Finally we can consider the superpotential terms with their soft-SUSY breaking pieces.

$$\Gamma_{SW} = \int dS \Big[\mu(\theta) H_u H_d$$

$$+ H_u Q y_u(\theta) U^c + H_d Q y_d(\theta) D^c$$

$$+ H_d L y_e(\theta) E^c \Big] + h.c.$$

In general the superpotential has the form

$\int dS\, W(\phi)$ where $W(\phi)$ is a function of chiral superfields only along with the SUSY breaking terms which we write as $\mu(\theta)$; $W(\phi, \mu, y)$ expanding in $\theta \Rightarrow$

$$W(\phi,\mu,y) = e^{-i\theta\partial\bar\theta} W\left(A^a + \theta\psi^a + \theta^2 F^a, \mu(1+\tfrac{1}{4}B\theta^2), y + \tfrac{1}{4}A\theta^2\right)$$

$$= e^{-i\theta\partial\bar\theta}\Big[W(A,\mu,y) + \frac{\partial W}{\partial A^a}\Big(\theta\psi^a + \theta^2 F^a\Big|\mu,y\Big)$$

$$+ \frac{\partial W}{\partial \mu}\Big|\tfrac{1}{4}\mu B\theta^2 + \frac{\partial W}{\partial y}\Big|\tfrac{1}{4}A\theta^2$$

$$+ \frac{1}{2}\frac{\partial^2 W}{\partial A^a \partial A^b}\Big|\theta\psi^a\theta\psi^b \Big]$$

So

$$\boxed{\int dS\, W(\phi,\mu,y)} = \int dS\, \theta^2 \Big[\frac{\partial W}{\partial A^a}\Big| F^a - \frac{1}{4}\psi^a\psi^b \frac{\partial^2 W}{\partial A^a \partial A^b}\Big| + \frac{1}{4}\mu B \frac{\partial W}{\partial \mu}\Big| + \frac{1}{4} A \frac{\partial W}{\partial y}\Big| \Big]$$

$$= \int d^4x \Big[-4 \frac{\partial W}{\partial A^a}\Big| F^a + \psi^a \frac{\partial^2 W}{\partial A^a \partial A^b}\Big| \psi^b - \mu B \frac{\partial W}{\partial \mu}\Big| - A\frac{\partial W}{\partial y}\Big| \Big]$$

For the MSSM $W = \mu H_u H_d + H_u \cdot \tilde{q}\, y_u\, \tilde{u}^c + H_d \cdot \tilde{q}\, y_d\, \tilde{d}^c + H_d \cdot \tilde{l}\, y_e\, \tilde{e}^c$

So letting $\psi \to \sqrt{2}\psi$, $\bar{\psi} \to \sqrt{2}\bar{\psi}$

$$\int dS\, W(\phi,\mu,y) = \int d^4x \Big[-4\mu (F_{H_u}\cdot H_d + H_u \cdot F_{H_d}) + 2\mu \tilde{H}_u \cdot \tilde{H}_d 2 - \mu B H_u \cdot H_d$$

$$-4 (F_{H_u}\cdot \tilde{q}\, y_u \tilde{u}^c + H_u \cdot F_q\, y_u \tilde{u}^c + H_u \cdot \tilde{q}\, y_u F_{u^c})$$

$$+2\tilde{H}_u \cdot q\, y_u \tilde{u}^c\, 2 + 2\tilde{H}_u \cdot \tilde{q}\, y_u u^c\, 2 + 2 H_u \cdot q\, y_u u^c \cdot 2$$

$$- H_u \cdot \tilde{q}\, A_u \tilde{u}^c$$

$$-4 (F_{H_d}\cdot \tilde{q}\, y_d \tilde{d}^c + H_d \cdot F_q\, y_d \tilde{d}^c + H_d \cdot \tilde{q}\, y_d F_{d^c})$$

$+2\tilde{H}_d\cdot q\, y_d\, \tilde{d}^c\cdot 2 + 2H_d\cdot \tilde{q}\, y_d\, \tilde{d}^c\cdot 2 + 2H_d\cdot q\, y_d\, d^c\cdot 2$

$- H_d\cdot\tilde{q}\, A_d\, \tilde{d}^c$

$-4\left(F_{H_d}\cdot\tilde{l}\, y_e\,\tilde{e}^c + H_d\cdot F_l\, y_l\,\tilde{e}^c + H_d\cdot\tilde{l}\, y_l\, F_{e^c}\right)$

$- H_d\cdot\tilde{l}\, A_l\,\tilde{e}^c \quad + 2\tilde{H}_d\cdot l\, y_l\,\tilde{e}^c\cdot 2 + 2\tilde{H}_d\cdot\tilde{l}\, y_l\, e^c\cdot 2 + 2H_d\cdot \tilde{l}\, y_l\, e^c\cdot 2\Big]$

Likewise for the anti-chiral superpotential.

