SUSY QED : 2 chiral matterfields with electric charge $\pm 1$ denoted $\phi_\pm$. Under $U(1)$ gauge transformations they transform according to their respective charges

$$\phi'_\pm = e^{\pm ie\Lambda}\phi_\pm \quad \text{(let's call } g=e\text{)}$$

$$\left(= e^{\pm ig\Lambda}\phi_\pm\right)$$

where $\Lambda$ is a chiral superfield $\bar{D}_{\dot\alpha}\Lambda = 0$.

The hermitean conjugate anti-chiral fields transform as

$$\bar\phi'_\pm = e^{\mp ig\bar\Lambda}\bar\phi_\pm = \bar\phi_\pm e^{\mp ig\bar\Lambda} \quad \text{(Abelian)}$$

where $\bar\Lambda$ is an anti-chiral superfield $D_\alpha\bar\Lambda = 0$

and we have left the $\pm$ subscript intact on $\phi$ $\phi_+^\dagger = \bar\phi_+$ ; $\phi_-^\dagger = \bar\phi_-$ .

Only a gauge invariant mass term can be made from $\phi_\pm$ — that is a 0 charge monomial

$$\Gamma_W = \frac{m}{4}\int dS\, \phi_+\phi_- + \frac{m}{4}\int d\bar S\, \bar\phi_+\bar\phi_-$$

The gauge invariant Kähler potential also has 2 terms

$$\bar{\Phi}_+ e^{gV} \Phi_+ \quad \text{and} \quad \Phi_- e^{-gV} \bar{\Phi}_- .$$

The gauge transformation of the Abelian super gauge field $V$ is given by

$$e^{gV'} = e^{+ig\bar{\Lambda}} e^{gV} e^{-ig\Lambda}$$

Since this is an Abelian $U(1)$ symmetry all factors commute so that

$$e^{gV'} = e^{gV + ig(\bar{\Lambda} - \Lambda)}$$

and hence $\boxed{V' = V + i(\bar{\Lambda} - \Lambda)}$

as usual for a $U(1)$ gauge field $\delta V = V' - V = i(\bar{\Lambda} - \Lambda)$ the variation is just the inhomogeneous term. So with this we see that, since $e^{-gV'} e^{gV'} = 1$,

$$e^{-gV'} = e^{+ig\Lambda} e^{-gV} e^{-ig\bar{\Lambda}},$$ which of

course in the $U(1)$ case the order is unimportant, but this is the necessary order in the non-Abelian case. So we see that

$$\bar{\phi}'_+ e^{gV'} \phi'_+ = \left(\bar{\phi}_+ e^{-ig\bar{\Lambda}}\right)\left(e^{+ig\bar{\Lambda}} e^{gV} e^{-ig\Lambda}\right)\left(e^{+ig\Lambda} \phi_+\right)$$
$$= \bar{\phi}_+ e^{gV} \phi_+$$

and likewise

$$\phi'_- e^{-gV'} \bar{\phi}'_- = \left(\phi_- e^{-ig\Lambda}\right)\left(e^{+ig\Lambda} e^{-gV} e^{-ig\bar{\Lambda}}\right)\left(e^{+ig\bar{\Lambda}} \bar{\phi}_-\right)$$
$$= \phi_- e^{-gV} \bar{\phi}_-$$

---

These are the only 2 possibilities, and the invariant Kähler Potential action terms are

$$\Gamma_K = \int dV \left[ Z_+ \bar{\phi}_+ e^{gV} \phi_+ + Z_- \phi_- e^{-gV} \bar{\phi}_- \right]$$

where $Z_\pm$ are normalization factors.

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Finally the U(1) gauge field strength kinetic energy terms are needed — The Abelian U(1) field strength spinors are

$$W_\alpha = \bar{D}\bar{D}\left[e^{-gV} D_\alpha e^{+gV}\right]$$
$$\bar{W}_{\dot{\alpha}} = DD\left[e^{+gV} \bar{D}_{\dot{\alpha}} e^{-gV}\right]$$

Since this is an abelian group we can simply take the derivatives and cancel the exponential factors to obtain

$$\boxed{\begin{aligned} W_\alpha &= +g\,\bar{D}\bar{D}D_\alpha V \\ \bar{W}_{\dot\alpha} &= -g\,DD\bar{D}_{\dot\alpha} V \end{aligned}}$$

