \\ \text{3 colours of } Q^a}}{3} + \underset{\substack{\uparrow \\ \text{L or Lepton}}}{1})\cdot F = 2F$$

$$\left.\begin{array}{l} + \frac{1}{2} \text{ for } H_u \\ + \frac{1}{2} \text{ for } H_d \end{array}\right\} = \frac{1}{2}\cdot H$$

$H$: 2 doublet Higgs in MSSM

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Note: For $F = 3$; $b_2 > 0$ So $\alpha_2$ is not asymptotically free for MSSM

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Finally for $g_1$:

$$b_1 = -\frac{1}{16\pi^2}\Big[3\,\cancel{C_2}^{\to 0}(U(1)) - \underset{\substack{\text{all}\\ \text{chiral}\\ \text{superfields}}}{\sum_i{}'} Y^2\Big]$$

$Y$ = hypercharge for each chiral superfield

$$\boxed{b_1 = +\frac{1}{16\pi^2}\Big[\frac{10}{3}F + \frac{1}{2}H\Big]}$$

since $T_S\delta^{ij} = \mathrm{Tr}[T^iT^j]$ for $U(1)$ is just the charge $T^i \to U(1)$ charge

So

$$T_S = Y^2 \text{ for each chiral superfield}$$

$$\left.\begin{aligned}
u &: Y = +1/6 \times 3\,\text{colors} \times F &&= \tfrac{1}{12}F\\
d &: Y = +1/6 \times 3\,\text{colors} \times F &&= \tfrac{1}{12}F\\
U^c &: Y = -\tfrac{2}{3} \times 3\,\text{colors} \times F &&= \tfrac{4}{3}F\\
D^c &: Y = +\tfrac{1}{3} \times 3\,\text{colors} \times F &&= \tfrac{1}{3}F\\
\nu &: Y = -\tfrac{1}{2} \times F &&= \tfrac{1}{4}F\\
e &: Y = -\tfrac{1}{2} \times F &&= \tfrac{1}{4}F\\
E^c &: Y = 1 \times F &&= 1F
\end{aligned}\right\}\ \frac{10}{3}F$$

$$\left.\begin{aligned}
H_u &: Y = +\tfrac{1}{2} \times 2 \leftarrow H_u^+,\ H_u^0 &&= \tfrac{1}{2}\\
H_d &: Y = -\tfrac{1}{2} \times 2 \leftarrow H_d^0,\ H_d^- &&= \tfrac{1}{2}
\end{aligned}\right\}\ \frac{1}{2}H$$

Note that the SUSY slopes $b_i$ are less than the SM slopes $b_i^{SM}$ for $SU(3)$ & $SU(2)$ & $b_1 > b_1^{SM}$ for $U(1)$.

$$b_3^{SM} = -\frac{1}{16\pi^2}\left[11 - \frac{4}{3}F\right]$$

$$b_2^{SM} = -\frac{1}{16\pi^2}\left[\frac{22}{3} - \frac{4}{3}F - \frac{1}{6}H\right].$$

$$b_1^{SM} = +\frac{1}{16\pi^2}\left[\frac{20}{9}F + \frac{1}{6}H\right]$$

---

So $\alpha_{3,2}$ will run more slowly but $\alpha_1$ more quickly. Again we will normalize the $U(1)$ coupling constant to its GUT normalization

$$g_1(GUT) = \sqrt{\frac{5}{3}}\, g_1 \Rightarrow \alpha_1(GUT) = \frac{5}{3}\alpha_1$$

So $b_1(GUT) = \frac{3}{5} b_1$.

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Now we have our measured values of $\alpha_{1,2,3}$ at the $\mu_1 = M_Z$ scale. At this "low" energy SUSY is already broken. Hence we can evolve the coupling constants according to the SM $\beta$-functions until ~~we~~ reach the SUSY breaking scale $M_{SUSY}$. After which the SUSY partners propagate and are no longer decoupled from the running of the coupling constants. For simplicity we

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will assume all the susy partner's effects occur at the same scale $M_{susy}$. Hence above $M_{susy}$ the couplings evolve according to the susy $\beta$-functions.