Before gathering all terms for the MSSM consider the generic SUSY gauge theory again and eliminate the auxiliary fields:

$$\Gamma = \int d^4x\,\mathcal{L} = \frac{1}{g^2}\int dS\, Z(\theta)\, \mathrm{Tr}[WW] + \frac{1}{\bar{g}^2}\int d\bar{S}\, \bar{Z}(\bar{\theta})\, \mathrm{Tr}[\bar{W}\bar{W}]$$

$$+ \frac{1}{16}\int dV\, Z_K(\theta,\bar{\theta})\, \bar{\Phi}\, e^{gV\cdot T}\,\Phi \qquad \text{field } U(1)$$

$$+ \int dS\, W(\phi,\mu,y) + \int d\bar{S}\, \bar{W}(\bar{\phi},\bar{\mu},\bar{y}) + \frac{1}{4}\xi\int dV\, V$$

yields: (where $\lambda\to\sqrt{4}\,\lambda$ $\quad D\to\sqrt{4}\,D$ $\quad Z = -\frac{2}{(16)^2}$

$\bar{\lambda}\to\sqrt{4}\,\bar{\lambda}$ $\quad g\to -2g$ $\quad Z_K = 1$

$\psi\to\sqrt{2}\,\psi$

$\bar{\psi}\to\sqrt{2}\,\bar{\psi}$)

$$\boxed{\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \tfrac{1}{2}D^iD^i + \tfrac{i}{2}\lambda\sigma^\mu \overleftrightarrow{D}_\mu \bar{\lambda} \\ &+\tfrac{1}{2}\left(M\lambda^i\lambda^i + \bar{M}\bar{\lambda}^i\bar{\lambda}^i\right) \\ &+(D_\mu A)^\dagger(D^\mu A) + \tfrac{i}{2}\bar{\psi}\bar{\sigma}^\mu\overleftrightarrow{D}_\mu\psi \\ &+\sqrt{2}g\left(A^\dagger(\lambda\cdot T)\psi + \bar{\psi}(\bar{\lambda}\cdot T)A\right) \\ &+F^\dagger_a F_a - g\left(A^\dagger_a T^i_{ab}A_b\right)D^i + \left(\xi D\right. \\ &-m^2A^\dagger_aA_a - m_K\bar{F}_aA_a - \bar{m}_K A^\dagger_a F_a \\ &+\Bigg(-4\frac{\partial W(A)}{\partial A^a}F^a + 2\psi^a\frac{\partial^2W(A)}{\partial A^a\partial A^b}\psi^b \\ &\qquad -\mu B_{ab}A^aA^b - A^aA^bA^cA_{abc} + h.c.\Bigg)\end{aligned}}$$

allowed in U(1) case ignore for now (see SQED)

Now the potential is

$$V = -F^\dagger_aF_a - \tfrac{1}{2}D^iD^i + g\left(A^\dagger_aT^i_{ab}A_b\right)D^i$$

$$+m^2A^\dagger_aA_a + m_K\bar{F}_aA_a + \bar{m}_K A^\dagger_aF_a$$

$$+4\frac{\partial W}{\partial A^a}F^a + 4\frac{\partial\bar{W}}{\partial\bar{A}^a}\bar{F}^a + \mu ABA + A_{abc}A^aA^bA^c$$

$$+\bar{\mu}A^\dagger\bar{B}A^\dagger + \bar{A}_{abc}\bar{A}^a\bar{A}^b\bar{A}^c$$

The D^i equation of motion is

$$\boxed{D^i = g\left(A^\dagger T^i A\right)}$$

& $F^\dagger_a$ eq. of motion

$$\boxed{F_a = m_K A_a + 4\frac{\partial \overline{W}}{\partial \overline{A}^a}}$$

& F_a eq. of motion

$$\boxed{F^\dagger_a = \overline{m}_K A^\dagger_a + 4\frac{\partial W}{\partial A^a}}$$