Since $V' = V + i(\bar{\Lambda} - \Lambda)$ we have that

$$W'_\alpha = g\bar{D}\bar{D}D_\alpha V' = g\bar{D}\bar{D}D_\alpha V + ig\bar{D}\bar{D}D_\alpha(\bar{\Lambda} - \Lambda)$$

$$= W_\alpha + ig\bar{D}\bar{D}D_\alpha\bar{\Lambda} - ig\bar{D}\bar{D}D_\alpha\Lambda$$

but $D_\alpha\bar{\Lambda} = 0$ & $\bar{D}\bar{D}D_\alpha = D_\alpha\bar{D}\bar{D} - 4i(\not{\partial}\bar{D})_\alpha$

So $\bar{D}\bar{D}D_\alpha\Lambda = 0$ as well &

$$\boxed{W'_\alpha = W_\alpha}$$

Similarly, $\boxed{\bar{W}'_{\dot\alpha} = \bar{W}_{\dot\alpha}}$. The gauge invariant kinetic energy terms are

$$\boxed{\Gamma_{sym} = \frac{Z}{g^2}\int dS\, W^\alpha W_\alpha + \frac{Z}{g^2}\int d\bar{S}\, \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}}$$

Hence we have the SQED action

$$\Gamma_{SQED} = \Gamma_{sym} + \Gamma_K + \Gamma_W$$

$$\Gamma_{sym} = \frac{Z}{g^2}\int dS\, W^\alpha W_\alpha + \frac{Z}{g^2}\int d\bar{S}\, \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}$$

$$\Gamma_K = \int dV \left[ Z_+ \bar\phi_+ e^{gV} \phi_+ + Z_- \phi_- e^{-gV} \bar\phi_- \right]$$

$$\Gamma_W = \frac{m}{4}\int dS\, \phi_+ \phi_- + \frac{m}{4}\int d\bar{S}\, \bar\phi_+ \bar\phi_-$$

with $W_\alpha = g \bar{D}\bar{D} D_\alpha V$

$$\bar{W}_{\dot\alpha} = -g DD\bar{D}_{\dot\alpha} V$$

Note that

$$\Gamma_{sym} = \frac{Z}{g^2}\int dS\, W^\alpha W_\alpha + \frac{Z}{g^2}\int d\bar{S}\, \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}$$

$$= Z\int dS (\bar{D}\bar{D} D^\alpha V)(\bar{D}\bar{D} D_\alpha V)$$

$$+ Z\int d\bar{S} (DD\bar{D}_{\dot\alpha} V)(DD\bar{D}^{\dot\alpha} V)$$

$$= Z\int dS\, \bar{D}\bar{D}\left[ D^\alpha V\, \bar{D}\bar{D} D_\alpha V \right]$$

$$+ Z\int d\bar{S}\, DD\left[ \bar{D}_{\dot\alpha} V\, DD\bar{D}^{\dot\alpha} V \right]$$

$$\Gamma_{sym} = Z\int dV\left[D^{\alpha}V\bar{D}\bar{D}D_{\alpha}V + \bar{D}_{\dot\alpha}VDD\bar{D}^{\dot\alpha}V\right]$$

$$\left(\begin{array}{c}\text{integrate}\\ \text{by parts}\end{array}\right) = Z\int dV\left[-VD^{\alpha}\bar{D}\bar{D}D_{\alpha}V - V\bar{D}_{\dot\alpha}DD\bar{D}^{\dot\alpha}V\right]$$

Using the $D, \bar{D}$ (anti-)commutators we have that

$$D\bar{D}\bar{D}D = \bar{D}DD\bar{D}$$

So the gauge field kinetic energy has the form

$$\boxed{\Gamma_{sym} = -2Z\int dV\left[VD\bar{D}\bar{D}DV\right]} \, .$$

---

Before expanding this action into component fields to see its QED content — let's take a look at the gauge transformations more closely to see that many of the gauge fields can be gauged away! This almost physical gauge is called the

Wess-Zumino gauge. In terms of superfields we have

$$V' = V + i(\bar\lambda - \lambda)$$

$$\phi'_{\pm} = e^{\pm ig\lambda}\phi_{\pm}$$

$$\bar\phi'_{\pm} = \bar\phi_{\pm}\, e^{\mp ig\bar\lambda}$$

Since $\Lambda, \bar{\Lambda}$ are chiral we can expand them

$$\Lambda(x,\theta,\bar\theta) = e^{-i\theta\not{\partial}\bar\theta}\left[\omega(x) + \theta^\alpha \zeta_\alpha(x) + \theta^2\sigma(x)\right]$$

$$\bar\Lambda(x,\theta,\bar\theta) = e^{+i\theta\not{\partial}\bar\theta}\left[\omega^\dagger(x) + \bar\theta_{\dot\alpha}\bar\zeta^{\dot\alpha}(x) + \bar\theta^2\sigma^\dagger(x)\right]$$