Introducing the running parameter $t$

$$\mu_1 = M_Z \; ; \; \mu = e^t M_Z \; ; \; \mu_{susy} = e^{t_{susy}} M_Z \equiv M_{susy}$$

So $\ln\left(\frac{\mu}{\mu_1}\right) = t$ and as usual $\mu \frac{d}{d\mu} = \frac{d}{dt}$

Hence running from $M_Z \to M_{susy}$ with the SM $\beta$'s $\Rightarrow$

$$\frac{1}{\alpha_i(t_{susy})} = \frac{1}{\alpha_i(M_Z)} - 8\pi b_i^{SM} t_{susy}$$

This gives us the new initial conditions at $M_{susy}$ to integrate up to high energy

$$\boxed{\begin{aligned} \frac{1}{\alpha_i(t)} &= \frac{1}{\alpha_i(t_{susy})} - 8\pi b_i (t - t_{susy}) \\ &= \frac{1}{\alpha_i(M_Z)} - 8\pi b_i t - 8\pi b_i^{SM} t_{susy} \\ &\qquad + 8\pi b_i t_{susy} \end{aligned}}$$

Using GUT normalization for the U(1) coupling constant $\quad g_1(GUT) = \sqrt{\frac{5}{3}}\, g_1 \;;\; \alpha_1(GUT) = \frac{5}{3}\alpha_1$

and $b_1(GUT) = \frac{3}{5} b_1$, we can evolve the coupling constants from the initial values at $M_Z \simeq 91\,\text{GeV}$

$$\begin{aligned} \alpha_3(M_Z) &= 0.1176 \qquad & M_Z = 91.1874 \\ \alpha_2(M_Z) &= 0.0336 & \text{GeV} \\ \alpha_1(GUT)(M_Z) &= \tfrac{5}{3}(0.0102) & (t_{M_Z} = 0) \end{aligned}$$

So

$$\frac{1}{\alpha_i(t)} = \begin{cases} \dfrac{1}{\alpha_i(M_Z)} - 8\pi b_i^{SM} t \,, & 0 \le t \le t_{susy} \\[2ex] \dfrac{1}{\alpha_i(t_{susy})} - 8\pi b_i (t - t_{susy}) \,, & t_{susy} \le t \end{cases}$$

where we use the $\alpha_1(GUT)$ & $b_1(GUT)$.

Suppose $M_{susy} = 200\,\text{GeV} \Rightarrow t_{susy} = \ln\left(\frac{M_{susy}}{M_Z}\right)$

and so on.

As seen in the graph — the SU(3), SU(2) and U(1) coupling constants unify at about $M_{GUT} = 2 \times 10^{16}\,\text{GeV}$ for $M_{susy} \sim 100 - 1000\,\text{GeV}$!

**Running of the MSSM Gauge Coupling Constants**

Now if the SU(3)XSU(2)XU(1) groups are embedded in a SU(5) or SO(10) GUT it is conventional to normalize the U(1) coupling to be $g_1(\mathrm{GUT}) = (5/3)^{(1/2)}\, g_{1.}$

Initial values of the fine structure constants are given at $M_Z := 91.1874$ GeV

There the inverse fine structure constants for U(1), SU(2) and SU(3) gauge coupling constants are

$$\alpha\mathrm{inverse}M_Z := \begin{pmatrix} \dfrac{\frac{3}{5}}{0.0102} \\ \dfrac{1}{0.0336} \\ \dfrac{1}{0.1176} \end{pmatrix}$$

The coefficients for their $\beta$ functions are given by the SM values below MSUSY

$$\mathrm{bSM} := \begin{pmatrix} \dfrac{41}{10} \\ \dfrac{-19}{6} \\ -7 \end{pmatrix} \cdot \frac{1}{16\cdot\pi^2}$$

After SUSY is broken the $\beta$ function coefficients are given by

$$\mathrm{bMSSM} := \begin{pmatrix} \dfrac{33}{5} \\ 1 \\ -3 \end{pmatrix} \cdot \frac{1}{16\cdot\pi^2}$$

The SUSY breaking scale is

$$\mathrm{MSUSY} := 100 \quad \mathrm{GeV}$$

Hence the running parameter tSUSY is

$$\mathrm{tSUSY} := \ln\left(\frac{\mathrm{MSUSY}}{M_Z}\right)$$

The renormalization group running of the inverse fine structure constants is described by the RGE equation

$$\alpha\mathrm{inverse}(t) := \begin{cases} \overrightarrow{\left(\overrightarrow{\alpha\mathrm{inverse}M_Z} - \overrightarrow{\left(8\cdot\pi\cdot\overrightarrow{\mathrm{bSM}\cdot t}\right)}\right)} & \text{if } t \le \mathrm{tSUSY} \\ \overrightarrow{\left(\overrightarrow{\left(\overrightarrow{\left(\overrightarrow{\alpha\mathrm{inverse}M_Z} - \overrightarrow{\left[8\cdot\pi\cdot\left(\overrightarrow{\mathrm{bSM}} - \overrightarrow{\mathrm{bMSSM}}\right)\cdot\mathrm{tSUSY}\right]}\right)} - \overrightarrow{\left(8\cdot\pi\cdot\overrightarrow{\mathrm{bMSSM}\cdot t}\right)}\right)}\right)} & \text{if } t \ge \mathrm{tSUSY} \end{cases}$$