$\Longrightarrow$

$$\boxed{\begin{aligned} V = &+\tfrac{1}{2} D^i D^i + F^\dagger_a F_a \\ &+ m^2 A^\dagger_a A_a + \mu ABA + \overline{\mu} A^\dagger \overline{B} A^\dagger \\ &+ A_{abc} A^a A^b A^c + \overline{A}_{abc} \overline{A}^a \overline{A}^b \overline{A}^c \end{aligned}}$$

$$\boxed{\begin{aligned} = &\tfrac{1}{2} g^2 \left(A^\dagger T^i A\right)\left(A^\dagger T^i A\right) \\ &+ \left[4\frac{\partial W}{\partial A^a} + \overline{m}_K A^\dagger_a\right]\left[m_K A_a + 4\frac{\partial \overline{W}}{\partial \overline{A}^a}\right] \\ &+ m^2 A^\dagger_a A_a + \mu ABA + \overline{\mu} A^\dagger \overline{B} A^\dagger \\ &+ A_{abc} A^a A^b A^c + \overline{A}_{abc} \overline{A}^a \overline{A}^b \overline{A}^c \end{aligned}}$$

Hence

$$\boxed{\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \tfrac{i}{2}\lambda\sigma^\mu \overleftrightarrow{D}_\mu\bar\lambda + \tfrac{1}{2}\left(M\lambda^2 + \bar{M}\bar\lambda^2\right)\\ &+ (D_\mu A)^\dagger\cdot(D^\mu A) + \tfrac{i}{2}\bar\psi\bar\sigma^\mu \overleftrightarrow{D}_\mu\psi\\ &+\sqrt{2}g\left(A^\dagger(\lambda\cdot T)\psi + \bar\psi(\bar\lambda\cdot T)A\right) + 2\cdot\bar\psi^a\frac{\partial^2\bar W(\bar A)}{\partial\bar A^a\partial\bar A^b}\bar\psi^b\\ &+2\cdot\psi^a\frac{\partial^2 W(A)}{\partial A^a\partial A^b}\psi^b \quad - V\end{aligned}}$$

If we set the soft susy breaking to zero — we have the most general susy invariant component gauge theory

$$\boxed{\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \tfrac{i}{2}\lambda\sigma^\mu \overleftrightarrow{D}_\mu\bar\lambda\\ &+ (D_\mu A)^\dagger\cdot(D^\mu A) + \tfrac{i}{2}\bar\psi\bar\sigma^\mu \overleftrightarrow{D}_\mu\psi\\ &+\sqrt{2}g\left[A^\dagger(\lambda\cdot T)\psi + \bar\psi(\bar\lambda\cdot T)A\right]\\ &+2\cdot\psi^a\frac{\partial^2 W(A)}{\partial A^a\partial A^b}\psi^b + 2\cdot\bar\psi^a\frac{\partial^2\bar W(\bar A)}{\partial\bar A^a\partial\bar A^b}\bar\psi^b\\ &-\tfrac{1}{2}\left[\xi^i + g\left(A^\dagger T^i A\right)\right]^2\\ &-16\,\frac{\partial\bar W(\bar A)}{\partial\bar A^a}\,\frac{\partial W(A)}{\partial A^a}\end{aligned}}$$

Sum over all fields gives factor of 2

[$\xi \to \xi$ in U(1) case only allowed there by gauge inv.]

Back to the MSSM:

$$\Gamma = \Gamma_{MSSM} + \Gamma_{MSSM}^{soft} = \Gamma_{SYM} + \Gamma_{SK} + \Gamma_{SW}$$

$$= \int d^4x \left(\mathcal{L}_{MSSM} + \mathcal{L}_{MSSM}^{soft} \right) = \int d^4x \, \mathcal{L}$$

Now let's first set all susy partner fields to zero to see that we obtain the 2-Higgs doublet SM:

$$\mathcal{L}_{SM} = \mathcal{L}\Big|_{\substack{susy \\ partners = 0}}$$

$$= \overset{\checkmark}{\mathcal{L}_{YM}} + \overset{\checkmark}{\mathcal{L}_F} + \mathcal{L}_\phi + \overset{\checkmark}{\mathcal{L}_{yuk}} + \overset{\checkmark}{\mathcal{L}_{SM}^{soft}}$$

with

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^i F^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G_{\mu\nu}^m G^{m\mu\nu}$$

$$\mathcal{L}_F = +i \bar{l}_F \bar{\sigma}^\mu D_\mu l_F + i \bar{q}_F \bar{\sigma}^\mu D_\mu q_F$$

$$+ i \bar{e}_F^c \bar{\sigma}^\mu D_\mu e_F^c + i \bar{u}_F^c \bar{\sigma}^\mu D_\mu u_F^c$$