recalling that $e^{\pm i\theta\not{\partial}\bar\theta} = 1 \pm i\theta\not{\partial}\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\partial^2$

we find

$$\boxed{\begin{aligned}(\bar\Lambda - \Lambda) &= (\omega^\dagger - \omega) - \theta^\alpha\zeta_\alpha + \bar\theta_{\dot\alpha}\bar\zeta^{\dot\alpha} - \theta^2\sigma + \bar\theta^2\sigma^\dagger \\ &\quad + i\theta\sigma^\mu\bar\theta\,\partial_\mu(\omega^\dagger + \omega) - \frac{i}{2}\bar\theta^2\theta\not{\partial}\bar\zeta \\ &\quad + \frac{i}{2}\theta^2\zeta\overleftarrow{\not{\partial}}\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\partial^2(\omega^\dagger - \omega).\end{aligned}}$$

The component expansion of the real or vector superfield (the gauge superfield) is

$$\begin{aligned}V(x,\theta,\bar\theta) &= C(x) + \theta\chi + \bar\theta\bar\chi + \frac{1}{2}\theta^2 M + \frac{1}{2}\bar\theta^2 M^\dagger \\ &\quad + \theta\sigma^\mu\bar\theta A_\mu + \frac{1}{2}\theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha} + \frac{1}{2}\bar\theta^2\theta^\alpha\lambda_\alpha \\ &\quad + \frac{1}{4}\theta^2\bar\theta^2 D\end{aligned}$$

Hence the U(1) gauge transformation

$V' = V + i(\bar{\Lambda} - \Lambda)$ yields the component transformations

$$\boxed{\begin{aligned}
&C' = C + i(\omega^{\dagger} - \omega) \\
&\chi'_{\alpha} = \chi_{\alpha} - i\xi_{\alpha} \quad ; \quad \bar{\chi}'^{\dot{\alpha}} = \bar{\chi}^{\dot{\alpha}} + i\bar{\xi}^{\dot{\alpha}} \\
&M' = M - 2i\sigma \quad ; \quad M'^{\dagger} = M^{\dagger} + 2i\sigma^{\dagger} \\
&A'_{\mu} = A_{\mu} - \partial_{\mu}(\omega^{\dagger} + \omega) \\
&\lambda'_{\alpha} = \lambda_{\alpha} + (\not{\partial}\bar{\xi})_{\alpha} \quad ; \quad \bar{\lambda}'^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}} + (\xi\overleftarrow{\not{\partial}})^{\dot{\alpha}} \\
&D' = D - i\partial^2(\omega^{\dagger} - \omega)
\end{aligned}}$$

We can always choose $\Lambda$ & $\bar{\Lambda}$ such that $C, \chi, \bar{\chi}, M$ and $M^{\dagger}$ are <u>gauged</u> <u>away</u>. That is, choose the field dependent gauge transformation parameters

$$\omega^\dagger - \omega = iC \qquad \Rightarrow \qquad C' = 0$$

$$\xi_\alpha = -i\chi_\alpha \qquad \Rightarrow \qquad \chi'_\alpha = 0$$

$$\bar{\xi}^{\dot\alpha} = +i\,\bar{\chi}^{\dot\alpha} \qquad \Rightarrow \qquad \bar{\chi}'^{\dot\alpha} = 0$$

$$\sigma = -\frac{i}{2} M \qquad \Rightarrow \qquad M' = 0$$

$$\sigma^\dagger = +\frac{i}{2} M^\dagger \qquad \Rightarrow \qquad M'^\dagger = 0$$

We still have the parameter $(\omega^\dagger + \omega)$ at our disposal. Only $A_\mu$ still transforms under this now restricted set of gauge transformation $\Lambda = \omega(x) = \Lambda^\dagger$.

So in the W-Z gauge

$$V = \theta\sigma^\mu\bar\theta\, A_\mu + \frac{1}{2}\theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}$$
$$+ \frac{1}{2}\bar\theta^2\theta^\alpha\lambda_\alpha + \frac{1}{4}\theta^2\bar\theta^2 D$$

and the gauge transformations of these remaining fields are

$$A'_\mu = A_\mu - 2\partial_\mu\omega = A_\mu - \partial_\mu(\omega^\dagger + \omega)$$

$$\lambda'_\alpha = \lambda_\alpha \; ; \; \bar\lambda'^{\dot\alpha} = \bar\lambda^{\dot\alpha} \; ; \; D' = D$$

