CH.2: Grand Unified Theories: The Georgi-Glashow SU(5) Model.

2.0. Introduction

Although this standard model describes all known facts of elementary particle physics it leaves unanswered many questions; that is there is "too much" arbitrariness in the model. For example:

1) Why a product of 3 groups? Why 3 families?

2) Why is $\alpha_{QCD} \simeq .3$; $\alpha_{SU(2)} \simeq .03$; $\alpha_{e-m} \simeq .007$ at $Q^2 \simeq 10\ GeV^2$; and $\sin^2\theta_W = \alpha_{e-m}/\alpha_{SU(2)}$?

3) Electric charge is not quantized. That is Y can be arbitrarily assigned; independently for each representation. Why $\pm$ 1, $\pm$ 1/3, $\pm$ 2/3 etc?

We hope to answer some of these questions by considering the standard model to be contained in a large single group.

2.1. The particle representations and the symmetric SU(5) Lagrangian

The Georgi-Glashow model is based on the group SU(5) which of course contains the SU(3) x SU(2) x U(1) group of the standard model. The $5^2 - 1 = 24$ generators of SU(5), T^i; i = 1...,24, obey the algebra

$$[T^i, T^j] = if_{ijk}T^k \tag{2.1.1}$$

It is quite tedious to list these f_{ijk} structure constants and it is simpler and equivalent to describe the group properties in terms of another basis of generators T^a_b: a, b = 1,...,5 with $T^a_a = 0$, with simpler commutation relations

$$[T^a_b, T^c_d] = \delta^a_d\, T^c_b - \delta^c_b\, T^a_d \tag{2.1.2}$$

where

$$T^a_b = S^{-1a}_{bi}\, T^i \tag{2.1.3}$$

and

$$T^i = S^{ib}_a\, T^a_b \tag{2.1.4}$$

$$= S^{ib}_a\, S^{-1a}_{bj} T^j$$

$$= \delta^i_j\, T^j$$

We will determine S for SU(2) and SU(3) to get the idea.

Next, we digress for a while to re-express our fundamental representation basis matrices of SU(2), SU(3) and now SU(5) generators in a more convenient basis. In general we define the (N^2-1) N x N traceless matrices

$$(T^a_b)^c_d \equiv \delta^c_b\delta^a_d - \frac{1}{N}\,\delta^a_b\delta^c_d\ ; \quad a,b,c,d = 1,\ldots,N \tag{2.1.5}$$

Note: there are $N^2 - 1$ matrices T^a_b since

$$T^a_a = 0 \quad . \tag{2.1.6}$$

The $N^2 - 1$ independent matrices T^a_b can be used as a basis for the SU(N) fundamental representation. For $a \neq b$ T^a_b is non-hermitean with

$$(T^a_b)^c_d = \delta^c_b \delta^a_d \quad . \tag{2.1.7}$$

Since $(T^a_b)^\dagger = (T^a_b)^T = T^b_a$ we can relate these T^a_b to our usual hermitean fundamental representation matrices

$$T^i (= T^{i\dagger}), \; i = 1,\ldots,N^2 - 1 \tag{2.1.8}$$

For each $a \neq b$, $a,b = 1,\ldots,N$ we have

$$\frac{1}{2}(T^a_b + T^b_a) \qquad \frac{N(N-1)}{2} \text{ hermitean matrices}$$

and

$$\frac{i}{2}(T^a_b - T^b_a) \qquad \frac{N(N-1)}{2} \text{ hermitean matrices} \quad . \tag{2.1.9}$$

These are non-diagonal. The remaining N-1 diagonal matrices of SU(N) (SU(N) is rank N-1) are given by the hermitean T^a_a (no summation over a); since $\sum_a T^a_a = 0$ only N-1 of these are independent. There is no generally accepted convention for identifying the T^i; $i = 1,\ldots,N^2-1$ with particular matrices above. To be concrete let's consider the SU(2) and SU(3) case where we used the Pauli matrices and Gell-Mann matrices for the fundamental representations. For SU(2): $T^a_b = \delta^c_b \delta^a_d - \frac{1}{2}\delta^a_b \delta^c_d$ a, b, c, d, = 1,2

that is $$T^1_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad ; \quad T^2_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(note for $(T^a_b)^c_d$ c labels rows and d labels columns by our conventions)

$$T^1_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad ; \quad T^2_2 = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (2.1.10)$$

Thus we see

$$T^1 = \frac{1}{2}\,\sigma^1 = \frac{1}{2}(T^1_2 + T^2_1), \quad T^1_2 = T^1 - iT^2$$

$$T^2 = \frac{1}{2}\,\sigma^2 = +\frac{i}{2}(T^1_2 - T^2_1), \; T^2_1 = T^1 + iT^2 \qquad (2.1.11)$$

$$T^3 = \frac{1}{2}\,\sigma^3 = T^1_1 = -T^2_2 \qquad = \frac{1}{2}(T^1_1 - T^2_2)$$

The SU(2) commutation relations can also be derived from the T^a_b commutation relations

$$[T^a_b, T^c_d]^e_g = (T^a_b)^e_f\,(T^c_d)^f_g - (T^c_d)^e_f\,(T^a_b)^f_g \qquad (2.1.12)$$

$$= [\delta^e_b\,\delta^a_f - \frac{1}{N}\,\delta^a_b\delta^e_f][\delta^f_d\delta^c_g - \frac{1}{N}\,\delta^c_d\delta^f_g] - (a \leftrightarrow c;\; d \leftrightarrow b)$$

$$= \delta^a_d[\delta^e_b\delta^c_g - \frac{1}{N}\,\delta^c_b\delta^e_g] - \delta^c_b[\delta^e_d\delta^a_g - \frac{1}{N}\delta^a_d\delta^e_g]$$

$$= \delta^a_d(T^c_b)^e_g - \delta^c_b(T^a_d)^e_g$$

yielding $$[T^a_b, T^c_d] = \delta^a_d T^c_b - \delta^c_b T^a_d] \qquad (2.1.13)$$

Thus for example in our SU(2) case

$$[T^1, T^2] = \frac{i}{4}[T^1_2 + T^2_1 , T^1_2 - T^2_1]$$

$$= - \frac{i}{2}[T^1_2, T^2_1] \qquad (2.1.14)$$

$$= - \frac{i}{2}[\delta^1_1 T^2_2 - \delta^2_2 T^1_1]$$

$$= + i(\frac{1}{2})(T^1_1 - T^2_2) = +i T^3 .$$

Thus we see that the SU(N) group is equivalently defined through the generators obeying the commutation relations

$$[T^a_b, T^c_d] = \delta^a_d T^c_b - \delta^c_b T^a_d . \qquad (2.1.15)$$

This is just a re-labeling of the generators. The fundamental representation is given by

$$(T^a_b)^c_d = \delta^c_b \delta^a_d - \frac{1}{N} \delta^a_b \delta^c_d \qquad (2.1.16)$$

above, any N x N SU(N) matrix can be made from this fundamental representation

$$U(\underline{\underline{\omega}}) = e^{+ig\omega^b_a T^a_b} \qquad (2.1.17)$$

where $\omega^b_a T^a_b = \omega^i T^i$ and $T^i = S^{ib}_a T^a_b$ and $(\omega^b_a T^a_b)^c_d = \omega^c_d$ and $\omega^b_a = \omega^i S^{ib}_a$.

where now the angles of rotation are given by the parameter matrix ω^b_a with $\omega^a_a = 0$. We can perform a similar analysis for SU(3)

$$T^1_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \qquad T^2_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;$$

$$T^1_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \; ; \qquad T^3_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ;$$

$$T^2_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \; ; \qquad T^3_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.1.18)$$

$$T^1_1 = \frac{1}{3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad T^2_2 = \frac{1}{3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \; ;$$

$$T^3_3 = \frac{1}{3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad .$$

So

$$\begin{aligned}
T^1 &= \tfrac{1}{2}\lambda^1 = \tfrac{1}{2}(T^1_2 + T^2_1) \; ; \qquad & T^2 &= \tfrac{1}{2}\lambda^2 = \tfrac{i}{2}(T^1_2 - T^2_1) \\
T^4 &= \tfrac{1}{2}\lambda^4 = \tfrac{1}{2}(T^1_3 + T^3_1) \; ; \qquad & T^5 &= \tfrac{1}{2}\lambda^5 = \tfrac{i}{2}(T^1_3 - T^3_1) \qquad (2.1.19) \\
T^6 &= \tfrac{1}{2}\lambda^6 = \tfrac{1}{2}(T^2_3 + T^3_2) \; ; \qquad & T^7 &= \tfrac{1}{2}\lambda^7 = \tfrac{i}{2}(T^2_3 - T^3_2) \\
T^3 &= \tfrac{1}{2}\lambda^3 = \tfrac{1}{2}(T^1_1 - T^2_2) \; ; \qquad & T^8 &= \tfrac{1}{2}\lambda^8 = -\sqrt{3}\; T^3_3 \quad .
\end{aligned}$$

Thus we can define a field which transforms according to the fundamental representation by

ψ^c; c = 1,...,N; the N-dimensional representation of SU(N) such that

$$\psi'^c = U^c(\underline{\omega})_d \psi^d$$
$$= \psi^c + ig(\omega^b_a T^a_b)^c_d \psi^d$$
$$(= \psi^c + ig(\omega^i T^i)_{cd} \psi^d) \tag{2.1.20}$$
$$= \psi^c + ig\omega^b_a \delta^c_b \delta^a_d \psi^d = \psi^c + ig\omega^c_d \psi^d \ .$$

as before, ψ^c is a contravariant vector, a (1,0) tensor.

The hermitean conjugate field transforms according to the N-dimensional representation called $\bar{N}$ of SU(N).

$$[\psi^\dagger]_c = [\psi^c]^\dagger \equiv \phi_c$$

$$\phi'_c = \phi_c - ig[\omega^b_a]^* \phi_d [T^a_b]^{c*}_d \tag{2.1.21}$$
$$= \phi_c - ig[\omega^b_a]^* [T^{a\dagger}_b]^d_c \phi_d \ .$$

Since $T^{a\dagger}_b = T^b_a$ and

$$\omega^b_a = \omega_i S^{ib}_a \rightarrow [\omega^b_a]^* = \omega^a_b$$

we have

$$\phi'_c = \phi_c - ig\omega^a_b [T^b_a]^d_c \phi_d = \phi_c - ig\,\omega^d_c\,\phi_d \tag{2.1.22}$$

ϕ_c is a covariant vector, a(0,1) tensor. So, for finite ω

$$\phi'_c = \phi_d U^\dagger(\underline{\omega})^d_c$$
$$= [e^{-ig\omega^b_a T^a_b}]^d_c \phi_d \ . \tag{2.1.23}$$

We can build higher dimensional representations of SU(N) from direct products of N and $\bar{N}$. For instance the $[N^2 - 1]$ dimensional adjoint representation is built from the product of $N \times \bar{N}$ by removing the trace. Let A^a_b be the field in the adjoint rep. with $A^a_a = 0$. We can write A as

$$A^a_b = \psi^a\phi_b - \frac{1}{N}\delta^a_b\psi^c\phi_c \quad . \tag{2.1.24}$$

Under a group transformation

$$\begin{aligned} A^{a'}_b &= \psi^{a'}\phi'_b - \frac{1}{N}\delta^a_b\psi^{c'}\phi'_c \\ &= U^a_c\psi^c\phi_d U^{\dagger d}_b - \frac{1}{N}\delta^a_b\, U^c_d\psi^d\phi_e U^{\dagger e}_c \\ &= U^a_c A^c_d U^{\dagger d}_b + \frac{1}{N}\delta^c_d U^a_c\psi^e\phi_e U^{\dagger d}_b \\ &\qquad - \frac{1}{N}\delta^a_b U^c_d\psi^d\phi_e U^{\dagger e}_c \\ &= U^a_c A^c_d U^{\dagger d}_b + \frac{1}{N}\delta^a_b\psi^c\phi_c - \frac{1}{N}\delta^a_b\psi^c\phi_c \end{aligned} \tag{2.1.25}$$

So

$$A^{a'}_b = U^a_c A^c_d U^{\dagger d}_b$$ just the homogeneous part of the

Y-M field transformation, for infinitesimal $\underline{\underline{\omega}}$

$$\begin{aligned} A^{a'}_b &= (\delta^a_c + ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^a_c)A^c_d(\delta^d_b - ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^d_b) \\ A^{a'}_b &= A^a_b + ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^a_c A^c_b - ig(\underline{\underline{\omega}}\cdot\underline{T})^d_b A^a_d \end{aligned} \tag{2.1.26}$$

$$= A^a_b + ig\,\omega^a_c A^c_b - ig\omega^d_b A^a_d$$

that is A^a_b is a (1,1) tensor.

Thus we see that this is a convenient notation for our Yang-Mills fields; for the fundamental representation matrices T we can form

$$\sum_{i=1}^{N^2-1} T^i A^i_\mu = \frac{1}{\sqrt{2}} \sum_{a,b=1}^{N} T^a_b A_{\mu a}^{\ b} \equiv \frac{1}{\sqrt{2}} A_\mu \quad \text{where} \tag{2.1.27}$$

A_μ on the RHS is a N x N matrix.

For example in the SU(2) case

$$T^i A^i_\mu = \frac{1}{2}\sigma^i A^i_\mu = \begin{bmatrix} \frac{1}{2}A^3_\mu & \frac{1}{2}(A^1_\mu - iA^2_\mu) \\ \frac{1}{2}(A^1_\mu + iA^2_\mu) & -\frac{1}{2}A^3_\mu \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}}A^3_\mu & W^+_\mu \\ W^-_\mu & -\frac{1}{\sqrt{2}}A^3_\mu \end{bmatrix} \tag{2.1.28}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} A^1_{\mu 1} & A^1_{\mu 2} \\ A^2_{\mu 1} & A^2_{\mu 2} \end{bmatrix} = \frac{1}{\sqrt{2}} A_\mu$$

and for SU(3) we find

$$A = \begin{bmatrix} \frac{A^3}{\sqrt{2}} + \frac{A^8}{\sqrt{6}} & \frac{1}{\sqrt{2}}(A^1 - iA^2) & \frac{1}{\sqrt{2}}(A^4 - iA^5) \\ \frac{1}{\sqrt{2}}(A^1 + iA^2) & -\frac{A^3}{\sqrt{2}} + \frac{A^8}{\sqrt{6}} & \frac{1}{\sqrt{2}}(A^6 - iA^7) \\ \frac{1}{\sqrt{2}}(A^4 + iA^5) & \frac{1}{\sqrt{2}}(A^6 + iA^7) & -\frac{2A^8}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} A^1_1 & A^1_2 & A^1_3 \\ A^2_1 & A^2_2 & A^2_3 \\ A^3_1 & A^3_2 & A^3_3 \end{bmatrix} \tag{2.1.29}$$

Note that $(T^a_b A^b_a)^c_d = A^c_d$ and

$$\begin{aligned} TrAA &= 2Tr(T^i A^i)^2 = 2A^i A^j \, TrT^i T^j \\ &= A^i A^i . \end{aligned} \tag{2.1.30}$$

Similarly we define the Yang-Mills field for SU(5) as a matrix

$$A_\mu \equiv T^a_b \, A^b_{\mu a} = \sqrt{2} \sum_{i=1}^{24} T^i A^i_\mu \quad . \tag{2.1.31}$$

Since SU(3) x SU(2) x U(1) is a subgroup of SU(5) each SU(5) representation will transform as a sum of SU(3) x SU(2) x U(1) representations. We can identify the SU(3) x SU(2) x U(1) fields in each SU(5) representation by embedding this subgroup in SU(5) according to

1) The SU(3) generators L^a_b ; a, b = 1, 2, 3 are given by

$$L^a_b \equiv T^a_b - \frac{1}{3} \delta^a_b \, T^c_c \qquad a,b,c, = 1,2,3 \tag{2.1.32}$$

2) The SU(2) generators S^a_b ; a,b = 1,2 are

$$S^a_b = T^{3+a}_{3+b} - \frac{1}{2} \delta^a_b \, T^{3+c}_{3+c} \quad ; \quad a,b,c = 1,2 \quad . \tag{2.1.33}$$

The L^a_b obey the SU(3) algebra

$$[L^a_b, L^c_d] = \delta^a_d \, L^c_b - \delta^c_b L^a_d \qquad a,b,c,d = 1,2,3 \quad . \tag{2.1.34}$$

and the S^a_b obey the SU(2) algebra

$$[S^a_b, S^c_d] = \delta^a_d S^c_b - \delta^c_b S^a_d \qquad a,b,c,d = 1,2 \quad . \tag{2.1.35}$$

[In general the indices will take values relevant to the group to which the object under discussion belongs].

3) The hypercharge generator Y is given by

$$Y \equiv -\frac{1}{3} \sum_{a=1}^{3} T^a_a + \frac{1}{2} \sum_{b=4}^{5} T^b_b \quad . \tag{2.1.36}$$

Since SU(5) is rank 4 there are 4 diagonal generators Y above and the third component of weak isospin S^3

$$S^3 \equiv \frac{1}{2}(S_1^1 - S_2^2) = \frac{1}{2}(T_4^4 - T_5^5). \tag{2.1.37}$$

The electric charge Q can be defined in terms of these two

$$Q = S^3 + Y = -\frac{1}{3}\sum_{a=1}^{3} T_a^a + T_4^4 \quad . \tag{2.1.38}$$

The other 2 diagonal generators are in the SU(3) as in QCD and can be defined as color hypercharge

$$Y^c \equiv \frac{1}{3}(L_1^1 + L_2^2 - 2L_3^3) \tag{2.1.39}$$

$$= -L_3^3$$

and the third component of color isospin

$$L^3 \equiv \frac{1}{2}(L_1^1 - L_2^2) \tag{2.1.40}$$

for instance. The 12 new generators T_b^a, T_a^b $a = 1,2,3$, $b = 4,5$ in SU(5) will relate flavor and color quantum numbers i.e. transform quarks into leptons.

