give the Z mass. A is an additional pseudoscalar particle (another Higgs particle) in the theory; it has mass

$$\boxed{m_A^2 \equiv \frac{-2b}{\sin 2\beta}}$$

Note: $\boxed{m_{H^\pm}^2 = m_A^2 + M_W^2}$

At tree level, the charged Higgs fields have mass

$$m_{H^\pm} \geq M_W \quad , \quad m_{H^\pm} \geq m_A$$

Finally consider the neutral scalar fields (real part) The scalar real part fields' potential is

$$\begin{aligned}
V_{Hs\,\text{mass}} = {} & \tfrac{1}{2} M_{u}^2 H_u^{R\,2} + \tfrac{1}{2} M_{Hd}^2 H_d^{R\,2} + b H_u^R H_d^R \\
& + \frac{g_2^2 + g_1^2}{8}\left[\tfrac{3}{2} v_u^2 H_u^{R\,2} + \tfrac{3}{2} v_d^2 H_d^{R\,2} \right] \\
& + \frac{g_2^2 - g_1^2}{4}\left[\tfrac{1}{4} v_d^2 H_u^{R\,2} + \tfrac{1}{4} v_u^2 H_d^{R\,2} + v_u v_d H_u^R H_d^R \right] \\
& - \frac{g_2^2}{2}\left[v_u v_d H_u^R H_d^R + \tfrac{1}{4} v_u^2 H_d^{R\,2} + \tfrac{1}{4} v_d^2 H_u^{R\,2} \right]
\end{aligned}$$

$$\boxed{V_{\substack{Hs\\ mass}} = \frac{1}{2}\begin{bmatrix} H_u^R & H_d^R \end{bmatrix} M_R^2 \begin{bmatrix} H_u^R \\ H_d^R \end{bmatrix}}$$

So

$$M_R^2 = \begin{bmatrix} \left.\dfrac{\partial^2 V_H}{\partial H_u^R \partial H_u^R}\right|_{vev} & \left.\dfrac{\partial^2 V_H}{\partial H_u^R \partial H_d^R}\right|_{vev} \\ \left.\dfrac{\partial^2 V_H}{\partial H_d^R \partial H_u^R}\right|_{vev} & \left.\dfrac{\partial^2 V_H}{\partial H_d^R \partial H_d^R}\right|_{vev} \end{bmatrix}$$

or from reading off the matrix elements directly from above

$$M_R^2 = \begin{bmatrix} \left(M_u^2 + \frac{3}{8}(g_2^2+g_1^2)v_u^2 + \frac{1}{8}(g_2^2-g_1^2)v_d^2 - \frac{1}{4}g_2^2 v_d^2\right) & \left(b + \frac{g_2^2-g_1^2}{4} v_u v_d - \frac{g_2^2}{2} v_u v_d\right) \\ \left(b + \frac{g_2^2-g_1^2}{4} v_u v_d - \frac{g_2^2}{2} v_u v_d\right) & \left(M_d^2 + \frac{3}{8}(g_2^2+g_1^2)v_d^2 + \frac{1}{8}(g_2^2-g_1^2)v_u^2 - \frac{1}{4}g_2^2 v_u^2\right) \end{bmatrix}$$

As previously use the minimization conditions to express M_R^2 as

$$\boxed{M_R^2 = \begin{bmatrix} (m_A^2\cos^2\beta + M_Z^2\sin^2\beta) & [-(m_A^2+M_Z^2)\sin\beta\cos\beta] \\ [-(m_A^2+M_Z^2)\sin\beta\cos\beta] & (m_A^2\sin^2\beta + M_Z^2\cos^2\beta) \end{bmatrix}}$$

As usual we can diagonalize the mass matrix with a similarity transformation but now with a different angle α

$$R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$R M_R^2 R^T R \begin{bmatrix} H_u^R \\ H_d^R \end{bmatrix} = \underset{\substack{\uparrow \\ \text{diagonal}}}{m_R^2} R \begin{bmatrix} H_u^R \\ H_d^R \end{bmatrix}$$

(Aside:

So consider RMR^T with $M^2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $B=C$

$$RM^2R^T = \begin{bmatrix} (A\cos^2\alpha + D\sin^2\alpha + 2B\sin\alpha\cos\alpha) & \begin{pmatrix} -A\cos\alpha\sin\alpha + D\cos\alpha\sin\alpha \\ + B(\cos^2\alpha - \sin^2\alpha) \end{pmatrix} \\ ((-A+D)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha)) & (A\sin^2\alpha + D\cos^2\alpha - 2B\sin\alpha\cos\alpha) \end{bmatrix}$$