where now we have relabelled the $\lambda, \bar{\lambda}, D$ fields in the W-Z gauge as

$$\lambda'_\alpha = \lambda_\alpha + (\gamma\bar{\zeta})_\alpha = \lambda_\alpha + i(\gamma\bar{\chi})_\alpha \longrightarrow \lambda_\alpha$$

$$\bar{\lambda}'^{\dot\alpha} = \bar{\lambda}^{\dot\alpha} + (\zeta\overleftarrow{\gamma})^{\dot\alpha} = \bar{\lambda}^{\dot\alpha} - i(\chi\overleftarrow{\gamma})^{\dot\alpha} \longrightarrow \bar{\lambda}^{\dot\alpha}$$

$$D' = D - i\gamma^2(\omega^\dagger - \omega) = D + \gamma^2 C \longrightarrow D$$

Likewise the matter fields in the W-Z gauge

$$\phi'_\pm = e^{\pm ig\Lambda}\phi_\pm = e^{\pm ig\omega}\phi_\pm$$

$$\bar{\phi}'_\pm = e^{\mp ig\bar{\Lambda}}\bar{\phi}_\pm = e^{\mp ig\omega}\bar{\phi}_\pm$$

transform under the restricted gauge transformations now with $\Lambda = \omega = \bar{\Lambda}$. Also after going to the W-Z gauge we re-labelled the matter fields $\phi'_\pm \to \phi_\pm$, $\bar{\phi}'_\pm \to \bar{\phi}_\pm$ then only the above $\Lambda = \omega = \bar{\Lambda}$ transformations remain.

The W-Z gauge is not a super-covariant gauge, however. SUSY transformations no longer commute with gauge transformations. If we allow (operator) field dependent gauge transformations, it can be shown that the algebra will again close and gauge transformations will commute with these covariant SUSY transformations (for details see B. DeWit & D.Z. Freedman, Physical Review D 12 page 2286).

Now in the W-Z gauge the $e^{\pm gV}$ factors simplify

$$e^{\pm gV} = 1 \pm g\,\theta\sigma^\mu\bar\theta A_\mu \pm \frac{g}{2}\theta^2\bar\theta\bar\lambda \pm \frac{g}{2}\bar\theta^2\theta\lambda$$

$$\pm \frac{g}{4}\theta^2\bar\theta^2 D + \frac{g^2}{4}\theta^2\bar\theta^2 A_\mu A^\mu$$

$$\left(\text{where we used } \theta\sigma^\mu\bar\theta\,\theta\sigma^\nu\bar\theta = \frac{1}{4}\theta^2\bar\theta^2\,\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu}\right)$$

Note also — even in the NON-Abelian case since $V'^i = V^i + ig(\bar\Lambda^i - \Lambda^i) + \cdots$ has the same inhomogeneous term to begin with we can always transform to the W-Z gauge in which $C^i = \chi^i = \bar\chi^i = M^i = M^{i+} = 0$ and $\lambda^i = \bar\lambda^i = \omega^i = \omega^{i+}$.

So let's expand the SQED action in the W-Z gauge. There is still the restricted gauge transformations possible as listed above

$$A'_\mu = A_\mu - 2\partial_\mu \omega$$

$$\lambda'_\alpha = \lambda_\alpha \; ; \; \bar\lambda'^{\dot\alpha} = \bar\lambda^{\dot\alpha} \; ; \; D' = D$$

$$\phi'_\pm = e^{\pm i g \omega} \phi_\pm$$

$$\bar\phi'_\pm = e^{\mp i g \omega} \bar\phi_\pm .$$

First consider the gauge field Kähler kinetic energy

$$\Gamma_{sym} = -2Z \int dV \, V D\bar D \bar D D V = \frac{Z}{g^2}\int dS \, W^\alpha W_\alpha + \frac{Z}{g^2}\int d\bar S \, \bar W_{\dot\alpha} \bar W^{\dot\alpha}$$

So consider

$$W_\alpha = \bar D \bar D \left[ e^{-gV} D_\alpha e^{+gV} \right] = g \bar D \bar D D_\alpha V$$

$$= g \bar D \bar D D_\alpha \left[ \theta \sigma^\mu \bar\theta A_\mu + \tfrac{1}{2}\theta^2 \bar\theta \bar\lambda + \tfrac{1}{2}\bar\theta^2 \theta \lambda + \tfrac{1}{4}\theta^2\bar\theta^2 D \right]$$