Towards SU(3) x SU(2) x U(1) decomposing the adjoint rep. A of SU(5) we can write A as

$$A = T_b^a A_a^b = \sum_{a,b=1}^{3} (T_b^a - \frac{1}{3}\delta_b^a \sum_{c=1}^{3} T_c^c) A_a^b$$

$$+ \sum_{a,b=1}^{2} (T_{b+3}^{a+3} - \frac{1}{2}\delta_b^a \sum_{c=1}^{2} T_{c+3}^{c+3}) A_{a+3}^{b+3}$$

$$+ \sum_{a,b=1}^{3} \frac{1}{3} T_b^b A_a^a + \frac{1}{2} \sum_{a,b=1}^{2} T_{a+3}^{a+3} A_{b+3}^{b+3} \tag{2.1.41}$$

$$+ \sum_{a=1}^{3} \sum_{b=4}^{5} T_b^a A_a^b + \sum_{a=4}^{5} \sum_{b=1}^{3} T_b^a A_a^b$$

So $A = L_b^a(A_a^b - \frac{1}{3}\delta_a^b A_c^c) + S_b^a(A_{a+3}^{b+3} - \frac{1}{2}\delta_a^b A_{c+3}^{c+3})$

$$- Y \sum_{a=1}^{3} A_a^a + \frac{1}{2} \sum_{b=4}^{5} T_b^b \underbrace{\left[\sum_{a=1}^{3} A_a^a + \sum_{c=4}^{5} A_a^a) \right]}_{=0}$$

$$+ \sum_{a=1}^{3} \sum_{b=4}^{5} T^a_b A^b_a + \sum_{a=4}^{5} \sum_{b=1}^{3} T^a_b A^b_a \quad . \tag{2.1.42}$$

Thus we define

$$G^b_a \equiv A^b_a - \frac{1}{3}\delta^b_a A^c_c \qquad a,b,c = 1,2,3 \tag{2.1.43}$$

$$A^{b+3}_{a+3} - \frac{1}{2}\delta^b_a A^{c+3}_{c+3} \equiv \begin{bmatrix} \frac{W^3}{\sqrt{2}} & W^+ \\ W^- & -\frac{W^3}{\sqrt{2}} \end{bmatrix}_{ba} \qquad a,b,c, = 1,2 \quad . \tag{2.1.44}$$

Further let the new vector bosons be called:

$$X_a = A^4_a \qquad \bar{X}^a = A^a_4$$
$$a = 1,2,3 \tag{2.1.45}$$
$$Y_a = A^5_a \qquad \bar{Y}^a = A^a_5$$

Finally we define

$$\sum_{a=1}^{3} A^a_a \equiv -\sqrt{\frac{6}{5}}\, B \tag{2.1.46}$$

(where recall $\quad (T^a_b)^c_d = \delta^c_b \delta^a_d - \frac{1}{5}\,\delta^a_b \delta^c_d \quad$ so that

$$T^1_1 = \frac{1}{5}(4,\ -1,\ -1,\ -1,\ -1),\ T^3_3 = \frac{1}{5}(-1,\ -1,\ 4,\ -1,\ -1)\ ,$$

$$T^5_5 = \frac{1}{5}\ (-1,\ -1,\ -1,\ -1,\ 4)\ ,$$

$$T^2_2 = \frac{1}{5}\ (-1,\ 4,\ -1,\ -1,\ -1),\ T^4_4 = \frac{1}{5}(-1,\ -1,\ -1,\ 4,\ -1))$$

and so

$$Y = -\frac{1}{3} \sum_{a=1}^{3} T^a_a + \frac{1}{2} \sum_{b=4}^{5} T^b_b$$

$$= \begin{bmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & 0 & \\ & & -\frac{1}{3} & & \\ & 0 & & +\frac{1}{2} & \\ & & & & +\frac{1}{2} \end{bmatrix} . \qquad (2.1.47)$$

We find

$$A = \left[\begin{array}{ccc|cc} [G^1_1 - \frac{2B}{\sqrt{30}}] & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\ G^2_1 & [G^2_2 - \frac{2B}{\sqrt{30}}] & G^2_3 & \bar{X}^2 & \bar{Y}^2 \\ G^3_1 & G^3_2 & [G^3_3 - \frac{2B}{\sqrt{30}}] & \bar{X}^3 & \bar{Y}^3 \\ \hline X_1 & X_2 & X_3 & [\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}] & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & [-\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}] \end{array}\right] \qquad (2.1.48)$$

We must check the SU(3) × SU(2) × U(1) transformation properties of these fields which are written in suggestive notation.

If the G^a_b are the gluons they should be in the 8 dimensional SU(3) adjoint representation while being SU(2) singlets and hypercharge neutral.

We designate the (SU(3),SU(2),U(1)) representations by their respective dimensionalities and their hypercharge.

The G^a_b should be (8,1,0) representations under the (SU(3),SU(2),U(1)) transformations, respectively. We check this by recalling the transformation properties of A under global SU(5) transformations

$$A^{a'}_b = A^a_b + ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^a_c A^c_b - ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^c_b A^a_c \qquad (2.1.49)$$

$$= A^a_b + ig\omega^c_d[\delta^a_c A^d_b - \delta^d_b A^a_c]$$

Now the SU(3) transformations are generated by the L^a_b so if we choose

$$\omega^a_b \equiv \begin{cases} \lambda^a_b & a,b=1,2,3 \\ 0 & \text{otherwise} \end{cases} \qquad (2.1.50)$$

with

$$\lambda^a_a = 0 \,, \qquad \text{then} \qquad \underline{\underline{\omega}}\cdot\underline{\underline{T}} = \omega^b_a T^a_b$$

$$= \lambda^b_a\left(T^a_b - \frac{1}{3}\,\delta^a_b T^c_c\right) = \lambda^b_a L^a_b \qquad (2.1.51)$$

Further we can isolate the SU(2) transformations by choosing

$$\omega^{a+3}_{b+3} = \begin{cases} s^a_b & a,b=1,2 \\ 0 & \text{otherwise} \end{cases} , \qquad (2.1.52)$$

with $s^a_a = 0$. Then

$$\underline{\underline{\omega}}\cdot\underline{\underline{T}} = \omega^{b+3}_{a+3}T^{a+3}_{b+3}$$

$$= s^b_a\left(T^{a+3}_{b+3} - \frac{1}{2}\,\delta^a_b T^{c+3}_{c+3}\right) = s^b_a S^a_b \quad . \tag{2.1.53}$$

The hypercharge transformations are generated by

$$Y = -\frac{1}{3}\sum_{a=1}^{3} T^a_a + \frac{1}{2}\sum_{b=4}^{5} T^b_b \quad .$$

So if we choose

$$\omega^a_b = \begin{cases} -\frac{1}{3}\,\theta\delta^a_b & \text{for} \quad a,b=1,2,3 \\ +\frac{1}{2}\,\theta\delta^a_b & \text{for} \quad a,b=4,5 \\ 0 & \text{otherwise}, \end{cases} \tag{2.1.54}$$

then

$$\underline{\underline{\omega}}\cdot\underline{\underline{T}} = Y\theta \;. \tag{2.1.55}$$

The remaining 12 parameters $\omega^a_4, \omega^a_5, \omega^4_a, \omega^5_a$, a=1,2,3, are associated with the 12 generators $T^a_{4,5}, T^{4,5}_a$ of SU(5) not in SU(3) × SU(2) × U(1) which rotate the electroweak fields into color fields. We now decompose the SU(5) fields in terms of their SU(3) × SU(2) × U(1) content. First we consider the gluons $G^a_b = A^a_b - \frac{1}{3}\,\delta^a_b A^c_c$ a,b,c=1,2,3.

Under SU(3) transformations

$$\begin{aligned} G^{a\,\prime}_b &= A^{a\,\prime}_b - \frac{1}{3}\,\delta^a_b A^{c\,\prime}_c \\ &= G^a_b + ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^a_c A^c_b - ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^c_b A^a_c \\ &\quad - \frac{1}{3}\,\delta^a_b \underbrace{\left[ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^c_d A^d_c - ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^d_c A^c_d\right]}_{=\;0} \\ &= G^a_b + ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^a_c\left[A^c_b - \frac{1}{3}\,\delta^c_b A^d_d\right] \\ &\quad - ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^c_b\left[A^a_c - \frac{1}{3}\,\delta^a_c A^d_d\right] \\ &\quad - \frac{1}{3}\,A^d_d \underbrace{\left[ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^a_b - ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^a_b\right]}_{=\;0} \end{aligned} \qquad (2.1.56)$$

yielding

$$\begin{aligned} G^{a\,\prime}_b &= G^a_b + ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^a_c G^c_b - ig(\underline{\underline{\lambda}}\cdot\underline{\underline{L}})^c_b G^a_c \\ &= G^a_b + ig\lambda^a_c G^c_b - ig\lambda^c_b G^a_c \end{aligned} \qquad (2.1.57)$$

Thus G^a_b transforms under SU(3) as the 8 dimensional adjoint representation.

Under SU(2) transformations $\omega^a_b = \begin{cases} S^{a-3}_{b-3} & \text{for } a,b=4,5 \\ 0 & \text{otherwise} \end{cases}$

$$G^{a\,\prime}_b = G^a_b + ig\omega^c_d[\delta^a_c A^d_b - \delta^d_b A^a_c] \qquad (2.1.58)$$

but a,b=1,2,3 and for SU(2) $\omega^{1,2,3}_d = 0$. So $G^{a\,\prime}_b = G^a_b$; the gluons are invariant under SU(2) transformations; they are singlets.

For hypercharge transformations

$$G^{a'}_b = G^a_b + ig\theta\ [-\frac{1}{3}\delta^c_d][\delta^a_c A^d_b - \delta^d_b A^a_c] = G^a_b\ . \qquad (2.1.59)$$

The G^a_b have zero hypercharge.

Thus we have confirmed that the G^a_b are indeed gluons. Under $(SU(3), SU(2), U(1))$ they transform as the (8,1,0) dimensional representations.

Next let's check that (X_a, Y_a) transforms as $(\bar{3}, 2, +5/6)$ under $(SU(3), SU(2), U(1))$. That is we want to check the transformation properties of A^a_b with a=4,5,b=1,2,3. Under SU(3) transformations

$$\omega^a_b = \begin{cases} \lambda^a_b & a,b=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

$$A^{a'}_b = A^a_b + ig\omega^c_d[\delta^a_c A^d_b - \delta^d_b A^a_c] = A^a_b - ig\lambda^c_b A^a_c\ . \qquad (2.1.60)$$

Thus X_b, Y_b are SU(3)$\bar{3}$'s.

Under SU(2) transformations

$$\omega^a_b = \begin{cases} s^{a-3}_{b-3} & a,b=4,5 \\ 0 & \text{otherwise} \end{cases}$$

$$A^{a'}_b = A^a_b + ig\omega^c_d[\delta^a_c A^d_b - \delta^d_b A^a_c] = A^a_b + igs^a_c A^c_b\ . \qquad (2.1.61)$$

Thus (X,Y) form an SU(2) doublet a 2.

Under hypercharge transformations

$$A_b^{a'} = A_b^a + ig\theta[+\frac{1}{3}\delta_d^c\delta_b^d A_c^a + \frac{1}{2}\delta_d^c\delta_c^a A_b^d]$$
$$= A_b^a + ig\theta\frac{5}{6}A_b^a \rightarrow \text{ the hypercharge } y = +\frac{5}{6} \quad . \tag{2.1.62}$$

Thus the doublet (X_a, Y_a) transforms as a $(\bar{3},2,+5/6)$. Since

$$\bar{X}^a = (X_a)^\dagger \text{ and } \bar{Y}^a = (Y_a)^\dagger \rightarrow (\bar{X}^a, \bar{Y}^a) \text{ transform as } (3,\bar{2},-5/6) .$$

The $W^o, W^\pm$ transforms as (1,3,0); the adjoint representation of SU(2), and B is a (1,1,0) a complete singlet.

To summarize the 24 dimensional representation of SU(5), A_b^a transforms under $\{SU(3), SU(2), U(1)\}$ according to

$$\begin{aligned} 24 &= (8,1,0) + (1,3,0) + (1,1,0) \\ A_b^a &= \quad G_b^a \qquad\quad W^\pm, W^o \qquad B \\ &\quad + (3,\bar{2},-5/6) + (\bar{3},2,+5/6) \\ &\qquad (\bar{X}^a, \bar{Y}^a) \qquad\quad (X_a, Y_a) \end{aligned} \tag{2.1.63}$$

The charge is $Q = S^3 + Y$; thus

Field	Q
G_b^a	0
W^o	0
$W^\pm$	± 1
B	0
X_a	$+\frac{4}{3}$
Y_a	$+\frac{1}{3}$
$\bar{X}^a$	$-\frac{4}{3}$
$\bar{Y}^a$	$-\frac{1}{3}$

(2.1.64)

Of course these are Yang-Mills fields and each has also the inhomogeneous form under local gauge transformations

$$A^{a'}_{b\mu} = A^{a}_{b\mu} + \sqrt{2}\,\partial_{\mu}\omega^{a}_{b} + ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^{a}_{c}A^{c}_{b\mu} - ig(\underline{\underline{\omega}}\cdot\underline{\underline{T}})^{c}_{b}A^{a}_{c\mu} \tag{2.1.65}$$

So for instance under SU(3) × SU(2) × U(1)

$$G^{a'}_{b\mu} = G^{a}_{b\mu} + \sqrt{2}\,\partial_{\mu}\lambda^{a}_{b} + ig\lambda^{a}_{c}G^{c}_{b\mu} - ig\lambda^{c}_{b}G^{a}_{c\mu} \tag{2.1.66}$$

<u>etc.</u>

The SU(5) gauge invariant pure-Yang-Mills part of the Georgi-Glashow model is

$$\mathcal{L}_{ym} = -\frac{1}{4}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\,F^{i}_{\mu\nu}F^{i\mu\nu} = -\frac{1}{4}(F_{\mu\nu})^{a}_{b}(F^{\mu\nu})^{b}_{a} \tag{2.1.67}$$

where as usual

$$\frac{1}{\sqrt{2}}F_{\mu\nu} \equiv \frac{T^{a}_{b}}{\sqrt{2}}F^{b}_{a\mu\nu} = T^{i}F^{i}_{\mu\nu}$$

$$= \frac{1}{\sqrt{2}}D_{\mu}A_{\nu} - \frac{1}{\sqrt{2}}D_{\nu}A_{\mu} = D_{\mu}T^{i}A^{i}_{\nu} - D_{\nu}T^{i}A^{i}_{\mu} \tag{2.1.68}$$

where

$$D_{\mu} = \partial_{\mu} - \frac{ig}{\sqrt{2}}A_{\mu} = \partial_{\mu} - igT^{i}A^{i}_{\mu}\ . \tag{2.1.69}$$

Now let's re-write the Yang-Mills field transformation properties in a more compact form. Recall

$$A^{a'}_{b\mu} = A^{a}_{b\mu} + ig\omega^{a}_{c}A^{c}_{b\mu} - ig\omega^{c}_{b}A^{a}_{c\mu} + \partial_{\mu}\omega^{a}_{b}\sqrt{2} \quad . \tag{2.1.70}$$

If we define the parameter matrix for some representation matrix T to be

$$\omega \equiv T^{a}_{b}\omega^{b}_{a} \qquad \text{i.e.,} \qquad (T^{a}_{b}\omega^{b}_{a})^{c}_{d} = \omega^{c}_{d} \quad \text{for the fundamental rep.} \tag{2.1.71}$$

and

$$A_{\mu} \equiv T^{a}_{b}A^{b}_{a\mu} \qquad \text{i.e.,} \qquad (T^{a}_{b}A^{b}_{a\mu})^{c}_{d} = A^{c}_{d\mu} \tag{2.1.72}$$

Then more cryptically in matrix notation the $A^{a}_{b\mu}$ equation is

$$A'_{\mu} = A_{\mu} + \sqrt{2}\,\partial_{\mu}\omega + ig[\omega,A_{\mu}] \tag{2.1.73}$$

Thus

$$\begin{aligned} F'_{\mu\nu} &= \partial_{\mu}A'_{\nu} - \partial_{\nu}A'_{\mu} - \frac{ig}{\sqrt{2}}[A'_{\mu},A'_{\nu}] \\ &= F_{\mu\nu} + ig\partial_{\mu}[\omega,A_{\nu}] - ig\partial_{\nu}[\omega,A_{\mu}] - ig[A_{\mu},\partial_{\nu}\omega] - ig[\partial_{\mu}\omega,A_{\nu}] \\ &\quad + \frac{g^{2}}{\sqrt{2}}[A_{\mu},[\omega,A_{\nu}]] + \frac{g^{2}}{\sqrt{2}}[[\omega,A_{\mu}],A_{\nu}] \\ &= F_{\mu\nu} + ig[\omega,\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}] + \frac{g^{2}}{\sqrt{2}}[\omega,[A_{\mu},A_{\nu}]] \end{aligned} \tag{2.1.74}$$

where we used the identity

$$[[A,B],C] = [A,[B,C]] - [B,[A,C]] \quad \text{to give}$$

$$F'_{\mu\nu} = F_{\mu\nu} + ig[\omega,F_{\mu\nu}] \tag{2.1.75}$$

Thus $F_{\mu\nu}$ transforms homogeneously as the adjoint representation. So

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}} [A_\mu, A_\nu] \tag{2.1.76}$$

In components this yields

$$F^a_{b\mu\nu} = \partial_\mu A^a_{b\nu} - \partial_\nu A^a_{b\mu} - \frac{ig}{\sqrt{2}} [A^a_{c\mu} A^c_{b\nu} - A^a_{c\nu} A^c_{b\mu}] . \tag{2.1.77}$$

We will write this in terms of SU(3) × SU(2) × U(1) fields later.