$$+ i \bar{d}_F^c \bar{\sigma}^\mu D_\mu d_F^c$$

$$\mathcal{L}_\phi = (D_\mu H_u)^\dagger (D^\mu H_u) + (D_\mu H_d)^\dagger (D^\mu H_d)$$

$$- V_\phi$$

$$V_\phi = \frac{1}{2}\left[\xi + g_1\left(\tfrac{1}{2}H_u^\dagger H_u\right) - g_1\left(\tfrac{1}{2}H_d^\dagger H_d\right)\right]^2$$
$$+\frac{1}{2}g_2^2\left[\left(H_u^\dagger \frac{\sigma^i}{2} H_u\right) + \left(H_d^\dagger \frac{\sigma^i}{2} H_d\right)\right]^2$$
$$+16|\mu|^2\left[H_u^\dagger H_u + H_d^\dagger H_d\right]$$

$$\mathcal{L}_{yuk} = 2H_u\cdot q\, y_u u^c\cdot 2 + 2H_d\cdot q\, y_d d^c\cdot 2 + 2H_d\cdot l\, y_e e^c\cdot 2$$
$$+2H_u^\dagger\cdot \bar{u}^c y_u^\dagger \bar{q}\cdot 2 + 2H_d^\dagger\cdot \bar{d}^c y_d^\dagger \bar{q}\cdot 2 + 2H_d^\dagger\cdot \bar{e}^c y_e^\dagger \bar{l}\cdot 2$$

$$\mathcal{L}_{SM}^{soft} = -m_{H_u}^2 H_u^\dagger H_u - m_{H_d}^2 H_d^\dagger H_d$$
$$-\mu B H_u\cdot H_d - \bar{\mu}\bar{B} H_u^\dagger\cdot H_d^\dagger$$
$$-\bar{m}_{KH_u} m_{KH_u} H_u^\dagger H_u - \bar{m}_{KH_d} m_{KH_d} H_d^\dagger H_d$$
$$-4\bar{m}_{KH_u} H_u^\dagger(\bar{\mu} H_d^\dagger) + 4 m_{KH_u}(\mu H_d) H_u$$
$$+4\bar{m}_{KH_d} H_d^\dagger(\bar{\mu} H_u^\dagger) - 4 m_{KH_d}(\mu H_u) H_d$$

Note: $\mathcal{L}_{SM}^{soft}$ just contributes Higgs potential terms to the Lagrangian.

Let's recall the SM in terms of 4-spinors

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$ Weyl Representation

Charge Conjugation

$$\Psi^c = C\,\bar\Psi^T$$

$$\bar\Psi^c = -\Psi^T C^{-1} \;\left(= \Psi^{c\dagger}\gamma^0\right)$$

where $C^{-1}\gamma_\mu C = -\gamma_\mu^T$

& so $C = -C^{-1} = -C^\dagger = -C^T = i\gamma^2\gamma^0$

As we showed earlier (p.-179- to -185-)

$$\Psi^c_{\substack{L\\R}} = C\,\bar\Psi^T_{\substack{R\\L}} \quad ; \quad \overline{\Psi^c_{\substack{L\\R}}} = -\Psi^T_{\substack{R\\L}}\,C^{-1}$$

and inverse

$$\Psi_{\substack{R\\L}} = C\left(\overline{\Psi^c_{\substack{L\\R}}}\right)^T \; ; \; \bar\Psi_{\substack{R\\L}} = \Psi^{c\,T}_{\substack{L\\R}}\,C$$

Now $\Psi_L = \frac{1}{2}(1-\gamma_5)\Psi_D = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$; $\bar\Psi_L = \begin{bmatrix} 0 & \bar\psi \end{bmatrix}$

$\Psi_R = \frac{1}{2}(1+\gamma_5)\Psi_D = \begin{bmatrix} 0 \\ \bar\chi \end{bmatrix}$; $\bar\Psi_R = \begin{bmatrix} \chi & 0 \end{bmatrix}$

with $\gamma_5 = +i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$

So

$$C = i\gamma^2\gamma^0 = \begin{bmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} i\sigma^2 & 0 \\ 0 & i\bar{\sigma}^2 \end{bmatrix}$$

and so

$$\Psi^c = C\bar{\Psi}^T = [\, i\gamma^2\gamma^0\gamma^0\Psi^* \,]$$

$$= \begin{bmatrix} 0 & i\sigma^2 \\ i\bar{\sigma}^2 & 0 \end{bmatrix} \begin{bmatrix} \bar{\psi} \\ \chi \end{bmatrix} = \begin{bmatrix} \chi \\ \bar{\psi} \end{bmatrix}$$