If $R M^2 R^T = M^2_{\text{diagonal}} = \begin{bmatrix} m_h^2 & 0 \\ 0 & m_H^2 \end{bmatrix}$

$\Rightarrow$

$$A\cos\alpha\sin\alpha - D\cos\alpha\sin\alpha = B(\cos^2\alpha - \sin^2\alpha)$$

$\Rightarrow$

$$\frac{A-D}{B} = \cot\alpha - \tan\alpha$$

$$\Rightarrow \boxed{\tan^2\alpha + \left(\frac{A-D}{B}\right)\tan\alpha - 1 = 0}$$

$$\boxed{\tan\alpha = \frac{D-A \pm \sqrt{(D-A)^2 + 4B^2}}{2B}}$$

Choose "$-$" since $B<0$ & $\tan\alpha > 0$

and

$$\boxed{\begin{aligned} m_h^2 &= A\cos^2\alpha + D\sin^2\alpha + B\sin 2\alpha \\ m_H^2 &= A\sin^2\alpha + D\cos^2\alpha - B\sin 2\alpha \end{aligned}}$$

Applied to the real components, we find

$$\boxed{\tan\alpha = \frac{(m_A^2 - M_Z^2)\cos 2\beta \mp \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2 2\beta}}{(m_A^2+M_Z^2)\sin 2\beta}}$$

here which was "$-$" root above

$0 \le \tan\alpha \le \infty$ so choose "$+$" Root

$$\boxed{\tan\alpha = \frac{(m_A^2 - M_Z^2)\cos 2\beta + \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2 2\beta}}{(m_A^2+M_Z^2)\sin 2\beta}}$$

The mass eigenvalues are

$$m_h^2 = \frac{1}{2}\left[(m_A^2 + M_Z^2) - \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2 \cos^2 2\beta}\right]$$

$$m_H^2 = \frac{1}{2}\left[(m_A^2 + M_Z^2) + \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2 \cos^2 2\beta}\right]$$

$$m_H^2 \geq m_h^2$$

and h denotes the lighter & H the heavier neutral scalar Higgs fields.

$$\begin{pmatrix} h \\ H \end{pmatrix} = R\begin{bmatrix} H_u^R \\ H_d^R \end{bmatrix} = \begin{bmatrix} H_u^R\cos\alpha + H_d^R\sin\alpha \\ H_d^R\cos\alpha - H_u^R\sin\alpha \end{bmatrix}$$

h has $(\text{mass})^2$ m_h^2 with $m_h^2 \leq m_H^2$
H has $(\text{mass})^2$ m_H^2

Now the expectation value of M_R^2 for any vector must lie between the eigenvalues

$$m_h^2 \leq \langle\psi|M_R^2|\psi\rangle \leq m_H^2$$

let $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow$

$$\langle \psi | M_R^2 | \psi \rangle = m_A^2 \left[a^2 \cos^2\beta + b^2 \sin^2\beta - ab\,2\sin\beta\cos\beta \right]$$
$$+ M_Z^2 \left[a^2 \sin^2\beta + b^2 \cos^2\beta - ab\,2\sin\beta\cos\beta \right]$$

So if $a = \cos\beta \, ; \, b = \sin\beta \Rightarrow$

$$\langle \psi | M_R^2 | \psi \rangle = m_A^2 \cos^2 2\beta$$

Hence $\boxed{m_h \leq m_A |\cos 2\beta| \leq m_H}$

And if $a = \sin\beta \, ; \, b = \cos\beta \Rightarrow$

$$\langle \psi | M_R^2 | \psi \rangle = M_Z^2 \cos^2 2\beta$$

Hence $\boxed{m_h \leq M_Z |\cos 2\beta| \leq m_H}$

Note if $\tan\beta = 1$ (i.e. $\beta = \frac{\pi}{4}$) $\cos 2\beta = 0 \Rightarrow$

$\boxed{m_h = 0}$

These are tree level predictions — radiative corrections will permit h to be much heavier than M_Z — otherwise LEP-II experiments would already rule out the MSSM!