Next we consider the matter field representations. Recall first the electron family. The fermions are the

$$\begin{array}{l} \nu_{eL}, e^-_L, u^a_L, d^a_L \\ e^-_R, u^a_R, d^a_R \end{array} \tag{2.1.78}$$

Thus we have 15 fields per family. Since we want to put the fields in one or two representations it is convenient to deal with all left-handed fields. So we take the charge conjugate fields for the right-handed e^-_R, u^a_R, d^a_R fields since they are left-handed. Under charge conjugation $\mathcal{C}$

$$\begin{aligned} \psi &\to (\mathbb{C}\psi\mathcal{C}^{-1} \equiv)\ \psi^c = C\bar{\psi}^T \\ \bar{\psi} &\to (\mathbb{C}\psi\mathcal{C}^{-1} \equiv)\ \bar{\psi}^c = -\psi^T C^{-1} \end{aligned} \tag{2.1.79}$$

where

$$C^{-1}\gamma_\mu C = -\gamma^T_\mu . \tag{2.1.80}$$

For our representation of the Dirac matrices

$$C = -C^{-1} = -C^\dagger = -C^T = i\gamma^2\gamma^o . \tag{2.1.81}$$

Thus

$$\begin{aligned}
\psi^c_{\substack{L\\R}} \equiv C\psi_{\substack{L\\R}}C^{-1} &= \gamma_{\mp}C\psi C^{-1} \\
&= \gamma_{\mp}\psi^c = \gamma_{\mp}C\bar{\psi}^T \\
&= C\gamma_{\mp}\bar{\psi}^T = C(\bar{\psi}\gamma_{\mp})^T \\
&= C\bar{\psi}^T_{\substack{R\\L}} \\
&\equiv \psi^c_{\substack{L\\R}} \quad .
\end{aligned} \tag{2.1.82}$$

Similarly

$$\begin{aligned}
\overline{\psi^c}_{\substack{L\\R}} \equiv C\bar{\psi}_{\substack{L\\R}}C^{-1} &= C\bar{\psi}C^{-1}\gamma_{\pm} = -\psi^T C^{-1}\gamma_{\pm} \\
&= -(\gamma_{\pm}\psi)^T C^{-1} \\
&= -\psi^T_{\substack{R\\L}}C^{-1} = \overline{\psi^c_{\substack{L\\R}}} \quad .
\end{aligned} \tag{2.1.83}$$

Since

$$\begin{aligned}
\overline{\psi^c_{\substack{L\\R}}} = \psi^{c\dagger}_{\substack{L\\R}}\gamma^o = \bar{\psi}^*_{\substack{R\\L}}C^{\dagger}\gamma^o &= \psi^T_{\substack{R\\L}}\gamma^{o*}C^{-1}\gamma^o \\
&= -\psi^T_{\substack{R\\L}}C^{-1}
\end{aligned} \tag{2.1.84}$$

hence

$$(\overline{\psi^c_{\substack{L\\R}}})^T = -C\psi_{\substack{R\\L}} \tag{2.1.85}$$

and so

$$\psi_{\substack{R\\L}} \;=\; C(\overline{\psi^c_{\substack{L\\R}}})^T \qquad (2.1.86)$$

and

$$\bar{\psi}_{\substack{R\\L}} \;=\; \psi^{cT}_{\substack{L\\R}}C \;. \qquad (2.1.87)$$

So instead of (ψ_L,ψ_R) as fundamental fields we could use the equivalent pairs $(\psi_L,\psi^c_L),(\psi_R,\psi^c_R),(\psi^c_R,\psi^c_L)$. Thus we replace e^-_R,u^a_R,d^a_R with e^+_L,u^{ca}_L,d^{ca}_L with

$$e^+_L \;=\; C\bar{e}^{-T}_R \;=\; e^{-c}_L \quad . \qquad (2.1.88)$$

Now under SU(3) × SU(2) × U(1) these fields transform as

d^c_L a$(\bar{3},1,1/3)$ since

d_R is a $(3,1,-1/3)$ and

$d^c_L \;=\; C\overline{d_R}^T$

the $\overline{d_R}$ transforms according to U^{-1} taking a 3 to $\bar{3}$ and a $-1/3$ to $+1/3$.

Recall

$$\ell_L \;=\; \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \text{ transforms like a } (1,2,-1/2);$$

however since for SU(2) a 2 and $\bar{2}$ are equivalent we can make a $\bar{2}$ out of ℓ_L

$$\begin{aligned}(i\sigma_2\ell_L)' &= i\sigma_2\ell_L + \tfrac{i}{2}\, g\; i\sigma_2\; \underline{\omega}\cdot\underline{\sigma}\; \ell_L \\ &= i\sigma_2\ell_L + \tfrac{i}{2}\, g\; i\sigma_2\; \underline{\omega}\cdot\underline{\sigma}(i\sigma_2)^{\dagger}\,(i\sigma_2\ell_L) \qquad (2.1.89)\\ &= i\sigma_2\ell_L - \tfrac{i}{2}\, g\; \underline{\omega}\cdot\underline{\sigma}^T(i\sigma_2\ell_L) \\ &= i\sigma_2\ell_L - \tfrac{i}{2}\, g(i\sigma_2\ell_L)\underline{\omega}\cdot\underline{\sigma} = (i\sigma_2\ell_L)e^{-\frac{ig}{2}(\underline{\omega}\cdot\underline{\sigma})} \quad .\end{aligned}$$

Thus $i\sigma_2\ell_L$ is a $\bar{2}$. So

$$i\sigma_2\ell_L = \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L \text{ is a } (1,\bar{2},-1/2).$$

Thus we see that the 5 fields d_L^c and $i\sigma_2\ell_L$ all transform as complex conjugate representations. The fundamental rep. of SU(5) is a 5, its complex conj. is $\bar{5}$ and is a good candidate for these fields: we must check to see if

$$\bar{5} = (\bar{3},1,+1/3) + (1,\bar{2},-1/2) \,. \tag{2.1.90}$$

Recall under SU(5) transformations if ψ_{La} is a $\bar{5}$ it transforms as

$$\psi'_{La} = \psi_{La} - ig\omega_a^b\psi_{Lb} \,. \tag{2.1.91}$$

Under SU(3) transformations

$$\omega_a^b = \begin{cases} \lambda_a^b & \text{for } a,b=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

giving

$$\begin{aligned} \psi'_{La} &= \psi_{La} - ig\lambda_a^b\psi_{Lb} && \text{for } a,b=1,2,3 \\ \psi'_{La} &= \psi_{La} && \text{for } a=4,5 \,. \end{aligned} \tag{2.1.92}$$

Thus $\psi_{L1,2,3}$ transforms as a $\bar{3}$ and $\psi_{L4,5}$ is a singlet under SU(3).

Under SU(2)

$$\omega_a^b = \begin{cases} s_{a-3}^{b-3} & \text{for } a,b=4,5 \\ 0 & \text{otherwise} \end{cases}$$

yielding

$$\psi'_{La} = \psi_{La} \qquad \text{for } a=1,2,3$$

$$\psi'_{La} = \psi_{La} - ig\, s_a^b \psi_{Lb} \qquad \text{for } a,b=4,5 \; . \tag{2.1.93}$$

Thus $\psi_{L\,\substack{1\\2\\3}}$ is an SU(2) singlet and $\psi_{L\,\substack{4\\5}}$ is a $\bar{2}$.

Finally under hypercharge

$$\omega_b^a = \begin{cases} -\frac{1}{3}\,\theta\delta_b^a & \text{for } a,b=1,2,3 \\ +\frac{1}{2}\,\theta\delta_b^a & \text{for } a,b=4,5 \\ 0 & \text{otherwise} \end{cases}$$

so

$$\begin{aligned} \psi'_{La} &= \psi_{La} - ig\,\theta\left(-\frac{1}{3}\right)\psi_{La} \qquad \text{for } a=1,2,3 \\ &= \psi_{La} + ig\theta y \psi_{La} \\ \psi'_{La} &= \psi_{La} - ig\left(+\frac{1}{2}\right)\theta\psi_{La} \qquad \text{for } a=4,5 \\ &= \psi_{La} + ig\,\psi_{La} y\theta \; . \end{aligned} \tag{2,1,94}$$

Thus

$\psi_{L\,\substack{1\\2\\3}}$ has hypercharge $+\frac{1}{3} = y$ and

$\psi_{L\,\substack{4\\5}}$ has hypercharge $-\frac{1}{2} = y$.

So we can write the $\bar{5}$ as a column vector

$$\psi_{La} \equiv \begin{bmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{bmatrix}_L \tag{2.1.95}$$

where under SU(3) × SU(2) × U(1)

ψ_{La};a=1,2,3 transforms as $(\bar{3},1,+1/3)$

and

ψ_{Lb};b=4,5 transforms as $(1,\bar{2},-1/2)$.

We are left with the 10 fields

$$e_L^+,\ u_L^{ca},\ u_L^a,\ d_L^a$$

The transformation properties of these under SU(3) × SU(2) × U(1) are

e_L^+ is a (1,1,1)

u_L^{ca} is a $(\bar{3},1,-2/3)$

$\begin{pmatrix} u_L^a \\ d_L^a \end{pmatrix}$ is a (3,2,+1/6)

since u_R is a (3,1,+2/3).

These fields have a chance to fit into the next highest tensor representation of SU(5) a 10. We must check if under SU(3) × SU(2) × U(1) the 10 of SU(5) decomposes into

$$10 = (\bar{3},1,-2/3) + (3,2,+1/6) + (1,1,1) \; . \qquad (2.1.96)$$

The 10 transformation properties are obtained by writing the 10 as the antisymmetric product of two 5's. Let ψ_L^{ab} be the 10 and A^a, B^b each be 5's;

$A^{a'} = A^a + ig\omega^a_b A^b$ and similarly for B^a, with

$\psi_L^{ab} = A^a B^b - A^b B^a$ we find

$\psi_L^{ab'} = A^{a'} B^{b'} - A^{b'} B^{a'}$ which gives the (2,0) tensor transformation equation

$$\psi_L^{ab'} = \psi_L^{ab} + ig\omega^a_c \psi_L^{cb} + ig\omega^b_c \psi_L^{ac} \; . \qquad (2.1.97)$$

Under SU(3) transformations

$$\omega^a_b = \begin{cases} \lambda^a_b & \text{for } a,b=1,2,3 \\ 0 & \text{otherwise} \end{cases} \; .$$

First consider a=1,2,3; b=4,5

$$\psi_L^{ab'} = \psi_L^{ab} + ig\lambda^a_c \psi_L^{cb} \qquad \begin{matrix} a,c=1,2,3 \\ b=4,5 \end{matrix} \; . \qquad (2.1.98)$$

This transforms as a 3 of SU(3). Next we consider ψ_L^{45}

$$\psi_L^{45'} = \psi_L^{45} \qquad \text{a SU(3) singlet} \; . \qquad (2.1.99)$$

Thus the remaining 3 fields $\varepsilon_{abc}\psi_L^{bc}$ a,b,c=1,2,3 must form a $\bar{3}$:

$$\phi_{La} \equiv \frac{1}{2}\varepsilon_{abc}\psi_L^{bc} \;;\rightarrow \psi_L^{bc} = \varepsilon^{bca}\phi_{La} \qquad \text{hence} \tag{2.1.100}$$

$$\phi'_{La} = \phi_{La} + \frac{1}{2}\varepsilon_{abc} ig\lambda_d^b\psi_L^{dc} + \frac{1}{2}\varepsilon_{abc} ig\lambda_d^c\psi_L^{bd}$$

$$= \phi_{La} + \frac{ig}{2}\lambda_d^b[\varepsilon_{abc}\varepsilon^{dce} + \varepsilon_{acb}\varepsilon^{cde}]\phi_{Le}$$

$$= \phi_{La} - ig\lambda_d^b\varepsilon_{abc}\varepsilon^{dec}\phi_{Le} \qquad \text{resulting in}$$

$$\phi'_{La} = \phi_{La} - ig\lambda_a^b\phi_{Lb} \qquad \text{a } \bar{3} \text{ of SU(3)} . \tag{2.1.101}$$

Under SU(2)

$$\omega_b^a = \begin{cases} s_{b-3}^{a-3} & \text{for a,b=4,5} \\ 0 & \text{otherwise} \end{cases} .$$

then for a=1,2,3; b=4,5

$$\psi_L^{ab'} = \psi_L^{ab} + igs_{c-3}^{b-3}\psi_L^{ac} \qquad \text{this is a 2 of SU(2)} . \tag{2,1,102}$$

For a=4; b=5

$$\psi_L^{45'} = \psi_L^{45} + igs_{c-3}^{1}\psi_L^{c5} + igs_{c-3}^{2}\psi_L^{4c} \tag{2,1.103}$$

but ψ_L^{ab} is antisymmetric so a ≠ b

$$\psi_L^{45'} = \psi_L^{45} + igs_1^1\psi_L^{45} + igs_2^2\psi_L^{45} \tag{2,1,104}$$

but $S_1^1 + S_2^2 = 0 = S_a^a$ for SU(2). So $\psi_L^{45'} = \psi_L^{45}$ an SU(2) singlet. Finally for $\phi'_{La} = \phi_{La}$ since a=1,2,3 only and $\omega_b^a = 0$ for these values; thus ϕ_{La} is also a SU(2) singlet.

Once again the hypercharge is given by

$$\omega_b^a = \begin{cases} -\frac{1}{3}\,\delta_b^a\theta & a,b=1,2,3 \\ +\frac{1}{2}\,\delta_b^a\theta & a,b=4,5 \\ 0 & \text{otherwise} . \end{cases}$$

So for a=1,2,3; b=4,5

$$\psi_L^{ab'} = \psi_L^{ab} + ig\theta\left(-\frac{1}{3} + \frac{1}{2}\right)\psi_L^{ab} \tag{2.1.105}$$

thus $y = +\frac{1}{6}$ for ψ_L^{ab}. For ψ_L^{45} we find

$$\psi_L^{45'} = \psi_L^{45} + ig\theta\left(\frac{1}{2} + \frac{1}{2}\right)\psi_L^{45} \tag{2.1.106}$$

So y = +1 for ψ_L^{45} . Finally

$$\begin{aligned} \phi'_{La} &= \phi_{La} + \frac{ig}{2}\,\varepsilon_{abc}\left(-\frac{1}{3} - \frac{1}{3}\right)\theta\psi^{bc} \\ &= \phi_{La} + ig\theta\left(-\frac{2}{3}\right)\phi_{La} \end{aligned} \tag{2.1.107}$$

thus ϕ_{La} has hypercharge $y = -\frac{2}{3}$. So we find indeed that

$$10 = (\bar{3},1,-2/3) + (3,2,+1/6) + (1,1,1)$$

where

ψ_L^{ab}, a=1,2,3, b=4,5 is the (3,2,+1/6)

ψ_L^{45} is the (1,1,1)

and $\phi_{La} = \frac{1}{2}\varepsilon_{abc}\psi_L^{bc}$ a,b,c=1,2,3 is the ($\bar{3}$,1,-2/3).

We can then relate the fields

$e_L^+, \begin{pmatrix} u_L^a \\ d_L^a \end{pmatrix}$ and u_L^{ca} to the 10 of SU(5) by

$$\psi_L^{ab} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3^c & -u_2^c & u^1 & d^1 \\ -u_3^c & 0 & u_1^c & u^2 & d^2 \\ +u_2^c & -u_1^c & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & e^+ \\ -d^1 & -d^2 & -d^3 & -e^+ & 0 \end{bmatrix}_L^{ab} \qquad (2.1.108)$$

The $1/\sqrt{2}$ is a convenient normalization factor. Thus we have incorporated the 15 quarks and leptons into two SU(5) representations (for each family) the $\bar{5}$ and the 10.

Following Langacker it is useful to display the fermion representations in a cryptic form

$$\bar{5}:\quad \begin{pmatrix} \nu_e & \\ & d^c_a \\ e^- & \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\mu & \\ & s^c_a \\ \mu^- & \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\tau & \\ & b^c_a \\ \tau^- & \end{pmatrix}_L \tag{2.1.109}$$

$$10:\quad \begin{pmatrix} & u^a & \\ e^+ & & u^c_a \\ & d^a & \end{pmatrix}_L \quad \begin{pmatrix} & c^a & \\ \mu^+ & & c^c_a \\ & s^a & \end{pmatrix}_L \quad \begin{pmatrix} & t^a & \\ \tau^+ & & t^c_a \\ & b^a & \end{pmatrix}_L$$

where we are in the weak basis (the superscript w is suppressed). The SU(2) doublets are arranged in columns. The SU(3) transformations act on the -script a. The remaining 12 generators of SU(5) mix adjacent columns.

We must now construct the SU(5) invariant kinetic energy terms for our two left-handed representations, recalling the SU(5) transformations

$$\begin{aligned} \psi'_{La} &= \psi_{La} - ig\omega^b_a\psi_{Lb} = \psi_{La} - ig(\omega^i T^i)^b_a\psi_{Lb} \\ \psi'^{ab}_L &= \psi^{ab}_L + ig\omega^a_c\psi^{cb}_L + ig\omega^b_c\psi^{ac}_L \\ &= \psi^{ab}_L + ig(\omega^i T^i)^a_c\psi^{cb}_L + ig(\omega^i T^i)^b_c\psi^{ac}_L \end{aligned} \tag{2.1.110}$$

Thus we define our covariant derivatives by

$$\begin{aligned}(D_\mu\psi_L)_a &\equiv [\partial_\mu\delta_a^b+ig(T^iA_\mu^i)_a^b]\psi_{Lb}\\ &= [\partial_\mu\delta_a^b + \frac{ig}{\sqrt{2}}(T_d^cA_{c\mu}^d)_a^b]\psi_{Lb}\\ &= [\partial_\mu\delta_a^b + \frac{ig}{\sqrt{2}}A_{a\mu}^b]\psi_{Lb}\end{aligned} \tag{2.1.111}$$

and

$$\begin{aligned}(D_\mu\psi_L)^{ab} &\equiv [\partial_\mu\delta_c^a\delta_d^b-ig(T^iA_\mu^i)_c^a\delta_d^b-ig(T^iA_\mu^i)_d^b\delta_c^a]\psi_L^{cd}\\ &= [\partial_\mu\delta_c^a\delta_d^b - \frac{ig}{\sqrt{2}}(T_f^eA_{e\mu}^f)_c^a\delta_d^b - \frac{ig}{\sqrt{2}}(T_f^eA_{e\mu}^f)_d^b\delta_c^a]\psi_L^{cd}\\ &= [\partial_\mu\delta_c^a\delta_d^b - \frac{ig}{\sqrt{2}}A_{c\mu}^a\delta_d^b - \frac{ig}{\sqrt{2}}A_{d\mu}^b\delta_c^a]\psi_L^{cd} \quad .\end{aligned} \tag{2.1.112}$$

We can check the $\bar{5}$ case to see that it remains a $\bar{5}$

$$\begin{aligned}(D_\mu\psi_L)'_a &= [\partial_\mu\delta_a^b + \frac{ig}{\sqrt{2}}A_{a\mu}^{b'}]\psi'_{Lb}\\ &= [\partial_\mu\delta_a^b + \frac{ig}{\sqrt{2}}(A_{a\mu}^b + \sqrt{2}\,\partial_\mu\omega_a^b + ig\omega_c^bA_{a\mu}^c - ig\omega_a^cA_c^b)](\psi_{Lb}-ig\omega_b^d\psi_{Ld})\\ &= (D_\mu\psi_L)_a - ig\omega_a^b(\partial_\mu\delta_b^c + \frac{ig}{\sqrt{2}}A_{b\mu}^c)\psi_{Lc} + \frac{(ig)^2}{\sqrt{2}}\omega_a^bA_{b\mu}^c\psi_{Lc}\\ &\quad + \frac{(ig)^2}{\sqrt{2}}[\omega_b^cA_{a\mu}^b - \omega_a^bA_{b\mu}^c - \omega_b^cA_{a\mu}^b]\psi_{Lc} \quad .\end{aligned} \tag{2.1.113}$$

So as expected

$$(D_\mu\psi_L)'_a = (D_\mu\psi_L)_a - ig\omega_a^b(D_\mu\psi_L)_b \tag{2,1.114}$$

$(D_\mu\psi_L)_a$ transforms as a $\bar{5}$ of SU(5) also. Thus the SU(5) invariant KE terms are

$$L_F = (\bar{\psi}_L)^a i(\not{D}\psi_L)_a - (\bar{\psi}_L)_{ab} i(\not{D}\psi_L)^{ab} \tag{2.1.115}$$

Notice that, in the last term in the above equation we are not overcounting due to the $1/\sqrt{2}$ normalization factor in ψ_L^{ab}. Expanding this somewhat

$$L_F = (\bar{\psi}_L)^a[i\not{\partial}\delta_a^b - \frac{g}{\sqrt{2}}\not{A}_a^b]\psi_{Lb}$$

$$- (\bar{\psi}_L)_{ab}[i\not{\partial}\delta_c^a\delta_d^b + \frac{g}{\sqrt{2}}\not{A}_c^a\delta_d^b + \frac{g}{\sqrt{2}}\not{A}_d^b\delta_c^a]\psi_L^{cd} \tag{2.1.116}$$

using the $\psi_L^{cd} = -\psi_L^{dc}$ we find

$$L_F = (\bar{\psi}_L)^a[i\not{\partial}\delta_a^b - \frac{g}{\sqrt{2}}\not{A}_a^b]\psi_{Lb} - (\bar{\psi}_L)_{ab}[i\not{\partial}\delta_c^a + \frac{2g}{\sqrt{2}}\not{A}_c^a]\psi_L^{cb} \tag{2.1.117}$$

As expected the gauge coupling for the 10 is 2g compared to g for the $\bar{5}$ since the 10 is twice the dimensionality.