(Hence if Ψ is a Majorana spinor Ψ_M this means it is self charge conjugate

$$\begin{bmatrix} \psi \\ \bar{\chi} \end{bmatrix} = \Psi_M = \Psi_M^c = C\bar{\Psi}_M^T = \begin{bmatrix} \chi \\ \bar{\psi} \end{bmatrix}$$

$$\Rightarrow \psi = \chi \; , \; \bar{\psi} = \bar{\chi}$$

$$\text{So} \quad \Psi_M = \begin{bmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{bmatrix}$$)

So

$$\psi_L^c \equiv (\Psi^c)_L = \begin{bmatrix} \chi \\ 0 \end{bmatrix} \; ; \; \psi_R^c = \begin{bmatrix} 0 \\ \bar{\psi} \end{bmatrix}$$

$$\bar{\psi}_L^c \equiv \left[(\psi^c)_L \right]^\dagger \gamma^0 \; ; \; \bar{\psi}_R^c = [\, \psi \;\; 0 \,]$$

$$= [\, 0 \;\; \bar{\chi} \,]$$

$$\Psi = \begin{pmatrix} \psi \\ \bar\chi \end{pmatrix} ; \quad \Phi = \begin{pmatrix} \phi \\ \bar\lambda \end{pmatrix}$$

So we can translate terms:

$$\boxed{\overline{\Psi}_L \gamma^\mu \Phi_L = \bar\psi \bar\sigma^\mu \phi}$$

$$\overline{\Psi}_R \gamma^\mu \Phi_R = \chi \sigma^\mu \bar\lambda$$

$$\overline{\Psi^c_L} \gamma^\mu \Phi^c_L = \bar\chi \bar\sigma^\mu \lambda = -\lambda \sigma^\mu \bar\chi$$

More directly $\quad \Phi^c \equiv \begin{pmatrix} \phi^c \\ \bar\lambda^c \end{pmatrix} = \begin{pmatrix} \lambda \\ \bar\phi \end{pmatrix} ; \quad \Psi^c \equiv \begin{pmatrix} \psi^c \\ \bar\chi^c \end{pmatrix} = \begin{pmatrix} \chi \\ \bar\psi \end{pmatrix}$

So

$$\overline{\Psi^c_L} \gamma^\mu \Phi^c_L = \overline{\psi^c} \bar\sigma^\mu \phi^c = \bar\chi \bar\sigma^\mu \lambda$$

$$\overline{\Psi}_R \gamma^\mu \Phi_R = \chi \sigma^\mu \bar\lambda = -\bar\lambda \bar\sigma^\mu \chi$$

Hence

$$\overline{\Psi^c_L} \gamma^\mu \Psi^c_L = \overline{\psi^c} \bar\sigma^\mu \psi^c = \bar\chi \bar\sigma^\mu \chi$$

$$= -\overline{\Psi}_R \gamma^\mu \Psi_R$$

Or more to the point

$$\overline{\Psi^c_L} \gamma^\mu \partial_\mu \Psi^c_L = \overline{\psi^c} \bar\sigma^\mu \partial_\mu \psi^c = \bar\chi \bar\sigma^\mu \partial_\mu \chi$$

$$= -\partial_\mu \overline{\Psi}_R \gamma^\mu \Psi_R$$

$$\partial_\mu \overline{\Psi^c_L} \gamma^\mu \Psi^c_L = \partial_\mu \overline{\psi^c} \bar\sigma^\mu \psi^c = \partial_\mu \bar\chi \bar\sigma^\mu \chi = -\overline{\Psi}_R \gamma^\mu \partial_\mu \Psi_R$$

So this simply allows us to write the $\mathcal{L}_F$ as in the SM

$$\bar{\Psi}_L \gamma^\mu \partial_\mu \Psi_L = \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi$$

$$\bar{\Psi}^c_L \gamma^\mu \partial_\mu \Psi^c_L = \bar{\psi}^c \bar{\sigma}^\mu \partial_\mu \psi^c$$