---

Now $\bar D \bar D = \left( -\frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i(\theta \partial\!\!\!/)_{\dot\alpha} \right)\left( -\frac{\partial}{\partial \bar\theta_{\dot\alpha}} + i(\theta \partial\!\!\!/)^{\dot\alpha} \right)$

$$= \frac{\partial^2}{\partial \bar\theta^2} - 2i\theta \partial\!\!\!/ \frac{\partial}{\partial \bar\theta} + \theta^2 \partial^2$$

Now

$$D_\alpha V = \left(\frac{\partial}{\partial\theta^\alpha} - i(\not{\partial}\bar\theta)_\alpha\right)\left[\theta\sigma^\mu\bar\theta A_\mu + \frac{1}{2}\theta^2\bar\theta\bar\lambda + \frac{1}{2}\bar\theta^2\theta\lambda + \frac{1}{4}\theta^2\bar\theta^2 D\right]$$

$$= (\sigma^\mu\bar\theta)_\alpha A_\mu + \theta_\alpha\bar\theta\bar\lambda + \frac{1}{2}\bar\theta^2\lambda_\alpha + \frac{1}{2}\theta_\alpha\bar\theta^2 D - \frac{i}{2}\bar\theta^2(\theta\sigma^\mu\bar\sigma^\nu)_\alpha\partial_\nu A_\mu - \frac{i}{4}\theta^2\bar\theta^2(\not{\partial}\bar\lambda)_\alpha$$

So

$$\boxed{\bar D\bar D D_\alpha V = e^{-i\theta\not{\partial}\bar\theta}\left[-2\lambda_\alpha - 2\theta_\alpha D - 2i(\theta\sigma^\mu\bar\sigma^\nu)_\alpha F_{\mu\nu} + 2i\theta^2(\not{\partial}\bar\lambda)_\alpha\right]}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

So the chiral field strength spinor in W-Z gauge is

$$\boxed{\begin{aligned} W_\alpha &= g\bar D\bar D D_\alpha V \\ &= -2g\, e^{-i\theta\not{\partial}\bar\theta}\left[\lambda_\alpha + \theta_\alpha D + i(\theta\sigma^\mu\bar\sigma^\nu)_\alpha F_{\mu\nu} - i\theta^2(\not{\partial}\bar\lambda)_\alpha\right]\end{aligned}}$$

The gauge kinetic terms follow

$$\int dS\, W^\alpha W_\alpha = 4g^2 \int dS \Big[ \theta^2 D^2 - \theta^2 2i\lambda \not{\partial} \bar{\lambda}$$

$$- 2(\theta\sigma^\mu\bar{\sigma}^\nu)^\alpha (\theta\sigma^\rho\bar{\sigma}^\sigma)_\alpha F_{\mu\nu} F_{\rho\sigma}$$

$$+ 2i(\theta\sigma^\mu\bar{\sigma}^\nu\theta) F_{\mu\nu} D + \underbrace{\cdots}_{\text{lower powers of } \theta} \Big]$$

---

Recall $F_{\mu\nu}\sigma^\mu\bar{\sigma}^\nu = F_{\mu\nu}[\eta^{\mu\nu} - i\sigma^{\mu\nu}] = -iF_{\mu\nu}\sigma^{\mu\nu}$

$\Rightarrow$

$$\theta\sigma^\mu\bar{\sigma}^\nu\theta F_{\mu\nu} = -i\, \cancel{\theta\sigma^{\mu\nu}\theta}^{\to 0} F_{\mu\nu}$$

$$\int dS\, W^\alpha W_\alpha = 4g^2 \int dS [\theta^2] \Big[ D^2 - 2i\lambda \not{\partial} \bar{\lambda}$$

$$+ 2(\theta\sigma^\mu\bar{\sigma}^\nu\sigma^\sigma\bar{\sigma}^\rho\theta) F_{\mu\nu} F_{\sigma\rho} \Big]$$

But $\theta\sigma^\mu\bar{\sigma}^\nu\sigma^\sigma\bar{\sigma}^\rho\theta F_{\mu\nu} F_{\sigma\rho} = -\theta\sigma^{\mu\nu}\sigma^{\rho\sigma}\theta F_{\rho\sigma}$

$$\theta\sigma^{\mu\nu}\sigma^{\rho\sigma}\theta = +\tfrac{1}{2}\theta^2 \,\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}]$$

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$$\boxed{\int dS\, W^\alpha W_\alpha = 4g^2 \int dS\, \theta^2 \Big[ D^2 - 2i\lambda \not{\partial} \bar{\lambda} - \mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}] F_{\mu\nu} F^{\rho\sigma} \Big]}$$

Now

$$\mathrm{Tr}\{\sigma^{\mu\nu}\sigma^{\rho\sigma}\} = 2\left[\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}\right]$$