Before studying the Higgs sector and SU(5) spontaneous breakdown to SU(3) × SU(2) × U(1), let's express this fermion kinetic energy Lagrangian in terms of the SU(3) × SU(2) × U(1) fields. First consider

$$(\bar{\psi}_L)^a A_a^b \, \psi_{Lb} = \left[\bar{d}^{c1} \, \bar{d}^{c2} \, \bar{d}^{c3} \, \bar{e}^- \; - \bar{\nu}_e \right]_L \times$$

$$\begin{bmatrix} (G_1^1 - \frac{2B}{\sqrt{30}}) & G_2^1 & G_3^1 & \bar{X}^1 & \bar{Y}^1 \\ G_1^2 & (G_2^2 - \frac{2B}{\sqrt{30}}) & G_3^2 & \bar{X}^2 & \bar{Y}^2 \\ G_1^3 & G_2^3 & (G_3^3 - \frac{2B}{\sqrt{30}}) & \bar{X}^3 & \bar{Y}^3 \\ X_1 & X_2 & X_3 & (\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & (-\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) \end{bmatrix}_\mu^T \gamma^\mu$$

$$\times \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix}_L \qquad (2.1.118)$$

$$= \left[\bar{d}^{c1}\bar{d}^{c2}\bar{d}^{c3}\bar{e}^{-} \ - \bar{\nu}_e\right]_L \times$$

$$\begin{bmatrix} (G^1_1 - \frac{2B}{\sqrt{30}}) & G^2_1 & G^3_1 & X_1 & Y_1 \\ G^1_2 & (G^2_2 - \frac{2B}{\sqrt{30}}) & G^3_2 & X_2 & Y_2 \\ G^1_3 & G^2_3 & (G^3_3 - \frac{2B}{\sqrt{30}}) & X_3 & Y_3 \\ \bar{X}^1 & \bar{X}^2 & \bar{X}^3 & (\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) & W^- \\ \bar{Y}^1 & \bar{Y}^2 & \bar{Y}^3 & W^+ & (-\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) \end{bmatrix}_\mu \gamma^\mu$$

$$\times \begin{pmatrix} d^c_1 \\ d^c_2 \\ d^c_3 \\ e^- \\ -\nu_e \end{pmatrix}_L \qquad (2.1,119)$$

$$= \left[\,\overline{d^{c1}}\,\overline{d^{c2}}\,\overline{d^{c3}}\,\bar{e}^- \quad - \bar{\nu}_e\,\right]_L$$

$$\times \left[\begin{array}{l}
\not{G}_1^a d_a^c - \frac{2\not{B}}{\sqrt{30}}\, d_1^c + \left[\not{X}_1 \not{Y}_1\right]\begin{pmatrix} e^- \\ -\nu_e \end{pmatrix} \\
\not{G}_2^a d_a^c - \frac{2\not{B}}{\sqrt{30}}\, d_2^c + \left[\not{X}_2 \not{Y}_2\right]\begin{pmatrix} e^- \\ -\nu_e \end{pmatrix} \\
\not{G}_3^a d_a^c - \frac{2\not{B}}{\sqrt{30}}\, d_3^c + \left[\not{X}_3 \not{Y}_3\right]\begin{pmatrix} e^- \\ -\nu_e \end{pmatrix} \\
\bar{\not{X}}^a d_a^c + \frac{3\not{B}}{\sqrt{30}}\, e^- + \frac{\not{W}^3}{\sqrt{2}}\, e^- - \not{W}^- \nu_e \\
\bar{\not{Y}}^a d_a^c - \frac{3\not{B}}{\sqrt{30}}\, \nu_e + \not{W}^+ e^- + \frac{\not{W}^3}{\sqrt{2}}\, \nu_e
\end{array}\right]_L$$

$$= \overline{d_L^c}{}^b \not{G}_b^a d_{La}^c - \frac{2}{\sqrt{30}}\, \overline{d_L^c}{}^a \not{B} d_{La}^c$$

$$+ \left[\bar{e}^- \;\; -\bar{\nu}_e\right]_L \begin{bmatrix} \frac{\not{W}^3}{\sqrt{2}} & + \not{W}^- \\ \not{W}^+ & - \frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L + \frac{3}{\sqrt{30}}\left(\bar{e}_L^- \not{B} e^- + \bar{\nu}_{eL} \not{B} \nu_{eL}\right) \tag{2,1,120}$$

$$+ \overline{d_L^c}{}^a \left[\not{X}_a \not{Y}_a\right] \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L + \left[\bar{e}^- \;\; -\bar{\nu}_e\right]_L \begin{pmatrix} \bar{\not{X}}^a \\ \bar{\not{Y}}^a \end{pmatrix} d_{La}^c$$

Now recall

$$\psi_L^c = C\bar{\psi}_R{}^T$$
$$\overline{\psi_L^c} = -\psi_R^T C^{-1} \qquad (2.1.121)$$

and $\quad C^{-1}\gamma_\mu C = -\gamma_\mu^T \ ; \qquad C = -C^{-1} = i\gamma^2\gamma^0$

so

$$1)\quad \overline{d_L^c}{}^b \gamma^\mu d_{La}^c = -d_{Rb}^T C^{-1}\gamma^\mu C\bar{d}_R^{Ta}$$
$$= +\, d_{Rb}^T \gamma^{\mu T} \bar{d}_R^{Ta} \qquad (2.1.122)$$
$$= -\,\bar{d}_R^a \gamma^\mu d_{Rb}$$

$$2)\quad \overline{\left(\bar{e}^- \ -\bar{\nu}_e\right)}_L \begin{bmatrix} \frac{\not{W}^3}{\sqrt{2}} & \not{W}^- \\ \not{W}^+ & -\frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L$$
$$= \bar{\ell}_L (i\sigma_2)^\dagger \begin{bmatrix} \frac{\not{W}^3}{\sqrt{2}} & \not{W}^- \\ \not{W}^+ & -\frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} (i\sigma_2)\ell_L \qquad (2.1.123)$$
$$= \bar{\ell}_L \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \frac{\not{W}^3}{\sqrt{2}} & \not{W}^- \\ \not{W}^+ & -\frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \ell_L$$

$$= \bar{\ell}_L \begin{bmatrix} -\frac{\not{W}^3}{\sqrt{2}} & -\not{W}^+ \\ -\not{W}^- & \frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} \ell_L$$

$$= -\bar{\ell}_L \begin{bmatrix} \frac{\not{W}^3}{\sqrt{2}} & \not{W}^+ \\ +\not{W}^- & -\frac{\not{W}^3}{\sqrt{2}} \end{bmatrix} \ell_L$$

$$= -\bar{\ell}_L \frac{1}{\sqrt{2}} \underline{\sigma}\cdot\underline{\not{A}}\, \ell_L$$ where recalling

$$\frac{1}{2}\,\underline{\sigma}\cdot\underline{A}^{\mu} = \frac{1}{\sqrt{2}} S^a_b A^{\mu b}_a$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{W^{\mu}_3}{\sqrt{2}} & W^{+\mu} \\ W^{-\mu} & -\frac{W^{\mu}_3}{\sqrt{2}} \end{bmatrix} . \qquad (2.1.124)$$

Note also for the d-quarks

$$\overline{d^c_L}\, {}^b\not{G}^a_b d^c_{La} = -\bar{d}^a_R \not{G}^a_b d_{Rb}$$

$$= -\bar{d}^a_R (T^c_d \not{G}^d_c)^a_b d_{Rb} \qquad (2.1.125)$$

$$= -\sqrt{2}\, \bar{d}^a_R (T^i \not{G}^i)^a_b d_{Rb}$$

$$= -\frac{\sqrt{2}}{2} \bar{d}^a_R (\underline{\lambda}\cdot\underline{\not{G}})^a_b d_{Rb}$$

yielding

$$(\bar{\psi}_L)^a \not{A}_a^b \psi_{Lb} = -\frac{\sqrt{2}}{2}\,\bar{d}_R(\underline{\lambda}\cdot\not{\underline{G}})d_R + \frac{2}{\sqrt{30}}\,\bar{d}_R\not{B}d_R$$

$$-\frac{1}{\sqrt{2}}\,\bar{\ell}_L\,\underline{\sigma}\cdot\not{\underline{A}}\,\ell_L + \frac{3}{\sqrt{30}}\,[\bar{\ell}_L\not{B}\ell_L] \tag{2.1.126}$$

$$+\,\overline{d_L^c}\,{}^a\left[-\not{Y}_a\ \not{X}_a\right]\ell_L + \bar{\ell}_L\begin{pmatrix}-\bar{\not{Y}}^a\\ \bar{\not{X}}^a\end{pmatrix}d^c_{La}$$

and of course (integrating by parts for the d-quarks)

$$\bar{\psi}_L^a i\not{\partial}\psi_{La} = +\,\bar{d}_R i\not{\partial}d_R + \bar{\ell}_L i\not{\partial}\ell_L \tag{2.1.127}$$

Next we consider the invariant term made from the SU(5) 10.

$$
\begin{aligned}
(\bar{\psi}_L)_{ab} \not{A}^a_c \psi_L^{cb} &= (\bar{\psi}^T_L)_{ba} \not{A}^a_c \psi_L^{cb} \\
&= \frac{1}{2}\,\mathrm{Tr}
\begin{bmatrix}
0 & \bar{u}^{c3} & -\bar{u}^{c2} & \bar{u}_1 & \bar{d}_1 \\
-\bar{u}^{c3} & 0 & \bar{u}^{c1} & \bar{u}_2 & \bar{d}_2 \\
\bar{u}^{c2} & -\bar{u}^{c1} & 0 & \bar{u}_3 & \bar{d}_3 \\
-\bar{u}_1 & -\bar{u}_2 & -\bar{u}_3 & 0 & \overline{e^+} \\
-\bar{d}_1 & -\bar{d}_2 & -\bar{d}_3 & -\overline{e^+} & 0
\end{bmatrix}_L \times \gamma_\mu
\end{aligned}
$$

$$
\times
\begin{bmatrix}
\left(G^1_1 - \frac{2B}{\sqrt{30}}\right) & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\
G^2_1 & \left(G^2_2 - \frac{2B}{\sqrt{30}}\right) & G^2_3 & \bar{X}^2 & \bar{Y}^2 \\
G^3_1 & G^3_2 & \left(G^3_3 - \frac{2B}{\sqrt{30}}\right) & \bar{X}^3 & \bar{Y}^3 \\
X_1 & X_2 & X_3 & \left(\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}\right) & W^+ \\
Y_1 & Y_2 & Y_3 & W^- & \left(-\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}\right)
\end{bmatrix}^\mu
$$

$$
\times
\begin{bmatrix}
0 & u^c_3 & -u^c_2 & u^1 & d^1 \\
-u^c_3 & 0 & u^c_1 & u^2 & d^2 \\
u^c_2 & -u^c_1 & 0 & u^3 & d^3 \\
-u^1 & -u^2 & -u^3 & 0 & e^+ \\
-d^1 & -d^2 & -d^3 & -e^+ & 0
\end{bmatrix}_L
\qquad (2.1.128)
$$

After tedious calculation we find

$$(\bar{\psi}_L)_{ab} \not{A}^a_c \psi^{cb}_L = \frac{1}{2}\{\overline{u^c_L}{}^a \not{G}^b_a u^c_{Lb} - \bar{u}_{La} \not{G}^a_b u^b_L - \bar{d}_{La} \not{G}^a_b d^b_L$$

$$+ \overline{u^c_L}{}^a \frac{4\not{B}}{\sqrt{30}} u^c_{La} - \bar{u}_{La} \frac{\not{B}}{\sqrt{30}} u^a_L - \bar{d}_{La} \frac{\not{B}}{\sqrt{30}} d^a_L - \bar{e}^+_L \frac{6\not{B}}{\sqrt{30}} e^+_L$$

$$- \overline{\left(\bar{u}_a \bar{d}_a\right)}_L \begin{pmatrix} \frac{\not{W}^3}{\sqrt{2}} & \not{W}^+ \\ \not{W}^- & -\frac{\not{W}^3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u^a \\ d^a \end{pmatrix}_L$$

$$+ \bar{X}^j_\mu[-\bar{d}_{Lj}\gamma^\mu e^+_L - \epsilon_{jik}\overline{u^c_L}{}^i \gamma^\mu u^k_L]$$

$$+ X_{\mu j}[-\bar{e}^+_L\gamma^\mu d^j_L + \epsilon^{jik}\bar{u}_{Li}\gamma^\mu u^c_{Lk}]$$

$$+ \bar{Y}^j_\mu[\bar{u}_{Lj}\gamma^\mu e^+_L - \epsilon_{jik}\overline{u^c_L}{}^i \gamma^\mu d^k_L]$$

$$+ Y_{\mu j}[\bar{e}^+_L\gamma^\mu u^j_L + \epsilon^{jik}\bar{d}_{Li}\gamma^\mu u^c_{Lk}]\} \qquad (2.1.129)$$

Now recall our charge conjugation formulae:

1) $$\overline{u^c_L}{}^a \not{G}^b_a u^c_{Lb} = -u^T_{Ra} C^{-1} \not{G}^b_a C \bar{u}^{Tb}_R = +u^T_{Ra} \not{G}^{b\,T}_a \bar{u}^{Tb}_R = -\bar{u}^b_R \not{G}^b_a u_{Ra}$$

2) $$\bar{e}^+_L \not{B} e^+_L = \bar{e}^{-c}_L \not{B} e^{-c}_L = -e^{-T}_R C^{-1} \not{B} C \bar{e}^{-T}_R = -\bar{e}^-_R \not{B} e^-_R$$

3) $$\overline{u^c_L}{}^a \not{B} u^c_{La} = -\bar{u}^a_R \not{B} u_{Ra} \,. \qquad (2.1.130)$$

Also

$$\overline{u^c_L}\, {}^a\not{G}^b_a u^c_{Lb} - \bar{u}_{La}\not{G}^a_b u^b_L = -\bar{u}^b_R \not{G}^b_a u_{Ra} - \bar{u}_{La}\not{G}^a_b u^b_L$$

$$= -\frac{\sqrt{2}}{2}\,\bar{u}^a_R(\underline{\lambda}\cdot\underline{\not{G}})^a_b u_{Rb} - \frac{\sqrt{2}}{2}\,\bar{u}_{La}(\underline{\lambda}\cdot\underline{\not{G}})^a_b u^b_L \qquad (2.1.131)$$

and similarly

$$-\bar{d}_{La}\not{G}^a_b d^b_L = -\frac{\sqrt{2}}{2}\,\bar{d}_{La}(\underline{\lambda}\cdot\underline{\not{G}})^a_b d^b_L\ . \qquad (2.1.132)$$

Thus finally

$$(\bar{\psi}_L)_{ab}\not{A}^a_c \psi^{cb}_L = \frac{1}{2}\{-\frac{\sqrt{2}}{2}\,\bar{u}^a_R(\underline{\lambda}\cdot\underline{\not{G}})^a_b u_{Rb} - \frac{\sqrt{2}}{2}\,\bar{u}_{La}(\underline{\lambda}\cdot\underline{\not{G}})^a_b u^b_L$$

$$-\frac{\sqrt{2}}{2}\,\bar{d}_{La}(\underline{\lambda}\cdot\underline{\not{G}})^a_b d^b_L - \frac{4}{\sqrt{30}}\,\bar{u}^a_R\not{B}u_{Ra} - \frac{1}{\sqrt{30}}\,\bar{u}_{La}\not{B}u^a_L$$

$$-\frac{1}{\sqrt{30}}\,\bar{d}_{La}\not{B}d^a_L + \frac{6}{\sqrt{30}}\,\bar{e}^-_R\not{B}e^-_R - \frac{1}{\sqrt{2}}\,\bar{q}_{aL}\,\underline{\sigma}\cdot\underline{\not{A}}\,q^a_L \qquad (2.1.133)$$

$$+\{\bar{X}^j_\mu[-\bar{d}_{Lj}\gamma^\mu e^+_L - \varepsilon_{jik}\bar{u}^{ci}_L\gamma^\mu u^k_L]$$

$$+\bar{Y}^j_\mu[\bar{u}_{Lj}\gamma^\mu e^+_L - \varepsilon_{jik}\bar{u}^{ci}_L\gamma^\mu d^k_L] + h.c.\}\}\ .$$