Particle Masses — Summary

A.) Gauge Fields : 1) Gluons G^m_μ : $\boxed{\text{Zero Mass}}$

2) Photon :

$$A_\mu = \frac{g_2 B_\mu + g_1 A^3_\mu}{\sqrt{g_1^2+g_2^2}} \quad : \boxed{\text{Zero Mass}}$$

$$= B_\mu \cos\theta_W + A^3_\mu \sin\theta_W \qquad (A^3_\mu = W^0_\mu)$$

3) Neutral Vector Boson : $Z_\mu = \dfrac{g_2 A^3_\mu - g_1 B_\mu}{\sqrt{g_1^2+g_2^2}}$

$$= A^3_\mu \cos\theta_W - B_\mu \sin\theta_W$$

$$\boxed{\begin{aligned} M_Z &= \tfrac{1}{2} v \sqrt{g_1^2+g_2^2} \\ &= \frac{M_W}{\cos\theta_W} \end{aligned}}$$

4) Charged Vector Bosons : $W^\pm_\mu = \dfrac{1}{\sqrt{2}}(A^1_\mu \mp i A^2_\mu)$

$$= \frac{1}{\sqrt{2}}(W^1_\mu \mp i W^2_\mu)$$

$$\boxed{M_W = \frac{g_2 v}{2}} \qquad v^2 = v_u^2 + v_d^2$$

$$\approx 246\,\text{GeV}$$

B) Fermion: quark & lepton masses as in SM

$$M^u = -\frac{1}{\sqrt{2}}\Gamma^u v \sin\beta \qquad (T_3 = +\tfrac{1}{2})$$

$$M^d = +\frac{1}{\sqrt{2}}\Gamma^d v \cos\beta \qquad (T_3 = -\tfrac{1}{2})$$

$$M^e = +\frac{1}{\sqrt{2}}\Gamma^e v \cos\beta \qquad (T_3 = -\tfrac{1}{2})$$

Diagonalize to find masses, mixing is given by CKM matrix.

C) Higgs Particles — enhanced scalar sector (8 real fields) as compared to SM (4 real fields)

1) Charged Higgs: $H^\pm = H_u^\pm \cos\beta + H_d^\pm \sin\beta$

$$\boxed{m_{H^\pm}^2 = m_A^2 + M_W^2}$$

2) Charged Goldstone Boson: $\Pi^\pm = H_d^\pm \cos\beta - H_u^\pm \sin\beta$

$\boxed{m_{\Pi^\pm} = 0}$ Goldstone Boson eaten by $W_\mu^\pm$ to give mass M_W.

3) Neutral Goldstone Boson: $\Pi^0 = H_d^I \cos\beta - H_u^I \sin\beta$

$\boxed{m_{\Pi^0} = 0}$ Goldstone Boson eaten by Z_μ to give mass M_Z.

C.4.) Neutral Pseudoscalar Higgs Particle:

$$A = H_u^I \cos\beta + H_d^I \sin\beta$$

$$\boxed{m_A^2 = \frac{-2b}{\sin 2\beta}}$$

$$\boxed{m_{H^\pm} \geq M_W \; ; \; m_{H^\pm} > m_A}$$

5.) Neutral Scalar Higgs Particles:

$$h = H_u^R \cos\alpha + H_d^R \sin\alpha$$

$$H = H_d^R \cos\alpha - H_u^R \sin\alpha$$

$$\tan\alpha = \frac{(m_A^2 - M_Z^2)\cos 2\beta + \sqrt{(m_A^2 + M_Z^2)^2 - 4 m_A^2 M_Z^2 \cos^2 2\beta}}{(m_A^2 + M_Z^2)\sin 2\beta}$$

$$\boxed{\begin{aligned} m_h^2 &= \tfrac{1}{2}\left[(m_A^2 + M_Z^2) - \sqrt{(m_A^2 + M_Z^2)^2 - 4 m_A^2 M_Z^2 \cos^2 2\beta}\right] \\ m_H^2 &= \tfrac{1}{2}\left[(m_A^2 + M_Z^2) + \sqrt{(m_A^2 + M_Z^2)^2 - 4 m_A^2 M_Z^2 \cos^2 2\beta}\right] \end{aligned}}$$

$$m_H^2 \geq m_h^2 \; ; \quad m_h \leq m_A |\cos 2\beta| \leq m_H$$

$$m_h \leq M_Z |\cos 2\beta| \leq m_H$$