To give

$$\boxed{\begin{aligned}\mathcal{L}_F &= i\bar{l}_F \bar{\sigma}^\mu D_\mu l_F + i\bar{q}_F \bar{\sigma}^\mu D_\mu q_F \\ &\quad + i\bar{e}^c_F \bar{\sigma}^\mu D_\mu e^c_F + i\bar{u}^c_F \bar{\sigma}^\mu D_\mu u^c_F \\ &\quad + i\bar{d}^c_F \bar{\sigma}^\mu D_\mu d^c_F \\ &= i\bar{l}_{LF}\, i\not{D}\, l_{LF} + i\bar{q}_{LF} \not{D} q_{LF} \\ &\quad + i\bar{e}^c_L \not{D} e^c_L + i\bar{u}^c_L \not{D} u^c_L \\ &\quad + i\bar{d}^c_L \not{D} d^c_L\end{aligned}}$$

Where as earlier

$$D_\mu l_{FL} = [\partial_\mu - \frac{ig_2}{2}\vec{\sigma}\cdot\vec{A}_\mu + \frac{ig_1}{2} B_\mu] l_{FL}$$

$$D_\mu q^b_{FL} = [(\partial_\mu - \frac{ig_2}{2}\vec{\sigma}\cdot\vec{A}_\mu - \frac{ig_1}{6} B_\mu)\delta^{ab} - \frac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)_{ab}] q^b_{FL}$$

$$D_\mu e^c_{FL} = (\partial_\mu - i g_1 B_\mu) e^c_{FL}$$

$$D^{ab}_\mu u^{cb}_{FL} = \left[\left(\partial_\mu + \frac{2i}{3} g_1 B_\mu\right)\delta^{ab} + \frac{i g_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T_{ab}\right] u^{cb}_{FL}$$

$$D^{ab}_\mu d^{cb}_{FL} = \left[\left(\partial_\mu - \frac{i}{3} g_1 B_\mu\right)\delta^{ab} + \frac{i g_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T_{ab}\right] d^{cb}_{FL}$$

Further the Yukawa terms must be considered: For instance consider the leptonic term

$$\bar{l}_{FL}\cdot\phi\, C\, \overline{e^c_{GL}}^T = \begin{bmatrix} 0 & \bar{l}_F \end{bmatrix}\cdot\phi \begin{bmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{bmatrix}\begin{bmatrix} 0 \\ \overline{e^c_G} \end{bmatrix}$$

$$= \bar{l}_F\cdot\phi\, i\sigma^2\, \overline{e^c_G} = +\bar{l}_F\, \overline{e^c_G}\,\phi$$

(i.e. $\overline{\psi^c_L}^T = \begin{bmatrix} 0 \\ i\sigma^2\bar{\chi} \end{bmatrix}$)

$$= +\overline{e^c_G}\,\bar{l}_F\cdot\phi \quad ✓$$

Thus we find

$$\boxed{\Gamma^e_{FG}\,\bar{l}_{FL}\,\phi\, C\, \overline{e^c_{GL}}^T = +\Gamma^e_{FG}\,\overline{e^c_G}\,\bar{l}_F\cdot\phi = -2\cdot 2\,\overline{e^c_G}\, y^\dagger_{\ell GF}\,\bar{l}_F\cdot\phi}$$

Define $-2 y^\dagger_{\ell GF}\cdot 2 = +\Gamma^e_{FG}$ & $\phi = H^\dagger_d$

So write this as

$$= \bar{l}_{FL}\,\Gamma^e_{FG}\, H^\dagger_d\, C\, \overline{e^c_{GL}}^T$$

$$\boxed{\begin{gathered} -2 y^\dagger_\ell\cdot 2 = \Gamma^{eT} \\ \Gamma^e = y^*_\ell(-2)\cdot 2 \end{gathered}}$$

Note below:

$$(\bar{l}_L \Gamma^e H_d^\dagger e_R)^\dagger = -(\bar{e}_R H_d \Gamma^{e\dagger} l_L)$$

Recall p. -185-

$$\bar{l}_{FL} \Gamma^e_{FG} H_d^\dagger C \bar{e}^{c\,T}_{GL} = \bar{l}_{FL} \Gamma_{FG} H_d e_{GR}$$

$$\boxed{= \sqrt{-2\cdot 2}\, \bar{e}^c y_l^\dagger \bar{l} \cdot H_d^\dagger = \bar{l}_L \Gamma^e H_d^\dagger e_R}$$

Similarly

$$\Gamma^d_{FG} \bar{q}_{FL} H_d^\dagger C \bar{d}^{c\,T}_{GL} = \sqrt{-2\cdot 2}\, d^c y_d^\dagger \bar{q} \cdot H_d^\dagger$$

$$= \bar{q}_L \Gamma^d H_d^\dagger d_R$$

$$\Gamma^u_{FG} \bar{q}_{FL} H_u^\dagger C \bar{u}^{c\,T}_{GL} = \sqrt{-2\cdot 2}\, \bar{u}^c y_u^\dagger \bar{q} \cdot H_u^\dagger$$

$$= \bar{q}_L \Gamma^u H_u^\dagger u_R$$

Now the Hermitian conjugates:

$\boxed{(-i\sigma^2)\ 0}$

$$-e^{c\,T}_{FL} \Gamma^{e\dagger}_{FG} H_d \cdot C l_{GL} = -\boxed{e^c_F\ 0} \Gamma^{e\dagger}_{FG} H_d \cdot \begin{bmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{bmatrix} \begin{bmatrix} l_G \\ 0 \end{bmatrix}$$

$$= \qquad = -e^c_F \Gamma^{e\dagger}_{FG} H_d \cdot l_G = \bar{e}_{FR} \Gamma^{e\dagger}_{FG} H_d l_{GL}$$

$$= H_d \cdot l\, \Gamma^{e\dagger T} e^c \quad (\Rightarrow -2y_l = \Gamma^{e*})$$

$$-d^{c\,T}_{FL} \Gamma^{d\dagger}_{FG} H_d C q_{GL} = d^c_F \Gamma^{d\dagger}_{FG} H_d q_G = 2 H_d \cdot q \cdot 2y^d d^c$$

$$= -2 d^c y^{d\,T} q \cdot H_d \cdot 2 \qquad = \bar{d}_{FR} \Gamma^{d\dagger}_{FG} H_d q_{GL}$$

$$-u^{c\,T}_{FL}\,\Gamma^{u\dagger}_{FG}\,H_u C q_{GL} = u^c_F\,\Gamma^{u\dagger}_{FG}\,H_u\,q_G$$

$$= 2u^c y^{u\dagger} H_u q \cdot 2 = \overline{u}_{FR}\,\Gamma^{u\dagger}_{FG}\,H_u\,q_{GL}$$

Hence the Yukawa Lagrangian can be written as

$$\mathcal{L}_{yuk} = 2H_u q\, y_u u^c \cdot 2 + 2H_d \cdot q\, y_d d^c \cdot 2 + 2H_d \cdot l\, y_e e^c \cdot 2$$
$$+ 2H_u^\dagger \cdot \overline{u}^c y_u^\dagger \overline{q} \cdot 2 + 2H_d^\dagger \cdot \overline{d}^c y_d^\dagger \overline{q} \cdot 2 + 2H_d^\dagger \cdot \overline{e}^c y_e^\dagger \overline{l} \cdot 2$$

$$= -u^{c\,T}_L\,\Gamma^{u\dagger} H_u C q_L - d^{c\,T}_L\,\Gamma^{d\dagger} H_d C q_L$$
$$- e^{c\,T}_L\,\Gamma^{e\dagger} H_d C l_L + \overline{q}_L\,\Gamma^u H_u^\dagger C\,\overline{u}^{c\,T}_L$$
$$+ \overline{q}_L\,\Gamma^d H_d^\dagger C\,\overline{d}^{c\,T}_L + \overline{l}_L\,\Gamma^e H_d^\dagger C\,\overline{e}^{c\,T}_L$$

$$= -\overline{u}_R\,\Gamma^{u\dagger} H_u q_L - \overline{d}_R\,\Gamma^{d\dagger} H_d q_L - \overline{e}_R\,\Gamma^{e\dagger} H_d l_L$$
$$+ \overline{q}_L\,\Gamma^u H_u^\dagger u_R + \overline{q}_L\,\Gamma^d H_d^\dagger d_R$$
$$+ \overline{l}_L\,\Gamma^e H_d^\dagger e_R$$

This is the 2-Higgs generalization of our previous SM Yukawa couplings see p.-185-
where $\phi \to H_d^\dagger$, $(\phi^\dagger \to H_d)$ and

$$\tilde{\phi} \to H_u^\dagger, (\tilde{\phi}^\dagger \to H_u)$$

Also just for completeness $\mathcal{L}_F$ can be written in terms of the left & right hand fields as in the SM also p.-184-

$$\mathcal{L}_F = i\bar{l}_L \not{D} l_L + i\bar{q}_L \not{D} q_L + i\bar{e}_R \not{D} e_R$$
$$+ i\bar{u}_R \not{D} u_R + i\bar{d}_R \not{D} d_R$$