The diagonal space-time derivative terms are

$$(\bar{\psi}_L)_{ab} i\not\partial \psi_L^{ab} = (\bar{\psi}_L^T)_{ba} i\not\partial \psi_L^{ab}$$

$$= \frac{1}{2} \mathrm{Tr} \begin{bmatrix} 0 & \overline{u^c}^3 & -\overline{u^c}^2 & \bar{u}_1 & \bar{d}_1 \\ -\overline{u^c}^3 & 0 & \overline{u^c}^1 & \bar{u}_2 & \bar{d}_2 \\ \overline{u^c}^2 & -\overline{u^c}^1 & 0 & \bar{u}_3 & \bar{d}_3 \\ -\bar{u}_1 & -\bar{u}_2 & -\bar{u}_3 & 0 & \overline{e^+} \\ -\bar{d}_1 & -\bar{d}_2 & -\bar{d}_3 & -\overline{e^+} & 0 \end{bmatrix}_L \times i\not\partial$$

$$\times \begin{bmatrix} 0 & u_3^c & -u_2^c & u^1 & d^1 \\ -u_3^c & 0 & u_1^c & u^2 & d^2 \\ u_2^c & -u_1^c & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & e^+ \\ -d^1 & -d^2 & -d^3 & -e^+ & 0 \end{bmatrix}_L$$

$$= \{-\overline{u_L^c}{}^a i\not\partial u_{La}^c - \widehat{\left(\bar{u}_a \bar{d}_a\right)}_L \, i\not\partial \begin{pmatrix} u^a \\ d^a \end{pmatrix}_L - \overline{e_L^+} i\not\partial e_L^+$$

$$= -\bar{u}_R^a i\not\partial u_{Ra} - \bar{q}_{aL} i\not\partial q_L^a - \bar{e}_R^- i\not\partial e_R^- \qquad (2.1.134)$$

$$= (\bar{\psi}_L)_{ab} i\not\partial \psi_L^{ab} \quad .$$

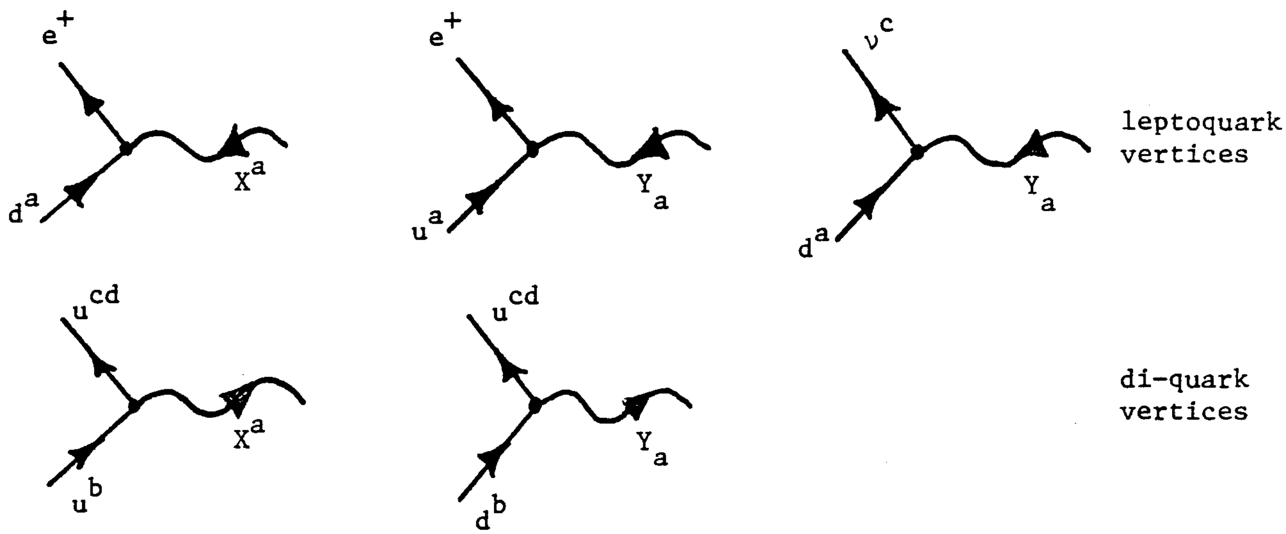
Putting all this together and regrouping the terms we get

$$
\begin{aligned}
L_F = {} & \bar{\ell}_L i\gamma^\mu[\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\not{\underline{A}}_\mu + \frac{3i}{\sqrt{60}} g B_\mu]\ell_L \\
& + \bar{q}_{aL} i\gamma^\mu[(+\partial_\mu - \frac{ig}{2}\underline{\sigma}\cdot\not{\underline{A}}_\mu - \frac{i}{\sqrt{60}} g B_\mu)\delta^a_b - \frac{ig}{2}(\underline{\lambda}\cdot\not{\underline{G}}_\mu)^a_b]q^b_L \\
& + \bar{e}^-_R i\gamma^\mu[\partial_\mu + \frac{i6g}{\sqrt{60}} B_\mu]e^-_R \\
& + \bar{u}_{aR} i\gamma^\mu[(\partial_\mu - \frac{i4g}{\sqrt{60}} B_\mu)\delta^a_b - \frac{ig}{2}(\underline{\lambda}\cdot\underline{G}_\mu)^a_b]u^b_R \\
& + \bar{d}_{aR} i\gamma^\mu[(+\partial_\mu + \frac{ig^2}{\sqrt{60}} B_\mu)\delta^a_b - \frac{ig}{2}(\underline{\lambda}\cdot\underline{G}_\mu)^a_b]d^b_R \\
& + \frac{g}{\sqrt{2}}\{\bar{X}^j_\mu[\bar{d}_{Lj}\gamma^\mu e^+_L - \bar{e}^-_L\gamma^\mu d^c_{Lj} + \varepsilon_{jik}\overline{u^c_L}\, i\gamma^\mu u^k_L] \qquad (2.1.135) \\
& + \bar{Y}^j_\mu[-\bar{u}_{Lj}\gamma^\mu e^+_L + \bar{\nu}_{eL}\gamma^\mu d^c_{Lj} + \varepsilon_{jik}\overline{u^c_L}\, i\gamma^\mu d^k_L] + \text{h.c.}\}
\end{aligned}
$$

Thus we see that the first 5 terms reproduce the SU(3) × SU(2) × U(1) standard model as long as we can identify $g = g_{SU(2)}$ $g' = \sqrt{\frac{3}{5}}\, g$ (this $\sqrt{3/5}$ comes from the normalization of the hypercharge Y) and $g_s = g$. At large momentum we can ignore the spontaneous symmetry of the SU(5) and indeed the coupling constants merge to be given by g (more on this later). The last two terms change quarks into leptons and vice-versa (i.e., they violate baryon and lepton number conservation).

2.2. Proton decay, unification mass, $\sin^2\theta w$, etc.

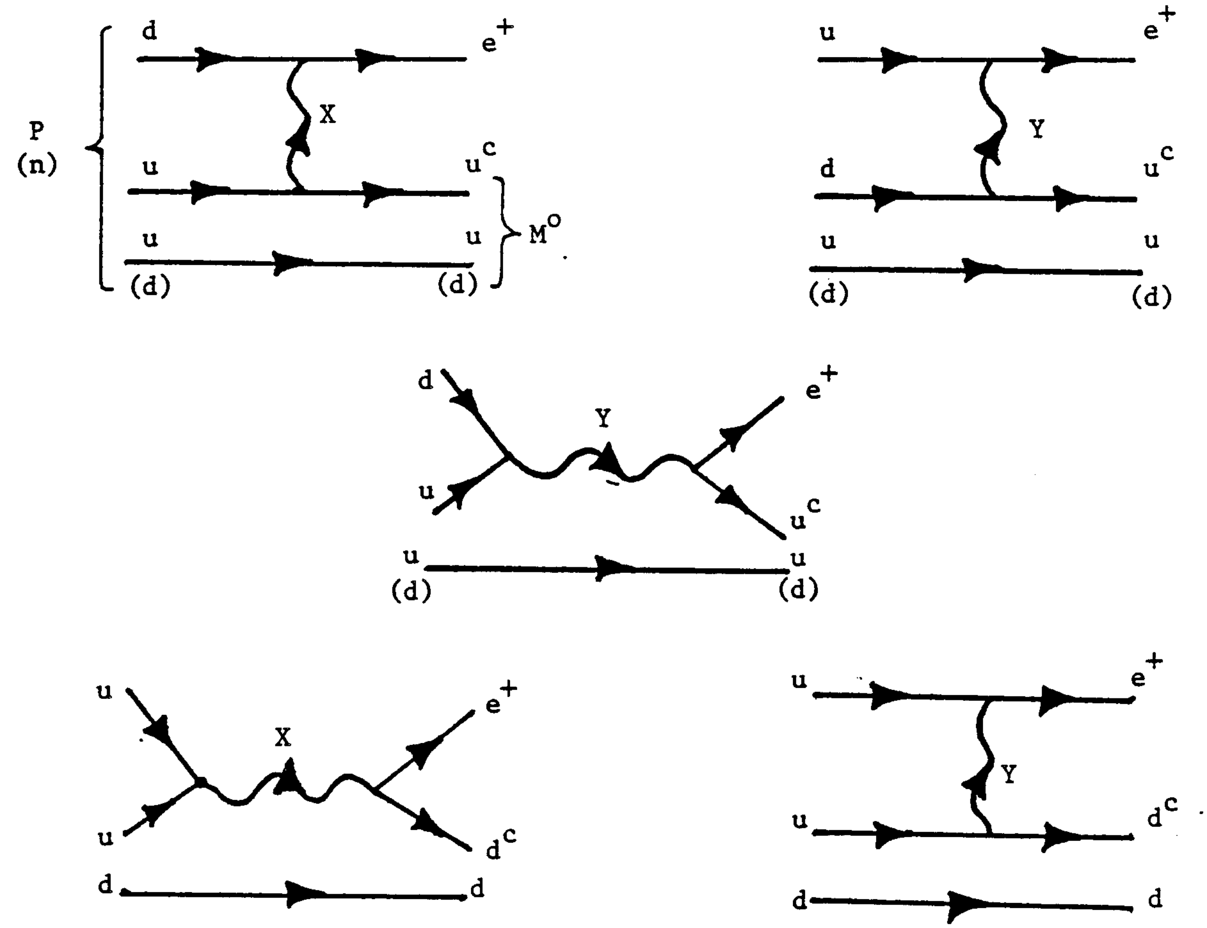
The vertices involving the X and Y bosons in L_F change the quarks into leptons and each other so the X and Y's are often called lepto-quark bosons or di-quark bosons. The Feynman diagrams corresponding to these vertices are



We can estimate the proton (neutron) decay rate from such an X or Y exchange.

The proton is a bound state of 2u and 1d quarks while the neutron is 2d and 1u quarks. Thus we have processes of the form $p \to e^+(\bar{q}q)$ where $(\bar{q}q)$ form a neutral meson such as $\pi^o, \rho^o, \omega, \eta$... Or $n \to e^+(\bar{u}d)$ where the $(\bar{u}d)$ can be π^-, ρ^-, etc.

The Feynman diagrams for such decay are



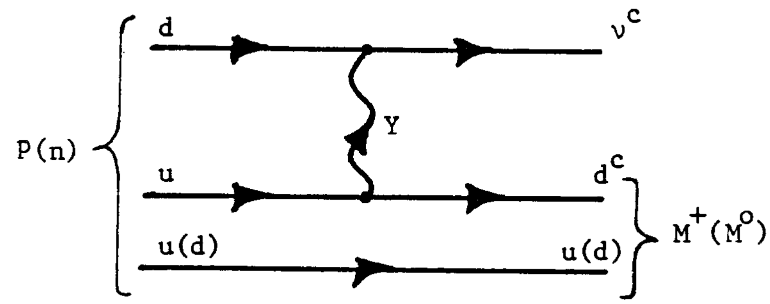
or the nucleon can decay into an anti-neutrino plus meson such as

$$p \to \bar{\nu} M^+ \qquad M^+ = \pi^+, \rho^+$$

or

$$n \to \bar{\nu} M^o \qquad M^o = \pi^o, \rho^o, \omega, \eta \text{ etc.}$$

for example



If we assume $M_x \sim M_y >> m_p$, the proton mass, then we can take the X,Y propagator to be $1/M_x^2$. Thus our Feynman amplitude goes as

$$\frac{g^2}{M_x^2} \sim \frac{\alpha_5}{M_x^2} \qquad \alpha_5 \equiv \frac{g^2}{4\pi} \quad . \tag{2.2.1}$$

then the proton lifetime should go as

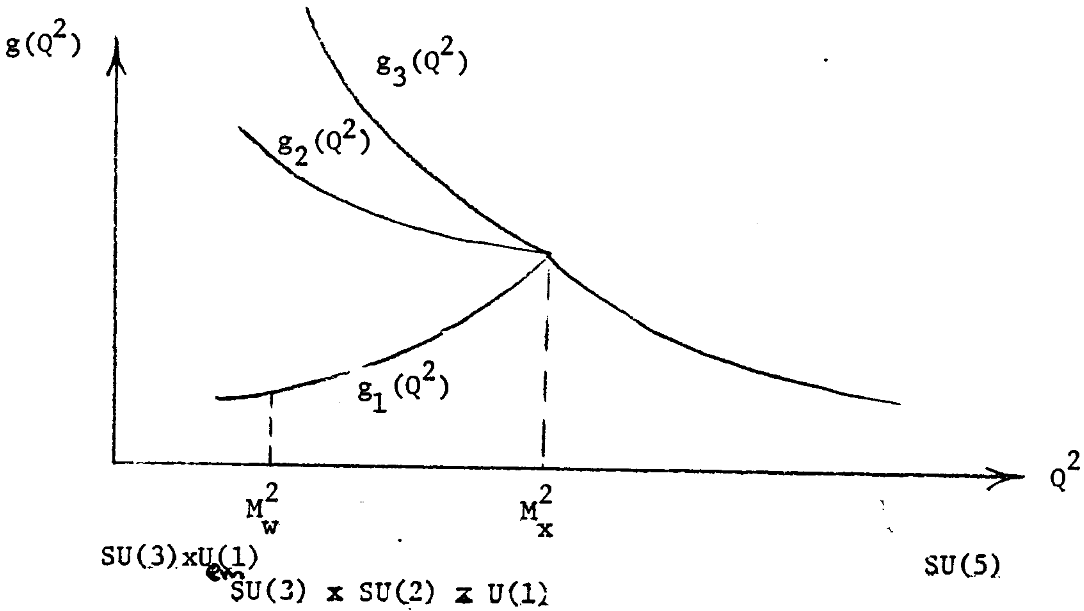
$$\tau_p \sim \frac{1}{\alpha_5^2} \frac{M_x^4}{m_p^5} \quad . \tag{2.2.2}$$

for $\alpha_5 \sim \alpha_{QED}$ and $\tau_p > 10^{30}$ yrs. the lower experimental limit we find

$$M_x \gtrsim 10^{14} \text{ GeV} \ . \tag{2.2.3}$$

We will next consider the spontaneous symmetry breaking of the SU(5) and independently determine M_x from knowing the low energy $g_s, g_{SU(2)}, g'$ and the RGE to be $\simeq 10^{14}$ GeV in agreement with our proton decay estimate.

There are two points of symmetry breakdown in the Georgi-Glashow model. First we must break the SU(5) symmetry to SU(3) x SU(2) x U(1) and then as usual we must break our SU(2) x U(1) down to $U_{em}(1)$. As drawn earlier we have the momentum behaviour for our coupling constants as shown below.



g_1, g_2, g_3 are the U(1), SU(2) and SU(3) effective coupling constants. Recall at energies > M_x we found that

$$g = g_2 = g_3 = \sqrt{\frac{5}{3}}\ g_1$$ all unification occurs at a single point M_x.

Remember

$$\sin^2\theta_w \equiv \frac{g'^2}{g^2 + g'^2} = \frac{g_1^2}{g_2^2 + g_1^2} = \frac{\frac{3}{5}}{1 + \frac{3}{5}} = \frac{3}{8} \qquad (2.2.4)$$

and

$$\frac{\alpha}{\alpha_s} = \frac{e^2}{g_s^2} = \frac{g^2 \sin^2\theta_w}{g_s^2} = \frac{g_2^2 \sin^2\theta_w}{g_3^2} = \frac{3}{8} \qquad (2.2.5)$$

experimentally $\sin^2\theta_w = .23 < 3/8$

and $$(Q^2 = 10\ \text{GeV}^2)\ \frac{\alpha}{\alpha_s} \simeq \frac{1}{25} < 3/8$$

But we know that $g_i = g_i(Q^2)$ and to compare our SU(5) predictions to these values of θ_w and α/α_5 found at low energy we must use the RGE. Recall our RGE for α, the fine structure constants

$$\lambda^2 \frac{\partial \bar{g}(\lambda^2,g)}{\partial \lambda^2} = \beta(\bar{g}) \quad \text{where } \bar{g}(1,g) = g \tag{2.2.6}$$

with

$$\beta_3 = -\frac{g_3^3}{32\pi^2}\left[11 - \frac{4}{3}F\right]$$

$$\beta_2 = -\frac{g_2^3}{32\pi^2}\left[\frac{22}{3} - \frac{4}{3}F\right] \tag{2.2.7}$$

$$\beta_1 = +\frac{g_1^3}{32\pi^2}\frac{20}{9}F \quad .$$

So in general for $\beta_i(\bar{g}_i) = b_i g_i^3$ and

$$\alpha_i(Q^2) = \frac{\bar{g}_i^2(\lambda^2)}{4\pi} \quad Q^2 = \lambda^2 M^2, M \text{ some mass scale or momentum scale} \tag{2.2.8}$$

then

$$\lambda^2 \frac{\partial \alpha_i}{\partial \lambda^2} = 8\pi b_i \alpha_i^2$$
$$= Q^2 \frac{\partial \alpha_i(Q^2)}{\partial Q^2} \tag{2.2.9}$$

with $\alpha_i(M^2) = \frac{g_i^2}{4\pi}$ we found

$$\alpha_i(Q^2) = \frac{\alpha_i(M^2)}{1 - 8\pi\alpha_i(M^2) b_i \ln Q^2/M^2} \tag{2.2.10}$$

That is

$$\frac{1}{\alpha_i(Q^2)} - \frac{1}{\alpha_i(M^2)} = -8\pi b_i \ln \frac{Q^2}{M^2} \quad . \tag{2.2.11}$$

Now for $M^2 = M_x^2$ in our SU(5) case as $Q^2 \simeq M_x^2$ we find $\alpha_i(Q^2) \simeq \alpha_i(M_x^2) = \frac{g_i^2}{4\pi}$. We find that θ_w and $\frac{\alpha}{\alpha_s}$ run, and are given by

$$\sin^2\theta_w = \frac{\bar{\alpha}_1}{\bar{\alpha}_1 + \bar{\alpha}_2} =$$

$$= \frac{\alpha_1}{\alpha_1 + \alpha_2 \left(\frac{1 - 8\pi\ \alpha_1 b_1\ \ln Q^2/M_x^2}{1 - 8\pi\ \alpha_2 b_2\ \ln Q^2/M_x^2} \right)} \tag{2.2.12}$$

$$\simeq \frac{\alpha_1}{\alpha_1 + \alpha_2\ [1 - 8\pi(\alpha_1 b_1 - \alpha_2 b_2) \ln Q^2/M_x^2}$$

$$\simeq \frac{\alpha_1}{\alpha_1 + \alpha_2} [1 + \frac{8\pi\ \alpha_2}{\alpha_1 + \alpha_2} [\alpha_1 b_1 - \alpha_2 b_2] \ln \frac{Q^2}{M_x^2}]$$

$$\sin^2\theta_w = \frac{3}{8} [1 + \frac{8\pi\ g^2}{4\pi(1 + \frac{3}{5})} [\frac{3}{5} b_1 - b_2] \ln \frac{Q^2}{M_x^2}] \tag{2.2.13}$$

Now $b_1 = \frac{1}{32\,\pi^2}\ \frac{20}{9}\ F$

$$b_2 = - \frac{1}{32\,\pi^2} (\frac{22}{3} - \frac{4}{3} F) \tag{2.2.14}$$

So $\frac{3}{5} b_1 - b_2 = \frac{11}{48\,\pi^2}$ independent of F.