where the energy scale of the effective coupling constant is given by

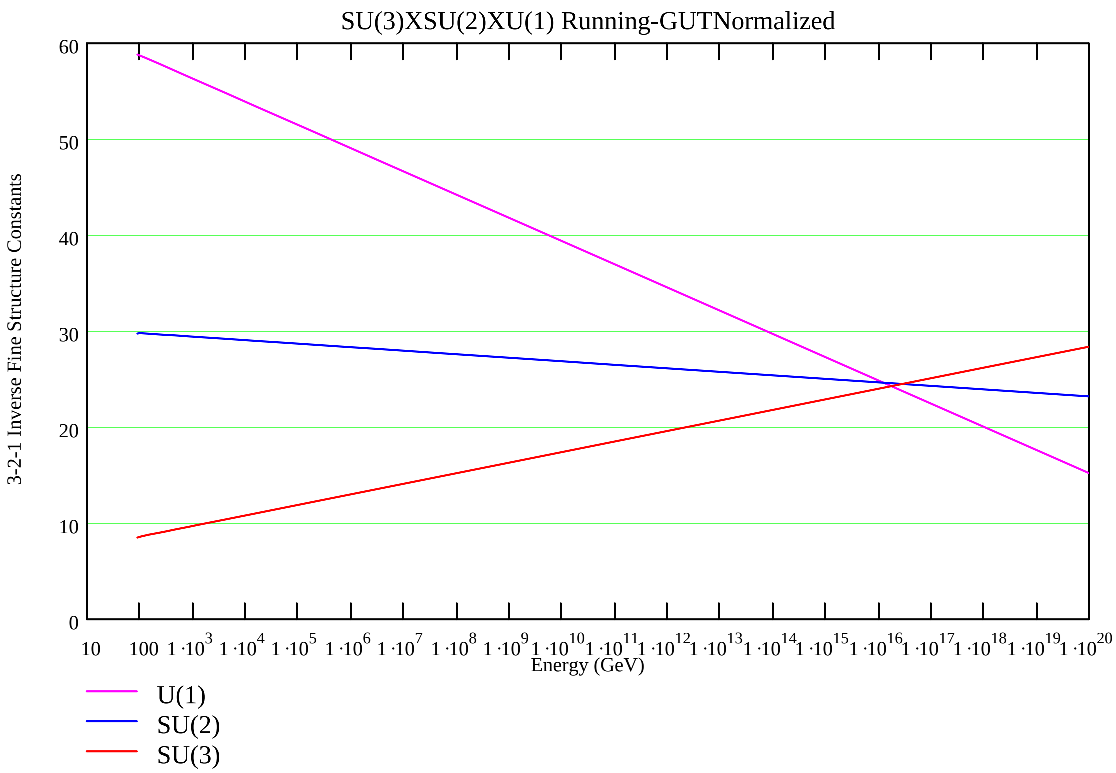
$$t := 0, 0.1 .. 42$$

$$Q(t) := M_Z\cdot e^t$$

$tSUSY = 0.092$

$Q(tSUSY) = 100$

The RGE running is now given by



$$\mathrm{MSUSY2} := 1000 \qquad \mathrm{tSUSY2} := \ln\left(\frac{\mathrm{MSUSY2}}{M_Z}\right)$$

$$\mathrm{tSUSY2} = 2.395$$

$$Q(\mathrm{tSUSY2}) = 1 \times 10^{3}$$

$$\alpha\mathrm{inverse}(t) := \begin{cases} \overrightarrow{\left(\overrightarrow{\alpha\mathrm{inverseM}_Z} - \overrightarrow{\left(8\cdot\pi\cdot\overrightarrow{\mathrm{bSM}}\cdot t\right)}\right)} & \text{if } t \le \mathrm{tSUSY2} \\ \overrightarrow{\left(\overrightarrow{\left(\overrightarrow{\left(\overrightarrow{\alpha\mathrm{inverseM}_Z} - \overrightarrow{\left[8\cdot\pi\cdot\left(\overrightarrow{\mathrm{bSM}} - \overrightarrow{\mathrm{bMSSM}}\right)\cdot\mathrm{tSUSY2}\right]}\right)} - \overrightarrow{\left(8\cdot\pi\cdot\overrightarrow{\mathrm{bMSSM}}\cdot t\right)}\right)}\right)} & \text{if } t \ge \mathrm{tSUSY2} \end{cases}$$