Finally we see the additional Higgs potential terms that come from $\mathcal{L}_{SM}^{soft}$

$$\mathcal{L}_{SM}^{soft} = -(m_{Hu}^2 + \bar{m}_{KHu} m_{KHu}) H_u^\dagger H_u$$
$$-(m_{Hd}^2 + \bar{m}_{KHd} m_{KHd}) H_d^\dagger H_d$$
$$-(\mu B + 4\mu m_{KHd} + 4\mu m_{KHu}) H_u \cdot H_d$$
$$+(\bar{\mu}\bar{B} + 4\bar{m}_{KHu}\bar{\mu} + 4\mu \bar{m}_{KHd}) H_d^\dagger \cdot H_u^\dagger$$

Hence the potential for the Higgs fields is

$$V_H = V_\phi - \mathcal{L}^{soft}_{SM}$$

$$= M^2_{Hu} H_u^\dagger H_u + M^2_{Hd} H_d^\dagger H_d$$

$$- b\, H_u \cdot H_d - \bar{b}\, H_u^\dagger \cdot H_d^\dagger$$

$$+ \frac{1}{2}\left\{ \xi + g_1 \; \frac{1}{2} H_u^\dagger H_u - g_1 \; \frac{1}{2} H_d^\dagger H_d \right\}^2$$

$$+ \frac{1}{2} g_2^2 \left[\left(H_u^\dagger \frac{\sigma^i}{2} H_u \right) + \left(H_d^\dagger \frac{\sigma^i}{2} H_d \right) \right]^2$$

where

$$M^2_{Hu} = |6\mu|^2 + m^2_{Hu} + \bar{m}_{KHu} m_{KHu}$$

$$M^2_{Hd} = |6\mu|^2 + m^2_{Hd} + \bar{m}_{KHd} m_{KHd}$$

$$-b = \mu B + 4\mu m_{KHu} + 4\mu m_{KHd}$$

$$-\bar{b} = \bar{\mu}\bar{B} + 4\bar{\mu}\,\bar{m}_{KHu} + 4\bar{\mu}\,\bar{m}_{KHd}$$

Note that $(H_u^\dagger \sigma^i H_u)^2 = (H_u^\dagger H_u)^2$

$(H_d^\dagger \sigma^i H_d)^2 = (H_d^\dagger H_d)^2$

&

$$(H_u^\dagger \sigma^i H_u)(H_d^\dagger \sigma^i H_d) = (H_u^\dagger H_u)(H_d^\dagger H_d) - 2|H_u \cdot H_d|^2$$

So

$$\begin{aligned}
V_H = {} & M_{H_u}^2 H_u^\dagger H_u + M_{H_d}^2 H_d^\dagger H_d - b\, H_u \cdot H_d - \bar{b}\, H_u^\dagger \cdot H_d^\dagger \\
& + \frac{1}{2}\xi^2 + \frac{g_1^2}{4\cdot 2}(H_u^\dagger H_u)^2 + \frac{g_1^2}{4\cdot 2}(H_d^\dagger H_d)^2 \\
& + \frac{g_1}{\sqrt{2}\sqrt{2}}\xi H_u^\dagger H_u - \frac{g_1}{\sqrt{2}\cdot\sqrt{2}}\xi H_d^\dagger H_d - \frac{g_1^2}{2\cdot 2}(H_u^\dagger H_u)(H_d^\dagger H_d) \\
& + \frac{g_2^2}{4\cdot 2}\left[(H_u^\dagger H_u)^2 + (H_d^\dagger H_d)^2\right] \\
& + \frac{g_2^2}{4\cdot 2}\left[2(H_u^\dagger H_u)(H_d^\dagger H_d) - 4|H_u \cdot H_d|^2\right]
\end{aligned}$$

$$\boxed{\begin{aligned}
V_H = {} & \left(M_{H_u}^2 + \frac{1}{\sqrt{2}}\frac{g_1}{\sqrt{2}}\xi\right) H_u^\dagger H_u + \left(M_{H_d}^2 - \frac{1}{\sqrt{2}}\frac{g_1}{\sqrt{2}}\xi\right) H_d^\dagger H_d \\
& - b\, H_u \cdot H_d - \bar{b}\, H_u^\dagger \cdot H_d^\dagger + \frac{1}{2}\xi^2 \\
& + \frac{g_2^2 + g_1^2}{4\cdot 2}\left[(H_u^\dagger H_u)^2 + (H_d^\dagger H_d)^2\right] \\
& + \frac{g_2^2 - g_1^2}{2\cdot 2}(H_u^\dagger H_u)(H_d^\dagger H_d) \\
& - \frac{g_2^2}{2}|H_u \cdot H_d|^2
\end{aligned}}$$