With $\alpha_3 b_3 - \alpha_2 b_2 = \frac{g^2}{4\pi} (b_3 - b_2) = \frac{g^2}{4\pi} (-\frac{11}{3}) \frac{1}{32\,\pi^2}$ we

find

$$\sin^2\theta_w = \frac{3}{8}[1 + (\frac{g^2}{4\pi}) \frac{55}{48\,\pi} \ln \frac{Q^2}{M_x^2}] \tag{2.2.15}$$

and

$$\frac{\bar{\alpha}_{QED}}{\bar{\alpha}_s} = \frac{\bar{e}^2}{\bar{g}_s^2} = \frac{\bar{g}_2^2 \sin^2\bar{\theta}_w}{\bar{g}_s^2}$$

$$= \frac{\bar{\alpha}_2}{\bar{\alpha}_3} \sin^2\bar{\theta}_w$$

$$= \frac{\alpha_2}{\alpha_3} \left[\frac{1 - 8\pi\alpha_3 b_3 \ln Q^2/M_x^2}{1 - 8\pi\alpha_2 b_2 \ln Q^2/M_x^2}\right] \sin^2\bar{\theta}_w \qquad (2.2.16)$$

$$= \frac{\alpha_2}{\alpha_3} \frac{3}{8} \left[1 + \left(\frac{g^2}{4\pi}\right) \frac{55}{48\pi} \ln \frac{Q^2}{M_x^2}\right] \left[1 - 8\pi(\alpha_3 b_3 - \alpha_2 b_2) \ln \frac{Q^2}{M_x^2}\right]$$

$$= \frac{3}{8}\left[1 + \left(\frac{g^2}{4\pi}\right) \left[\frac{55}{48\pi} + \frac{11}{12\pi}\right] \ln \frac{Q^2}{M_x^2}\right]$$

yielding

$$\frac{\bar{\alpha}_{QED}}{\bar{\alpha}_s} = \frac{3}{8} \left[1 + \left(\frac{g^2}{4\pi}\right) \frac{33}{16\pi} \ln \frac{Q^2}{M_x^2}\right] \qquad (2.2.17)$$

and rewriting $\sin^2\theta_w$ we have

$$\sin^2\theta_w = \frac{1}{6} + \frac{5}{9} \frac{\bar{\alpha}_{QED}(Q^2)}{\bar{\alpha}_s(Q^2)} \qquad (2.2.18)$$

In general we find from QCD that

$$\frac{\bar{\alpha}_{QED}}{\bar{\alpha}_s} \simeq \frac{1}{25} \text{ at } Q^2 = 10 \text{ GeV}^2$$

i.e. $\bar{\alpha}_s = \dfrac{12\pi}{25 \ln Q^2/\Lambda^2}$ $\quad \Lambda \simeq .25\text{GeV} < (c,t)$ threshold

and $\alpha_{QED} \simeq \alpha(0)_{QED} \simeq \dfrac{1}{137.04}$

This implies

$$\sin^2\theta_w \simeq .19 \text{ at } Q^2 = 10 \text{ GeV}^2$$

This is close to the measured value of 0.23. This yields

$$M_x \simeq 10^{16} \text{ GeV and } \tau_p \geq 10^{37} \text{ yrs.}$$

Since $\tau_p \sim M_x^4$ it is very sensitive to corrections for M_x. If we more carefully apply the RGE and include Higgs and threshold effects

$$M_x \simeq 10^{14} \text{ GeV and } \tau_p \simeq 10^{30} \text{ yrs.}$$

and $\sin^2\theta_w \simeq .21$.

Before discussing the spontaneous symmetry breaking in more detail, let's review what we have done so far in building the SU(5) Georgi-Glashow Model.

Review of the SU(5) Georgi-Glashow Model

We have built our grand unified theory on the SU(5) group. This group has 24 generators which in the fundamental representation are given by

$(T^a_b)^c_d = \delta^c_b\delta^a_d - \frac{1}{5}\,\delta^a_b\delta^c_d$ (a,b,c,d = 1,...,5) obeying the defining algebra of SU(5)

$[T^a_b, T^c_d] = \delta^a_d T^c_b - \delta^c_b T^a_d$. The fundamental 5 dimensional representation is defined by the transformation property

$$\psi'^c = \psi^c + ig(\omega^b_a T^a_b)^c_d\,\psi^d \qquad (2.2.19)$$

$$= \psi^c + ig\,\omega^c_d\psi^d$$

and the fundamental $\bar{5}$ dimensional representation is defined by the transformation properties of the complex conjugate of ψ namely

$$\phi_c = (\psi^c)^\dagger$$

and

$$\phi'_c = \phi_c - ig(\omega^b_a T^a_b)^d_c\,\phi_d = \phi_c - ig\,\omega^d_c\phi_d. \qquad (2.2.20)$$

[ω^a_b are the 24 parameters describing the SU[5] transformations $\omega^a_a = 0$]. All higher dimensional tensor representations transform as the appropriate products of the 5 and $\bar{5}$ representations. In particular the matter fields were taken to belong to the $\bar{5}$ and 10 dimensional representation of SU(5) that is (for each generation)

$$\bar{5}:\quad \psi_{La} \equiv \begin{bmatrix} d^c_1 \\ d^c_2 \\ d^c_3 \\ e^- \\ \bar{\nu}_e \end{bmatrix}_L \qquad 10:\ \psi^{ab}_L = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u^c_3 & -u^c_2 & u^1 & d^1 \\ -u^c_3 & 0 & u^c_1 & u^2 & d^2 \\ u^c_2 & -u^c_1 & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & e^+ \\ -d^1 & -d^2 & -d^3 & -e^+ & 0 \end{bmatrix}^{ab}_L \qquad (2.2.21)$$

where under SU(5)

$$\psi'_{La} = \psi_{La} - ig\omega_a^b \psi_{Lb} \qquad (2.2.22)$$

and

$$\psi_L'^{ab} = \psi_L^{ab} + ig\omega_c^a \psi_L^{cb} + ig\omega_c^b \psi_L^{ac} \quad . \qquad (2.2.23)$$

The SU(3) x SU(2) x U(1) generators were embedded in the SU(5) generators according to

1) for SU(3) transformations generated by L_b^a with parameters λ_b^a we choose

$$\omega_b^a = \begin{cases} \lambda_b^a & \text{if } a,b = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda_a^a = 0$ then $\omega_a^b T_b^a = \lambda_a^b L_b^a$,

2) for SU(2) transformations generated by S_b^a with parameters s_b^a we choose

$$\omega_b^a = \begin{cases} s_{b-3}^{a-3} & a,b = 4,5 \\ 0 & \text{otherwise} \end{cases}$$

with $s_a^a = 0$. Then $\omega_a^b T_b^a = s_a^b S_b^a$

3) The U(1) hypercharge transformations are generated by

$$Y = -\frac{1}{3} \sum_{a=1}^{3} T_a^a + \frac{1}{2} \sum_{b=4}^{5} T_b^b$$

So we choose

$$\omega_b^a = \begin{cases} -\frac{1}{3}\, \theta \delta_b^a & \text{for } a,b = 1,2,3 \\ +\frac{1}{2}\, \theta \delta_b^a & \text{for } a,b = 4,5 \\ 0 \text{ otherwise} \end{cases}$$

Then

$$\omega_a^b T_b^a = \theta Y \quad .$$

So under the SU(3) x SU(2) x U(1) transformations the $\bar{5}$ and 10 have the properties

1) $\bar{5}$: a) ψ_{La}, a = 1,2,3 transforms as $(\bar{3}, 1, +\frac{1}{3})$

b) ψ_{La}: a = 4,5 transforms as $(1, \bar{2}, -\frac{1}{2})$

2) 10: a) ψ_L^{ab}; a = 1,2,3, b = 4,5 transforms as $(3,2, +\frac{1}{6})$

b) $\psi_L^{4,5}$ transforms as $(1,1, +1)$

c) $\phi_{La} \equiv \frac{1}{2}\,\varepsilon_{abc}\,\psi_L^{bc}$, a,b,c, = 1,2,3 transforms as $(\bar{3},1, -\frac{2}{3})$

verifying the choice of notation.

In addition we introduced the 24 gauge fields $A^a_{\mu b}$ which transform as the adjoint transformation under SU(5) (for global SU(5) trans.); for local transformations we have the inhomogeneous term also to give

$$\begin{aligned} A'^a_{b\mu} &= A^a_{b\mu} + \sqrt{2}\,\partial_\mu\,\omega^a_b \\ &\quad + ig\omega^a_c A^c_{b\mu} - ig\omega^c_b A^a_{c\mu} \quad . \end{aligned} \tag{2.2.24}$$

Under the SU(3) x SU(2) x U(1) transformations we found the transformation properties of $A^a_{b\mu}$

$$\begin{aligned} 24 &= (8, 1,0) \quad + \quad (1,3,0) \quad + \quad (1,1,0) \\ A^a_{b\mu} &= G^a_{b\mu} \quad + \quad W^\pm_\mu,\ W^o_\mu \quad + \quad B_\mu \\ &\quad + (3, \bar{2}, -\tfrac{5}{6}) \quad + \quad (\bar{3}, 2, +\tfrac{5}{6}) \\ &\quad + (\bar{X}^a, \bar{Y}^a) \quad + \quad (X_a, Y_a) \end{aligned} \tag{2.2.25}$$

where we summarized this decomposition in the matrix notation

$$A^a_{b\mu} = \begin{bmatrix} (G^1_1 - \frac{2B}{\sqrt{30}}) & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\ G^2_1 & (G^2_2 - \frac{2}{\sqrt{30}}B) & G^2_3 & \bar{X}^2 & \bar{Y}^2 \\ G^3_1 & G^3_2 & (G^3_3 - \frac{2}{\sqrt{30}}B) & \bar{X}^3 & \bar{Y}^3 \\ X_1 & X_2 & X_3 & (\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & (-\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) \end{bmatrix}^{ab}_{\mu} \qquad (2.2.26)$$

The SU(5) gauge invariant Lagrangian making up the Georgi-Glashow model will consist of 4 pieces

$$L^{SU(5)} = L_{ym} + L_F + L_\phi + L_{yuk} \qquad (2.2.27)$$

where so far we have discussed L_F and L_{ym} with

$$L_{ym} = -\frac{1}{4}\,\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \qquad (2.2.28)$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu] \qquad (2.2.29)$$

and

$$A'_\mu = A_\mu + \sqrt{2}\,\partial_\mu \omega + ig\,[\omega, A_\mu] \qquad (2.2.30)$$

implying

$$F'_{\mu\nu} = F_{\mu\nu} + ig[w, F_{\mu\nu}] \qquad (2.2.31)$$

thus leaving Tr $F_{\mu\nu}F^{\mu\nu}$ invariant. The fermion kinetic energy terms are given in terms of the covariant derivative

$$(D_\mu \psi_L)_a \equiv [\partial_\mu \delta^b_a + \frac{ig}{\sqrt{2}} A^b_{a\mu}]\psi_{Lb} \qquad (2.2.32)$$

and

$$(D_\mu \psi_L)^{ab} \equiv [\partial_\mu \delta^a_c \delta^b_d - \frac{ig}{\sqrt{2}} A^a_{c\mu}\, \delta^b_d - \frac{ig}{\sqrt{2}} A^b_{d\mu}\, \delta^a_c]\psi^{cd}_L \qquad (2.2.33)$$

thus

$$L_F = (\bar{\psi}_L)^a \, i(\not{D}\, \psi_L)_a - (\bar{\psi}_L)_{ab}\, i(\not{D}\, \psi_L)^{ab}$$

$$= (\bar{\psi}_L)^a \, [i\not{\partial}\delta_a^b - \frac{g}{\sqrt{2}} \not{A}_a^b]\psi_{Lb} \qquad (2.2.34)$$

$$-(\bar{\psi}_L)_{ab}\, [i\not{\partial}\delta_c^a + \frac{2g}{\sqrt{2}} \not{A}_c^a]\psi_L^{cb}$$

Under local SU(5) transformation $L_F' = L_F$ as desired. Before studying the Higgs sector we expanded the L_F in terms of the SU(3) x SU(2) x U(1) fields and found after a tedious amount of algebra:

$$L_F = \bar{\ell}_L i\, \gamma^\mu [\partial_\mu - \frac{ig}{2} \underline{\sigma}\cdot\underline{A}_\mu + \frac{3i}{\sqrt{60}} g\, B_\mu]\ell_L$$

$$+ \bar{q}_{aL}\, i\gamma^\mu [(\partial_\mu - \frac{ig}{2} \underline{\sigma}\cdot\underline{A}_\mu - \frac{i}{\sqrt{60}} gB_\mu)\delta_b^a - \frac{ig}{2} (\underline{\lambda}\cdot\underline{G}_\mu)_b^a] q_L^b$$

$$+ \bar{e}_R^-\, i\gamma^\mu [(\partial_\mu + \frac{i6g}{\sqrt{60}} B_\mu] e_R^- \qquad (2.2.35)$$

$$+ \bar{u}_{aR}\, i\gamma^\mu [\partial_\mu - \frac{i4g}{\sqrt{60}} B_\mu)\delta_b^a - \frac{ig}{2} (\underline{\lambda}\cdot\underline{G}_\mu)_b^a] u_R^b$$

$$+ \bar{d}_{aR}\, i\gamma^\mu [(\partial_\mu + \frac{2ig}{\sqrt{60}} B_\mu)\delta_b^a - \frac{ig}{2}(\underline{\lambda}\cdot\underline{G}_\mu)_b^a] d_R^b$$

$$+ \frac{g}{\sqrt{2}} \{\bar{X}_\mu^j [\ldots] + \bar{Y}_\mu^j [\ldots] + \text{h.c.}\} \quad .$$

Thus the first set of terms are just those of our standard SU(3) x SU(2) x U(1) GWS model but with $g_{SU(2)} = g_{SU(3)} = g' \sqrt{5/3} = g$

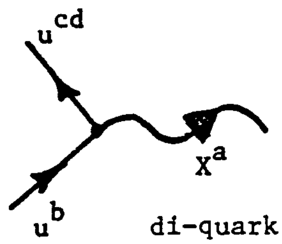
So

$$L_F = L^{GWS}(g = g_{SU(3)} = g_{SU(2)} = \sqrt{5/3}\, g_{U(1)})$$

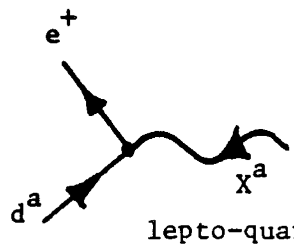
$$+ \frac{g}{\sqrt{2}} \{\bar{X}_\mu^j [\bar{d}_{L_j} \gamma^\mu e_L^+ - \bar{e}_L^- \gamma^\mu d_{L_j}^c + \varepsilon_{jik}\, \bar{u}_L^{ci} \gamma^\mu u_L^k] \qquad (2.2.36)$$

$$+ \bar{Y}_\mu^j [-\bar{u}_{L_j} \gamma^\mu e_L^+ + \bar{\nu}_{eL} \gamma^\mu d_{L_j}^c + \varepsilon_{jik} \bar{u}_L^{ci} \gamma^\mu d_L^k] + \text{h.c.}\}$$

From the last set of terms we are able to guestimate the nucleon lifetime. Since these terms lead to baryon number changing processes such as

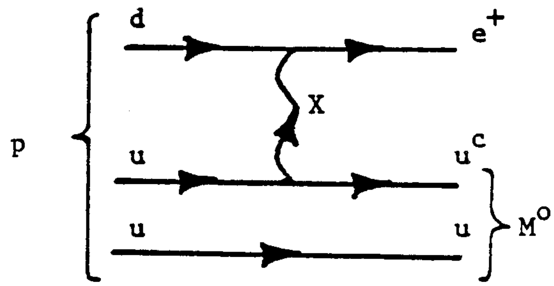


di-quark process



lepto-quark process

we have typical proton decay processes such as



Thus we could have $p \to e^+ \begin{pmatrix} \pi^o \\ \rho^o \\ \omega \\ \eta \end{pmatrix} \ldots$

Since when we do spontaneously break the SU(5) symmetry the X and Y IVB will have a large mass we can estimate such a process ,just as we related the GWS theory to the old 4-fermi theory, by replacing the X,Y propagators with $\frac{1}{M_x^2}$ for low energies. The proton decay amplitude will go as

$$\frac{g^2}{M_x^2} \sim \frac{\alpha_5}{M_x^2} \qquad \alpha_5 \equiv \frac{g^2}{4\pi} \tag{2.2.37}$$

Thus the proton lifetime should go as

$$\tau_p \sim \frac{1}{\alpha_5^2} \frac{M_x^4}{m_p^5} \quad . \tag{2.2.38}$$

We can independently estimate M_x from the assumption that the SU(3) x SU(2) x U(1) coupling constants all unify at the same energy, M_x,and use the RGE t relate our high energy (L_F) predictions for $\sin^2\theta_w$ and $\frac{\alpha_{em}}{\alpha_s}$ to the measured "low" energy values and hence obtain M_x and τ_p. For energies > M_x we found

$$g = g_2 = g_3 = \sqrt{\frac{5}{3}}\, g_1 . \tag{2.2.39}$$

Thus at high energy (> M_x)

$$\sin^2\theta_w \equiv \frac{g_1^2}{g_1^2 + g_2^2} = \frac{3}{8} \tag{2.2.40}$$

$$\frac{\alpha}{\alpha_s} \equiv \frac{e^2}{g_3^2} = \frac{g_2^2 \sin^2\theta_w}{g_3^2} = \frac{3}{8} .$$