So

$$F_{\mu\nu}F_{\rho\sigma}\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = 2F_{\mu\nu}F^{\mu\nu} - 2F_{\mu\nu}F^{\nu\mu}$$

$$- 2i F_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$= 4F_{\mu\nu}F^{\mu\nu} - 4i F_{\mu\nu}\tilde{F}^{\mu\nu}$$

where we call $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

So

$$\int dS\, W^\alpha W_\alpha = 4g^2\int dS\,\theta^2\left[D^2 - 2i\lambda\not{\partial}\bar{\lambda}\right.$$

$$\left. - 4F_{\mu\nu}F^{\mu\nu} + 4i F_{\mu\nu}\tilde{F}^{\mu\nu}\right]$$

$$= -16g^2\int d^4x\left[D^2 - 2i\lambda\not{\partial}\bar{\lambda} - 4F_{\mu\nu}F^{\mu\nu}\right.$$

$$\left. + 4i F_{\mu\nu}\tilde{F}^{\mu\nu}\right]$$

$$= -2\cdot(16)^2 g^2\int d^4x\left[\frac{1}{32}D^2 - \frac{i}{16}\lambda\not{\partial}\bar{\lambda}\right.$$

$$\left. - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} + \frac{i}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]$$

$$= \int dS\, W^\alpha W_\alpha$$

So finally

$$\boxed{\int d S\, W^\alpha W_\alpha = -2(16)^2 g^2 \int d^4x \left[\frac{1}{32}D^2 - \frac{i}{16}\lambda \not{\partial} \bar{\lambda} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} + \frac{i}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]}$$

and likewise the conjugate terms are

$$\boxed{\int d\bar{S}\, \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} = -2(16)^2 g^2 \int d^4x \left[\frac{1}{32}D^2 + \frac{i}{16}\lambda \overleftarrow{\not{\partial}} \bar{\lambda} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} - \frac{i}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]}$$

So we find

$$\int dS\, W^\alpha W_\alpha + \int d\bar{S}\, \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

$$= -2(16)^2 g^2 \int d^4x \left[\frac{1}{16}D^2 - \frac{i}{16}(\lambda \overleftrightarrow{\not{\partial}} \bar{\lambda}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]$$

also we have

$$\int dS\, W^\alpha W_\alpha - \int d\bar{S}\, \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

$$= -2(16)^2 g^2 \int d^4x \underbrace{\left[\frac{-i}{16}\partial_\mu(\lambda\sigma^\mu\bar{\lambda}) + \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]}_{\text{total divergence}}$$

$= 0$ (in U(1) case.)

Thus we see how the $\Theta$-term appears in the action in the non-abelian case. In perturbation theory in all cases we will ignore this term (it is zero in U(1) case)

So

$$\boxed{\Gamma_{SYM} = -2(16)^2 Z \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{16} (\lambda \overleftrightarrow{\not{\partial}} \bar{\lambda}) + \frac{1}{16} D^2 \right]}$$

Next we consider the gauge invariant Kähler potential terms

$$\Gamma_K = \int dV \left[ Z_+ \bar{\Phi}_+ e^{gV} \Phi_+ + Z_- \Phi_- e^{-gV} \bar{\Phi}_- \right]$$

in the W-Z gauge we expand the exponentials

$$= \int dV \Big[ Z_+ \bar{\Phi}_+ \Phi_+ \Big[ 1 + g\,\theta \sigma^\mu \bar{\theta} A_\mu + \frac{g}{2} \theta^2 \bar{\theta} \bar{\lambda} + \frac{g}{2} \bar{\theta}^2 \theta \lambda + \frac{g}{4} \theta^2 \bar{\theta}^2 D + \frac{g^2}{4} \theta^2 \bar{\theta}^2 A_\mu A^\mu \Big]$$

$$+ Z_- \bar{\Phi}_- \Phi_- \Big[ 1 - g\,\theta \sigma^\mu \bar{\theta} A_\mu - \frac{g}{2} \theta^2 \bar{\theta} \bar{\lambda} - \frac{g}{2} \bar{\theta}^2 \theta \lambda - \frac{g}{4} \theta^2 \bar{\theta}^2 D + \frac{g^2}{4} \theta^2 \bar{\theta}^2 A_\mu A^\mu \Big] \Big]$$

$$\Gamma_K = \int dV \Big[ Z_+ \left( e^{+i\theta\gamma\bar\theta}(\bar A_+ + \bar\theta\bar\psi_+ + \bar\theta^2 \bar F_+)\right) \times$$