Our renormalization group analysis told us that

$$Q^2 \frac{\partial\alpha_i(Q^2)}{\partial Q^2} = 8\pi b_i \alpha_i^2 \quad \text{where}$$

$$Q^2 = \lambda^2 M_x^2 \tag{2.2.41}$$

$$\beta_i(g_i) = b_i g_i^3 \quad \text{and} \quad \alpha_i(Q^2) = \frac{\bar{g}_i^2(\lambda^2)}{4\pi}$$

with the initial conditions that

$$\alpha_i(M_x^2) = \frac{g_i^2}{4\pi} \qquad g_i = \begin{cases} \left.\begin{matrix} g \\ g \end{matrix}\right\} & i = 2,3 \\ \sqrt{3/5}\,g \,\} & i = 1 \end{cases} . \tag{2.2.42}$$

The solution to this equation is

$$\frac{1}{\alpha_i(Q^2)} - \frac{1}{\alpha_i(M_x^2)} = -8\pi\, b_i \ln \frac{Q^2}{M_x^2}$$

and

$$b_3 = -\frac{1}{32\pi^2}\left[11 - \frac{4}{3}F\right] \tag{2.2.43}$$

$$b_2 = -\frac{1}{32\pi^2}\left[\frac{22}{3} - \frac{4}{3}F\right] \qquad , \quad F = \#\text{ of families}$$

$$b_1 = +\frac{1}{32\pi^2}\left(\frac{20}{9}F\right)$$

Putting all this together we found

$$\sin^2\theta_w \simeq \frac{3}{8}\left[1 + \left(\frac{g^2}{4\pi}\right)\frac{55}{48\pi}\ln\frac{Q^2}{M_x^2}\right] \quad (2.2.44)$$

$$\frac{\bar{\alpha}_{QED}}{\bar{\alpha}_s} \simeq \frac{3}{8}\left[1 + \left(\frac{g^2}{4\pi}\right)\frac{33}{16\pi}\ln\frac{Q^2}{M_x^2}\right]$$

Thus

$$\sin^2\theta_w \simeq \frac{1}{6} + \frac{5}{9}\frac{\bar{\alpha}_{QED}(Q^2)}{\bar{\alpha}_s(Q^2)} \quad (2.2.45)$$

From QCD we find $\frac{\bar{\alpha}_{QED}}{\bar{\alpha}_s} \simeq \frac{1}{25}$ at $Q^2 = 10$ GeV^2. Thus $\sin^2\theta_w \simeq 0.19$ at $Q^2 = 10$ GeV^2 from our above analyses. This is close to the exp. 0.23 value. This yields $M_x \approx 10^{16}$ and $\tau_p \geq 10^{37}$ yrs. After a more careful RG analysis including Higgs and heavy quark thresholds we find

$$M_x \approx 10^{14} \text{ GeV} \quad ; \quad \tau_p \simeq 10^{30} \text{ yrs.}$$

$$\sin^2\theta_w \simeq .21 \quad .$$

2.3 Spontaneous symmetry breaking and fermion masses

The SU(5) symmetry will be spontaneously broken to SU(3) x SU(2) x U(1) by allowing the SU(3) x SU(2) x U(1) invariant components of an adjoint (24 dim) representation of Higgs fields get a large ~ $0(10^{14}$ GeV) vacuum expectation value. Let's call this Higgs field ϕ^a_b. Then the Higgs doublet that breaks the SU(2) x U(1) → $U_{em}(1)$ will get a vacuum expectation value of 0(100 GeV). Since the 5 of SU(5) contains under SU(3) x SU(2) x U(1) transformations a weak doublet

$$5 = (3,1,-\tfrac{1}{3}) + (1,2,+\tfrac{1}{2})$$

we embed our usual Higgs in a 5 of SU(5) of Higgs fields, call it H^a. The $(3,1,-\frac{1}{3})$ fields in H^a will also mediate proton decay and so will have to be made very massive. Thus consider the two Higgs fields ϕ^a_b a 24 of SU(5) and H^a a 5 of SU(5). As with the gauge fields we can represent ϕ as a matrix

$$\phi^c_d = (\phi^b_a\, T^a_b)^c_d$$

$$\equiv \begin{bmatrix} (H^1_1 - \frac{2}{\sqrt{30}} H_B) & H^1_2 & H^1_3 & \bar{H}^1_X & \bar{H}^1_Y \\ H^2_1 & (H^2_2 - \frac{2}{\sqrt{30}} H_B) & H^2_3 & \bar{H}^2_X & \bar{H}^2_Y \\ H^3_1 & H^3_2 & (H^3_3 - \frac{2}{\sqrt{30}} H_B) & \bar{H}^3_X & \bar{H}^3_Y \\ H_{X1} & H_{X2} & H_{X3} & (\frac{H^o}{\sqrt{2}} + \frac{3H_B}{\sqrt{30}}) & H^+ \\ H_{Y1} & H_{Y2} & H_{Y3} & H^- & (-\frac{H^o}{\sqrt{2}} + \frac{3B}{\sqrt{30}}) \end{bmatrix}_{cd} \qquad (2.3.1)$$

where

$$24 = (8,\ 1,\ 0) \quad + \quad (1,\ 3,\ 0) \quad + \quad (1,\ 1,\ 0)$$

$$\phi^a_b = H^a_b \quad + \quad H^\pm,\ H^o \quad + \quad H_B$$

$$+ \quad (3, \bar{2},\ -5/6) \quad + \quad (\bar{3}, 2,\ +5/6)$$

$$+ \quad (\bar{H}^a_X \ \ \bar{H}^a_Y) \quad + \quad (H_{Xa},\ H_{Ya}) \tag{2.3.2}$$

Similarly the 5 can be written as

$$H^a = \begin{bmatrix} H^1 \\ H^2 \\ H^3 \\ \phi^+ \\ \phi^o \end{bmatrix} \tag{2.3.3}$$

where under SU(3) x SU(2) x U(1) transformations

$$H^{1,2,3} \text{ is a } (3, 1, -\tfrac{1}{3})$$

and

$$\begin{pmatrix} \phi^+ \\ \phi^o \end{pmatrix} \text{ is a } (1, 2, +\tfrac{1}{2}) . \tag{2.3.4}$$

As usual we construct the SU(5) gauge covariant derivatives for these fields. Under SU(5) transformations

$$H'^a = H^a + ig\ \omega^a_b H^b \tag{2.3.5}$$

and

$$\phi'^a_b = \phi^a_b + ig\ \omega^a_c \phi^c_b - ig\ \omega^c_b \phi^a_c \tag{2.3.6}$$

Thus

$$(D_\mu H)^a \equiv [\partial_\mu \delta^a_b - \frac{ig}{\sqrt{2}} A^a_{b\mu}]H^b \tag{2.3.7}$$

and

$$(D_\mu \phi)^a_b \equiv [\partial_\mu \delta^a_c \delta^d_b - \frac{ig}{\sqrt{2}} A_{\mu c}{}^a \delta^d_b + \frac{ig}{\sqrt{2}} A_{\mu b}{}^d \delta^a_c]\phi^c_d \tag{2.3.8}$$

In addition to the covariant kinetic energy terms

$$(D^\mu H)^\dagger_a \; (D_\mu H)^a \; , \; (D^\mu \phi)^b_a \; (D_\mu \phi)^a_b \tag{2.3.9}$$

we can also make quadratic, cubic and quartic invariants from these fields. As usual we have mass terms which are quadratic

$$\begin{aligned} \phi^b_a \phi^a_b &= \mathrm{Tr}\phi^2 \\ H^\dagger_a H^a &= H^\dagger H \end{aligned} \tag{2.3.10}$$

we can also make quartic terms from these

$$\begin{matrix} (\mathrm{Tr}\phi^2)^2 & \text{also} \quad \phi^b_a \phi^a_c \phi^c_d \phi^d_b = \mathrm{Tr}\phi^4 \text{ is SU(5) inv. as} \\ (H^\dagger H)^2 & \end{matrix} \tag{2.3.11}$$

are $H^\dagger H \; \mathrm{Tr}\phi^2$ and $H^\dagger_a \phi^a_b \phi^b_c H^c = H^\dagger \phi\phi H$.

Also there are cubic invariants

$$\mathrm{Tr}\phi^3 \quad \text{and} \quad H^\dagger \phi H \quad . \tag{2.3.12}$$

So we have exhausted all possible invariants up to and including dimension 4 (renormalizable) in the fields ϕ and H, hence L_ϕ is given by

$$\begin{aligned} L_\phi &= (D_\mu H)^\dagger_a (D^\mu H)^a + \frac{1}{2}(D_\mu \phi)^b_a (D^\mu \phi)^a_b - V \\ &= (D_\mu H)^\dagger (D^\mu H) + \frac{1}{2} \mathrm{Tr}[(D_\mu \phi) D^\mu \phi)] - V \end{aligned} \tag{2.3.13}$$

where the SU(5) invariant potential is given by

$$\begin{aligned} V = &- \frac{\mu^2}{2} \mathrm{Tr}\phi^2 + \frac{a}{4}(\mathrm{Tr}\phi^2)^2 + \frac{b}{2} \mathrm{Tr}\phi^4 + \frac{c}{3} \mathrm{Tr}\phi^3 \\ &- \frac{\nu^2}{2} H^\dagger H + \frac{\lambda}{4} (H^\dagger H)^2 \\ &+ \alpha H^\dagger H \, \mathrm{Tr}\phi^2 + \beta H^\dagger \phi^2 H + \gamma H^\dagger \phi H \end{aligned} \tag{2.3.14}$$

with $\mu^2 > 0$, $\nu^2 > 0$ and the "wrong" sign on the mass terms is already made explicit so as to spontaneously break the SU(5) → SU(3) x SU(2) x U(1) → SU(3) x $U_{em}(1)$. In order to simplify our calculations we will impose a discrete $\phi \to -\phi$ symmetry at the SU(5) level thus $C = \gamma = 0$. Since it is discrete when it is spontaneously broken no Goldstone bosons will arise. Now we must find the absolute minimum of this potential since we must perturb about it. Since we desire to break to SU(3) x $U_{em}(1)$ we must give SU(3) singlets and charge-zero fields vacuum values. Thus in the 24 this corresponds to H_B and H^o and in the 5 ϕ^o. We define the vacuum values (to agree with BEGN: Buras, Ellis, Gaillard and Nanopaulos, Nucl. Phys. B135 (1978) 66) to be

$$\langle 0|H|0\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{v_o}{\sqrt{2}} \end{bmatrix} \qquad \underline{\text{i.e.}}\ \langle 0|\phi_o|0\rangle = \frac{1}{\sqrt{2}} v_o$$

$$\langle 0|\phi|0\rangle = \begin{bmatrix} v & & & & 0 \\ & v & & & \\ & & v & & \\ & & & (-\frac{3}{2} - \frac{1}{2}\varepsilon)v & \\ 0 & & & & (-\frac{3}{2} + \frac{1}{2}\varepsilon)v \end{bmatrix} \tag{2.3.15}$$

$\underline{\text{i.e.}}$ $\langle 0|H_B|0\rangle = -\frac{\sqrt{30}}{2}\ v$

$\langle 0|H^o|0\rangle = -\frac{1}{\sqrt{2}}\ \varepsilon v$.

To find the minimum we must differentiate V

$$L\delta_a^b + \left.\frac{\partial V}{\partial \phi_b^a}\right|_{v,\varepsilon,v_o} = 0 \qquad \left.\frac{\partial V}{\partial H_a^\dagger}\right|_{v,\varepsilon,v_o} = 0 \quad \text{and} \tag{2.3.16}$$

$$\frac{\partial}{\partial L}(V + L\phi_a^a) = \phi_a^a = 0$$

where L is the lagrange multiplier enforcing the $\phi_a^a = 0$ constraint.

0) $\phi_a^a = 0$

$$\text{1)}\quad \Big[-\mu^2 \phi_a^b + a\ \text{Tr}\phi^2\phi_a^b + 2b\phi_c^b\phi_d^c\phi_a^d + L\delta_a^b + 2\alpha\ H^\dagger H\phi_a^b + \beta[H_a^\dagger\phi_c^b H^c + H_c^\dagger\phi_a^c H^b]\Big]_{\substack{\phi = \langle\phi\rangle \\ H = \langle H\rangle}} = 0 \tag{2.3.17}$$

$$2)\quad -\frac{\nu^2}{2} H^a + \frac{\lambda}{2}(H^{\dagger}H)H^a + \alpha H^a \,\mathrm{Tr}\phi^2 + \beta(\phi^2 H)^a \Big|_{\substack{\phi = \langle\phi\rangle \\ H = \langle H\rangle}} = 0 \quad .$$

Thus we have 3 equations for v, v_o, ε, from 2) above we find

$$\text{iii)}\quad \nu^2 = \frac{\lambda v_o^2}{2} + 15\alpha v^2 + \alpha\varepsilon^2 v^2 \tag{2.3.18}$$

$$+ \frac{9}{2}\beta v^2 - 3\beta\varepsilon v^2 + \frac{\beta}{2}\varepsilon^2 v^2 = 0$$

We can solve for L by taking trace of 1 then we can eliminate it from our equations using $\phi^a_a = 0$ this implies

$$5L + 2b\,\mathrm{Tr}[\phi^3] + 2\beta\, H^{\dagger}\phi H = 0 \tag{2.3.19}$$

Hence

$$\mu^2\phi^b_a = a\,\mathrm{Tr}\phi^2\phi^b_a + 2\alpha H^{\dagger}H\phi^b_a$$

$$+ 2b(\phi\phi\phi)^b_a - \frac{2}{5}b\delta^b_a \mathrm{Tr}\phi^3$$

$$+ \beta[H^{\dagger}_a\phi^b_c H^c + H^{\dagger}_c\phi^c_a H^b] - \frac{2}{5}\beta H^{\dagger}\phi H\delta^b_a \tag{2.3.20}$$

for a = b = 1 we find

$$\text{a)}\quad \mu^2 = a[\frac{15}{2} + \frac{\varepsilon^2}{2}]v^2 + \alpha\, v_o^2 + 2bv^2$$

$$- \frac{2}{5}b[3 - (\frac{3}{2} + \frac{\varepsilon}{2})^3 - (\frac{3}{2} - \frac{\varepsilon}{2})^3]v^2 \tag{2.3.21}$$

$$+ \frac{2}{5}\beta\,\frac{v_o^2}{2}(\frac{3}{2} - \frac{\varepsilon}{2})$$

for a = b = 4 we find

$$\text{b)}\quad -(\tfrac{3}{2} + \tfrac{\varepsilon}{2})v\mu^{2} = -a[\tfrac{15}{2} + \tfrac{\varepsilon^2}{2}]v^2\ (\tfrac{3}{2} + \tfrac{\varepsilon}{2})v$$

$$- \alpha v_o^2(\tfrac{3}{2} + \tfrac{\varepsilon}{2})v + 2b(-1)(\tfrac{3}{2} + \tfrac{\varepsilon}{2})^3 v^3$$

$$- \tfrac{2}{5}\, b[3 - (\tfrac{3}{2} + \tfrac{\varepsilon}{2})^3 - (\tfrac{3}{2} - \tfrac{\varepsilon}{2})^3]v^3 \qquad (2.3.22)$$

$$+ \tfrac{2}{5}\beta\, \frac{v_o^2}{2}\, (\tfrac{3}{2} - \tfrac{\varepsilon}{2})v$$

for a = b = 5 we find

$$\text{c)}\quad -(\tfrac{3}{2} - \tfrac{\varepsilon}{2})v\mu^2 = -a(\tfrac{15}{2} + \tfrac{\varepsilon^2}{2})\ v^2(\tfrac{3}{2} - \tfrac{\varepsilon}{2})v$$

$$-\alpha v_o^2(\tfrac{3}{2} - \tfrac{\varepsilon}{2})v - 2b(\tfrac{3}{2} - \tfrac{\varepsilon}{2})^3 v^3 \qquad (2.3.23)$$

$$- \tfrac{2}{5}b[3 - (\tfrac{3}{2} + \tfrac{\varepsilon}{2})^3 - (\tfrac{3}{2} - \tfrac{\varepsilon}{2})^3]v^3$$

$$+ \tfrac{2}{5}\beta\, \frac{v_o^2}{2}\, (\tfrac{3}{2} - \tfrac{\varepsilon}{2})v$$

$$+ 2\beta\, \frac{v_o^2}{2}\, (-1)(\tfrac{3}{2} - \tfrac{\varepsilon}{2})v$$

Of course we can obtain condition c by adding -(3 x (a) + (b)) this is just the vanishing trace condition. So we have

$$\text{a)}\quad \mu^2 = a[\tfrac{15}{2} + \tfrac{\varepsilon^2}{2}]v^2 + \alpha\ v_o^2 + 2bv^2 \qquad (2.3.24)$$

$$- \tfrac{2}{5}\, b[3 - (\tfrac{3}{2} + \tfrac{\varepsilon}{2})^3 - (\tfrac{3}{2} - \tfrac{\varepsilon}{2})^3]v^2$$

$$+ \tfrac{2}{5}\, \beta\, \frac{v_o^2}{2}\, (\tfrac{3}{2} - \tfrac{\varepsilon}{2})$$

$$\text{b)}\quad (\tfrac{3}{2}+\tfrac{\varepsilon}{2})\mu^2 = a[\tfrac{15}{2}+\tfrac{\varepsilon^2}{2}]\, v^2(\tfrac{3}{2}+\tfrac{\varepsilon}{2}) + \alpha v_o^2(\tfrac{3}{2}+\tfrac{\varepsilon}{2})$$

$$+ 2b(\tfrac{3}{2}+\tfrac{\varepsilon}{2})^3\, v^2 \tag{2.3.25}$$

$$+ \tfrac{2}{5}\, b[3 - (\tfrac{3}{2}+\tfrac{\varepsilon}{2})^3 - (\tfrac{3}{2}-\tfrac{\varepsilon}{2})^3]v^2$$

$$- \tfrac{2}{5}\,\beta\,\frac{v_o^2}{2}\,(\tfrac{3}{2}-\tfrac{\varepsilon}{2}) \quad .$$

Add these 2 and simplify to obtain our 3 equations determining v, v_o, ε. Thus our 3 equations for v, v_o, ε are

$$\text{i)}\quad [1+\frac{1}{50}\frac{\beta v_o^2}{bv^2}]\varepsilon + \frac{1}{100}\frac{\beta v_o^2}{bv^2}\varepsilon^2 - \frac{1}{25}\varepsilon^3 = \frac{3}{20}\frac{\beta v_o^2}{bv^2}$$

(2,3.26)

$$\text{ii)}\quad \mu^2 = \frac{15}{2}av^2 + \frac{7}{2}bv^2 + \alpha v_o^2 + \frac{3}{10}\beta v_o^2$$

$$+ \frac{av^2}{2}\varepsilon^2 + \frac{9}{10}bv^2\,\varepsilon^2 - \frac{\beta}{10}v_o^2\varepsilon$$

$$\text{iii)}\quad \nu^2 = \frac{\lambda v_o^2}{2} + 15\,\alpha v^2 + \frac{9}{2}\beta v^2 - 3\beta ev^2$$

$$+ (\alpha + \frac{\beta}{2})v^2\varepsilon^2 \quad .$$

Let's investigate these solutions first by ignoring the SU(2) x U(1) breaking <u>i.e.</u> setting $v_o = \varepsilon = 0$ and $\beta = \alpha = 0$ and so we are left with eq. ii only

$$\mu^2 = \frac{v^2}{2}(15a + 7b) \tag{2.3.27}$$

This mass should be very large since the X, Y gauge bosons have mass squared

$$M_x^2 = M_y^2 = \frac{25}{8}g^2v^2 \tag{2.3.28}$$

Thus $v^2 \sim M_x^2$. (Further the potential can be shown to be positive for

$a > -\frac{7b}{15}$ and that $b > 0$ for this SU(3) x SU(2) x U(1) extrema to be an absolute minimum). Next we could add the pure H potential, but keeping $\alpha = \beta = 0$ still, in order to break SU(2) x U(1) down to $U_{em}(1)$. Then $\varepsilon = 0$ but $v_o \neq 0$ and we find still $\mu^2 = \frac{v^2}{2}(15a + 7b)$ and now

$$\nu^2 = \frac{\lambda v_o^2}{2} \qquad (2.3.29)$$

with the W and Z masses given by $M_W^2 = \frac{g^2 v_o^2}{4}$. So while

$$v \sim 0(M_X)$$

$$v_o \sim 0(M_W) \quad .$$

Unfortunately the colorful Higgs fields H^a a = 1,2,3 and H_Y^a a = 1,2,3 mix so that one linear combination is eaten by the Y gauge boson to give it mass but the orthogonal combination remains massless; further these can result in proton decay - yielding too short a life-time. Thus we indeed need the mixed ϕ-H potential terms which solve this difficulty.

When we add in all terms to the potential it can be shown for $b > 0$, $\beta < 0$; $a > -\frac{7b}{15}$ that the SU(3) x $U_{em}(1)$ is an abs. minimum and for $\alpha,\beta \to 0$ this solution should yield our previous v, v_o solutions. So $\varepsilon \to 0$ as $\alpha,\beta \to 0$ and $\varepsilon \ll 1$. Thus equation i yields

$$\varepsilon = \frac{3}{20} \frac{\beta v_o^2}{b v^2} + 0\left(\frac{v_o^4}{v^4}\right) \ll 1. \qquad (2.3.30)$$

while μ^2 is large so we can neglect the $\beta\varepsilon$ and ε^2 terms yielding

$$\mu^2 = \frac{15}{2} a v^2 + \frac{7}{2} b v^2 + \alpha v_o^2 + \frac{3}{10} \beta v_o^2 \quad . \qquad (2.3.31)$$

The last 2 terms being a small perturbation to our previous μ^2 equation. The third equation, ignoring the highest order correction, yields

$$\nu^2 = \frac{\lambda v_o^2}{2} + 15\,\alpha v^2 + \frac{9}{2}\beta v^2 - 3\beta\varepsilon v^2 \quad . \qquad (2.3.32)$$

The first thing we notice is that the W, Z masses will be of order ν and we do not want ν to be very large thus we will require a delicate cancellation amongst the v^2 (very large) terms in order to keep ν^2 small. This adjustment or "fine tuning" of the α,β parameters is an unasthetic aspect of the model and is known as the hierarchy problem or naturalness problem. In order to keep the hierarchy of masses $M_x >> M_w$ or $v >> v_o$ we must fine-tune the parameters in the theory to one part in $\frac{v^2}{v_o^2} \sim 10^{24}$! Further when radiative corrections to the potential are taken into account this fine-tuning of the parameters persists and will involve re-adjusting the cancellations in each order of perturbation theory. However supersymmetric theories, since their renormalization properties are improved compared to ordinary field theories, will obviate this fine-tuning problem in each order of perturbation theory.

Once the parameters are adjusted in the tree approximation the resulting hierarchy will be preserved as long as the supersymmetry is maintained.

Adjusting the parameters so that $v_o << v$, we find that previously massless, colorful Higgs field now has $[\text{mass}]^2 \sim \beta v^2 \sim 0\left(M_x^2\right)$.

Although this Higgs exchange leads to proton decay we assume its contribution is small compared to the X,Y exchange since the Yukawa couplings are in general small compared to the gauge coupling.

This will complete our discussion of L_ϕ. It is left as an exercise to calculate the various mass matrices in this sector! Finally we study the Yukawa interaction of the Higgs fields and fermions and find the fermion masses. Since the $\bar{5}$ and 10 are both left-handed fields we first review what a mass term looks like for these. Recall a mass term mixes the left and right handed fields and our right handed fields are charge conjugate fields

$$\overline{\psi_R^c} = -\psi_L^T C^{-1} = \psi_L^T C \quad . \tag{2.3.33}$$

Thus the mass terms are of the form

$$\overline{\psi_R^c}\phi_L + \text{h.c.} = \overline{\psi_R^c}\phi_L + \bar{\phi}_L\psi_R^c$$

$$= \psi_L^T C\phi_L + \bar{\phi}_L C\bar{\psi}_L^T \quad . \tag{2.3.34}$$

The Yukawa terms are of the same form with Higgs scalars; for the SU(5) we have the bilinear matter fields

$$\psi_{La}^T \, C\psi_{Lb} + \text{h.c.}$$

$$\psi_{La}^T \, C\psi_L^{bc} + \text{h.c.} \tag{2.3.35}$$

$$\psi_L^{Tab} \, C\psi_L^{cd} + \text{h.c.}$$

These products decompose into irreducible representations according to

$$\bar{5} \times \bar{5} = \overline{10} + \overline{15}$$

$$\bar{5} \times 10 = 5 + 45 \tag{2.3.36}$$

$$10 \times 10 = \bar{5} + \overline{45} + 50$$

We note that none of these products contains the adjoint representation 24; thus there is no Yukawa coupling to the ϕ_b^a Higgs field. This is good since $v \sim M_x$ and this would imply some of our matter fields would have too large a mass.

In order to obtain a more realistic fermion mass spectrum we must introduce additional Higgs fields i.e. the 45 to couple to these bilinear forms. However, for "simplicity" we will consider our model with only the 5 and 24 of Higgs. The only Yukawa couplings are then

$$\psi^T_{La}\ C\ \psi^{ab}_L\ H^\dagger_b + h.c.$$

and

$$\epsilon_{abcde}\ \psi^{Tab}_L\ C\ \psi^{cd}_L\ H^e + h.c. \qquad (2.3.37)$$

where ϵ_{abcde} is the 5 dimensional totally anti-symmetric tensor. It is this 2nd term which will also contribute to proton decay. Once again we can introduce inter-generation mixing so that the most general Yukawa lagrangian has the form

$$L_{yuk} = \gamma_{mn}\ \psi^T_{mLa}\ C\ \psi^{ab}_{nL}\ H^\dagger_b + \Gamma_{mn}\epsilon_{abcde}\ \psi^{Tab}_{mL}\ C\ \psi^{cd}_{nL}\ H^e + h.c. \qquad (2.3.38)$$

where $\Gamma_{mn} = \Gamma_{nm}$ since $\psi^{Tab}_{mL}\ C\ \psi^{cd}_{nL}\ \epsilon_{abcde} H^e$ is m-n symmetric.

The fermion mass terms are obtained from $\langle 0|H^a|0\rangle = \frac{1}{\sqrt{2}} v_o\ \delta^a_5$. Studying first the $\bar{5}$ x 10 term

$$\gamma_{mn}\ \psi^T_{mLa}\ C\ \psi^{a5}_{nL}\ \frac{1}{\sqrt{2}}\ v_o$$

$$= \frac{v_o}{\sqrt{2}}\ \gamma_{mn}\ \overline{\psi^c_{mRa}}\ \psi^{a5}_{nL} \qquad (2.3.39)$$

Now

$$\psi_{La} = \begin{pmatrix} d^c_i \\ e^- \\ -\nu_e \end{pmatrix}_L \ ; \quad \text{So} \qquad (2.3.40)$$

$$\psi^{ca}_R = C\ \bar{\psi}^T_{La}$$

$$= \begin{bmatrix} C\ \bar{d}^{cTi}_L \\ C\ \bar{e}^{-T}_L \\ -C\bar{\nu}^T_{eL} \end{bmatrix} \qquad \text{but } C\ \bar{d}^{cT}_L = d_R \qquad (2.3.41)$$

$$\text{and } C\bar{\nu}^T_{eL} = \nu^c_R$$

$$C\bar{e}^{-T}_L = C\bar{e}^{+c\,T}_L = e^+_R$$

So

$$\psi_R^{ca} = \begin{bmatrix} d^i \\ e^+ \\ -\nu^c \end{bmatrix}_R \quad ; \tag{2.3.42}$$

$$\overline{\psi^c_{mRa}}\, \psi^{a5}_{nL} = \frac{1}{\sqrt{2}} (\bar{d}_{mRi}\, d^i_{nL} + \overline{e^+_{mR}}\, e^+_{nL}) \quad .$$

The mass term becomes

$$\gamma_{mn}\, \psi^T_{mLa}\, C\, \psi^{a5}_{nL}\, \frac{1}{\sqrt{2}} v_o + \text{h.c.}$$

$$= \frac{1}{2} v_o \gamma_{mn} [\bar{d}_{mRi} d^i_{nL} + \overline{e^+_{mR}}\, e^+_{nL}] + \text{h.c.} \tag{2.3.43}$$

$$= \bar{d}_R\, M^d\, d_L + \overline{e^+_R}\, M^e\, e^+_L + \text{h.c.}$$

where

$$M^d_{mn} = M^e_{mn} \equiv \frac{v_o}{2} \gamma_{mn}. \tag{2.3.44}$$

(M^d here is the adjoint of the d-mass matrix in the GWS model). Thus SU(5) with the minimal Higgs fields requires the d-quark and electron masses to be the same. That is

$$\begin{aligned} m_d &= m_e \\ m_s &= m_\mu \\ m_b &= m_\tau \quad . \end{aligned} \tag{2.3.45}$$

This result should be interpreted, as with the equality of coupling constants result, as a prediction at $Q^2 \geq M_x^2$. The RG analysis can be applied to the fermion masses to obtain for $Q^2 < M_x^2$ [BEGN]

$$\ln \left[\frac{m^{d,s,b}(Q^2)}{m_{e,\mu,\tau}(Q^2)} \right] = \ln \left[\frac{m^{d,s,b}(M_x^2)}{m_{e,\mu,\tau}(M_x^2)} \right]$$

$$+ \frac{4}{11 - \frac{4F}{3}} \ln \left[\frac{\alpha_3(Q^2)}{\alpha_5(M_x^2)}\right] + \frac{3}{4F} \ln \left[\frac{\alpha_1(Q^2)}{\alpha_5(M_x^2)}\right] \quad . \qquad (2.3.46)$$

This yields m_b = 5.3 GeV (for QCD Λ = 300 MeV) where the accepted value is around 5 GeV. For $m_s \simeq$ 470 MeV which is a bit high compared to the 150 - 300 MeV accepted values. The mass of the lighter d quark is quite uncertain so a meaningful comparison is unattainable, however, the ratio $m_d/m_s \simeq \frac{1}{24}$ is obtainable from meson, baryon mass spectra. The SU(5) prediction is

$$\frac{m_d}{m_s} = \frac{m_e}{m_\mu} = \frac{1}{207} \text{ ; quite a difference} \quad .$$

Thus the minimal Higgs SU(5) model is not the most satisfactory model; one can introduce a 45 of Higgs in addition to the 5 to yield a correct prediction for m_d/m_s.

Next we consider the other mass term

$$\begin{aligned}
&\Gamma_{mn}\, \varepsilon_{abcd5}\, \psi^{Tab}_{mL}\, C\psi^{cd}_{nL} \frac{v_o}{\sqrt{2}} + \text{h.c.} \\
&= \frac{4v_o}{\sqrt{2}} \Gamma_{mn}\, \varepsilon_{abc45}\, \psi^{Tab}_{mL}\, C\psi^{c4}_{nL} + \text{h.c.} \\
&= \frac{4v_o}{\sqrt{2}} \Gamma_{mn}\, \varepsilon_{abc}\, \psi^{Tab}_{mL}\, C\psi^{c4}_{nL} + \text{h.c.} \\
&= \frac{4v_o}{\sqrt{2}} \Gamma_{mn}\, \varepsilon_{ijk}\, \psi^{Tij}_{mL}\, Cu^{k}_{nL} + \text{h.c.} \\
&= \frac{8v_o}{\sqrt{2}} \Gamma_{mn}\, u^{cT}_{mL_i}\, Cu^{i}_{nL} + \text{h.c.} \\
&= \frac{8v_o}{\sqrt{2}} \Gamma_{mn}\, \bar{u}_{mRi}\, u^{i}_{nL} + \text{h.c.} \\
&= \bar{u}_R M^u u_L + \text{h.c.}
\end{aligned} \qquad (2.3.47)$$

where

$$M^u_{mn} = \frac{8v_o}{\sqrt{2}} \Gamma_{mn} = M^u_{nm} \quad . \tag{2.3.48}$$

Thus we can combine these fermion mass terms to yield

$$L_{mass} = \bar{u}^w_R M^u u^w_L + \bar{d}^w_R M^d d^w_L$$
$$+ \bar{e}^{+w}_R M^e e^{+w}_L + \text{h.c.} \tag{2.3.49}$$

where with minimal Higgs

$M^u = M^{uT}$; $M^d = M^e$, otherwise they are arbitrary

and the superscript W reminds us we are in the weak interaction basis not the mass eigenstate basis. As in the GWS model we can diagonalize these 3 x 3 matrices by L,R matrices,

$$M_{diag.} = \begin{pmatrix} m_1 & & 0 \\ & m_2 & \\ 0 & & m_3 \end{pmatrix} = A_L^\dagger M A_R \tag{2.3.50}$$

for $M^{u,d,e}$. As previously the $A_{L,R}$ are determined up to diagonal phase matrices $K_{L,R}$ by diagonalizing $MM^\dagger$, $M^\dagger M$. The phase differences $K_L^\dagger K_R$ are determined by demanding $M_{diag.}$ to be real and positive. So $K_L = K_R = K$ is arbitrary and can be used to put the matrices in a convenient conventional form.

Recall in the standard GWS model only the K-M matrix was observable

$A_{KM} = A_L^{d\dagger} A_L^u$. In general we will be able to observe the $A_R^{u,d}$ and A_L^{e+} in the SU(5) model through the lepto-quark and di-quark interactions and when the Higgs structure is non-minimal. However, for our simplified case of the 5 and 24 of Higgs only we have $M^d = M^e$ so that $A^d_{L,R} = A^{e+}_{L,R}$. Also $M^u = M^{uT}$ so

$$(A_L^{u\dagger} M^u A_R^u = M^u_{diag.})^T$$
$$= M^u_{diag.} = A_R^{uT} M^u A_L^{u*}$$
$$= A_L^{u\dagger} M^u A_R^u \tag{2.3.51}$$

this implies $A_R^u = A_L^{u*}K$ where K is a diagonal matrix of phases uniquely determined by $M_{diag.}$ being real and positive. (2.3.52)

Since, if $A_R^u = A_L^{u*}K$, then

$$A^{u\dagger} = K^\dagger A_R^{uT}$$

So

$$\begin{aligned} A_L^{u\dagger} M^u A_R^u &= K^\dagger A_R^{uT} M^u A_L^{u*} K \\ &= K^\dagger M^u_{diag.} K = M^u_{diag.} \\ &= A_R^{uT} M^u A_L^{u*} \quad . \end{aligned} \tag{2.3.53}$$

Thus we see that the charged weak current, since it involves left-handed fields will still only depend on the K-M matrix ,

$$A_{KM} = A_L^{d\dagger} A_L^u \quad . \tag{2.3.54}$$

While the A_{KM} and phases K =

$$K = \begin{bmatrix} e^{-i\phi_1} & & 0 \\ & e^{-i\phi_2} & \\ 0 & & e^{-i\phi_3} \end{bmatrix} \tag{2.3.55}$$

(only 2 phases: one overall phase is arbitary) will be observable in the di-quark interactions. Recall

$$\begin{aligned} u_L^w &= A_L^u u_L \qquad & d_L^w &= A_L^d d_L \\ u_R^w &= A_R^u u_R \qquad & d_R^w &= A_R^d d_R \\ e_L^{w+} &= A_L^{e+} e_L^+ = A_L^d e_L^+ & & \\ e_R^{w+} &= A_R^{e+} e_R^+ = A_R^d e_R^+ & & \end{aligned} \tag{2.3.56}$$

Note for the charge conjugate fields we must define the appropriate mass eigenstates i.e.

$$d_L^{wc} \equiv C\, d_R^{wT}$$

$$= (A_R^{d\dagger})^T C \bar{d}_R^T \qquad (2.3.57)$$

$$\equiv (d_L^c\, A_R^{d\dagger})$$

That is

$$d_L^c \equiv C\, \bar{d}_R^T \qquad (2.3.58)$$

it is not $A_L^{d\dagger} d_L^{wc}$.

So in terms of the mass eigenstates the X and Y interaction terms in L_F become

$$\frac{g}{\sqrt{2}} \{\bar{X}_\mu^j [\bar{d}_{L_j} \gamma^\mu e_L^+ - \bar{e}_L^- \gamma^\mu d_{L_j}^c + \varepsilon_{jik}\, \bar{u}_L^{ci} \gamma^\mu K u_L^k\,]$$

$$+ \bar{Y}_\mu^j [+ \bar{\nu}_{e_L} \gamma^\mu d_{L_j}^c - \bar{u}_{L_j} \gamma^\mu A_{KM}^+ e_L^+ \qquad (2.3.59)$$

$$+ \varepsilon_{jik}\, \bar{u}_L^{ci}\, \gamma^\mu K A_{KM}^+\, d_L^k] + \text{h.c.}\} \qquad .$$

Thus we have investigated the minimal SU(5) GUT. For additional features of the model or for a more detailed account of proton decay, Langacker's review article provides an excellent account as well as the references contained therein.