$$\times \left( e^{-i\theta\gamma\bar\theta}(A_+ + \theta\psi_+ + \theta^2 F_+)\right) \times$$

$$\times \Big[ 1 + g\,\theta\sigma^\mu\bar\theta A_\mu + \tfrac{g}{2}\theta^2\bar\theta\bar\lambda + \tfrac{g}{2}\bar\theta^2\theta\lambda$$

$$+ \tfrac{g}{4}\theta^2\bar\theta^2(D + g A_\mu A^\mu)\Big]$$

$$+ (+ \to -,\ g \to -g) \Big]$$

---

Only the $\theta^2\bar\theta^2$ terms survive the vector ($\int dV$) integration — so isolating these terms

$$\Gamma_K^+ = \int dV\, Z_+ \bar\Phi_+ e^{gV} \Phi_+$$

$$= \int dV\, Z_+ \Big\{ \tfrac{g}{4}\theta^2\bar\theta^2 (D + g A_\mu A^\mu)\bar A_+ A_+$$

$$+ \tfrac{g}{2}\bar\theta^2\theta\lambda\,\bar A_+\,\theta\psi_+ + \tfrac{g}{2}\theta^2\bar\theta\bar\lambda\,\bar\theta\bar\psi_+ A_+$$

$$+ g\,\theta\sigma^\mu\bar\theta A_\mu \Big[ (i\theta\gamma\bar\theta\,\bar A_+)A_+ - i\bar A_+(\theta\gamma\bar\theta)A_+$$

$$+ \bar\theta\bar\psi_+\theta\psi_+ \Big]$$

$$+ \theta^2\bar\theta^2 \Big( \bar F_+ F_+ + \partial_\mu \bar A_+ \partial^\mu A_+ + \tfrac{i}{4}\psi_+ \overleftrightarrow{\gamma}\bar\psi_+$$

$$- \tfrac{1}{4}\partial^2(\bar A_+ A_+)\Big)\Big\}$$

Likewise for $+ \to -$, $g \to -g$ in $\Gamma_K^-$

As usual $\theta\sigma^\mu\bar\theta\,\theta\sigma^\nu\bar\theta = -\theta^\alpha\theta^\beta\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$

$$= \frac{1}{4}\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\alpha}\sigma^{\nu\,\alpha\dot\alpha}$$

$$= \frac{1}{4}\theta^2\bar\theta^2\, 2\eta^{\mu\nu}$$

$$\theta\lambda\,\theta\psi_+ = -\theta^\alpha\theta^\beta\lambda_\alpha\psi_{+\beta} = -\frac{1}{2}\theta^2\lambda\psi_+$$

$$\bar\theta\bar\lambda\,\bar\theta\bar\psi_+ = -\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\bar\lambda_{\dot\alpha}\bar\psi_{+\dot\beta} = -\frac{1}{2}\bar\theta^2\bar\lambda\bar\psi_+$$

$$\theta\sigma^\mu\bar\theta\;\bar\theta\bar\psi_+\,\theta\psi_+ = -\theta^\alpha\theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\bar\psi_{+\dot\beta}\psi_{+\alpha}$$

$$= \frac{1}{4}\theta^2\bar\theta^2\epsilon^{\beta\alpha}\epsilon^{\dot\beta\dot\alpha}\sigma^\mu_{\beta\dot\alpha}\;\bar\psi_{+\dot\beta}\psi_{+\alpha}$$

$$= +\frac{1}{4}\theta^2\bar\theta^2\,\psi_+\sigma^\mu\bar\psi_+$$

So we have

$$\boxed{\begin{aligned}\Gamma_K^+ = \int dV\,\theta^2\bar\theta^2 Z_+ \Big\{ & \bar F_+ F_+ + \frac{i}{4}\psi_+(\not\partial - i\frac{g}{2}\not A)\bar\psi_+ \\ & -\frac{i}{4}(\partial_\mu + \frac{ig}{2}A_\mu)\psi_+\sigma^\mu\bar\psi_+ \\ & + (\partial_\mu - i\frac{g}{2}A_\mu)\bar A_+(\partial^\mu + \frac{ig}{2}A^\mu)A_+ \\ & + \bar A_+\frac{g}{4}D A_+ \quad -\frac{1}{4}g\lambda\psi_+\bar A_+ - \frac{1}{4}g\bar\lambda\bar\psi_+ A_+ \Big\}\end{aligned}}$$

and $\Gamma_K^-$ is obtained from above with

$+ \to -$, $g \to -g$

So we obtain

$$\Gamma_K = \int d^4x \Big\{ 16 Z_+ \Big( (\partial_\mu - i\tfrac{g}{2} A_\mu)\bar{A}_+ (\partial^\mu + \tfrac{ig}{2} A^\mu) A_+$$

$$+ \tfrac{i}{4} \psi_+ \sigma^\mu (\partial_\mu - \tfrac{ig}{2} A_\mu) \bar{\psi}_+$$

$$- \tfrac{i}{4} (\partial_\mu + \tfrac{ig}{2} A_\mu) \psi_+ \sigma^\mu \bar{\psi}_+$$

$$+ \bar{F}_+ F_+ + D \tfrac{g}{4} \bar{A}_+ A_+$$

$$- \tfrac{g}{4} (\lambda \psi_+ \bar{A}_+ + \bar{\lambda} \bar{\psi}_+ A_+) \Big)$$

$$+ 16 Z_- \Big( (\partial_\mu + i\tfrac{g}{2} A_\mu) \bar{A}_- (\partial^\mu - \tfrac{ig}{2} A^\mu) A_-$$

$$+ \tfrac{i}{4} \psi_- \sigma^\mu (\partial_\mu + \tfrac{ig}{2} A_\mu) \bar{\psi}_-$$

$$- \tfrac{i}{4} (\partial_\mu - \tfrac{ig}{2} A_\mu) \psi_- \sigma^\mu \bar{\psi}_-$$

$$+ \bar{F}_- F_- - D \tfrac{g}{4} A_- \bar{A}_-$$

$$+ \tfrac{g}{4} (\lambda \psi_- \bar{A}_- + \bar{\lambda} \bar{\psi}_- A_-) \Big) \Big\}$$

This just looks like a usual $U(1)$ gauge theory with gauge covariant derivatives

$$D_\mu A_\pm \equiv \left(\partial_\mu \pm i\frac{g}{2}A_\mu\right)A_\pm$$

$$D_\mu \bar{A}_\pm \equiv \left(\partial_\mu \mp i\frac{g}{2}A_\mu\right)\bar{A}_\pm$$

$$= \left[D_\mu A_\pm\right]^\dagger$$

$$D_\mu \psi_\pm \equiv \left(\partial_\mu \pm i\frac{g}{2}A_\mu\right)\psi_\pm$$

$$D_\mu \bar{\psi}_\pm \equiv \left(\partial_\mu \mp i\frac{g}{2}A_\mu\right)\bar{\psi}_\pm$$

$$= \left(D_\mu \psi_\pm\right)^\dagger$$

and $A_\mu$, $\lambda$, $\bar{\lambda}$, $D$ are in the adjoint representation of $U(1)$ — which just means they are (global) $U(1)$ singlets and in the $W$-$Z$ gauge we have residual gauge transformations

$$A'_\mu = A_\mu - \partial_\mu(\omega^\dagger + \omega) = A_\mu - 2\partial_\mu \omega$$

$$\lambda' = \lambda \; ; \; \bar{\lambda}' = \bar{\lambda} \; , \; D' = D$$

$$A'_\pm = e^{\pm ig\omega} A_\pm \; ; \; \psi'_\pm = e^{\pm ig\omega}\psi_\pm$$

$$\bar{A}'_\pm = e^{\mp ig\omega}\bar{A}_\pm \; ; \; \bar{\psi}'_\pm = e^{\mp ig\omega}\bar{\psi}_\pm$$

and $F'_\pm = e^{\pm igw} F_\pm$ ; $\bar{F}'_\pm = e^{\mp igw} \bar{F}_\pm$ .

Hence we can write the action as

$$\boxed{\begin{aligned}\Gamma_K = 16 Z_+ \int d^4x \Big\{ & D_\mu \bar{A}_+ D^\mu A_+ + \frac{i}{4} \psi_+ \overleftrightarrow{\not{D}} \bar{\psi}_+ \\ & + \bar{F}_+ F_+ + D \frac{g}{4} \bar{A}_+ A_+ \\ & - \frac{g}{4} (\lambda \psi_+ \bar{A}_+ + \bar{\lambda} \bar{\psi}_+ A_+) \Big\} \\ + 16 Z_- \int d^4x \Big\{ & D_\mu \bar{A}_- D^\mu A_- + \frac{i}{4} \psi_- \overleftrightarrow{\not{D}} \bar{\psi}_- \\ & + \bar{F}_- F_- - D \frac{g}{4} A_- \bar{A}_- \\ & + \frac{g}{4} (\lambda \psi_- \bar{A}_- + \bar{\lambda} \bar{\psi}_- A_-) \Big\}\end{aligned}}$$

Finally we have the usual looking mass terms

$$\boxed{\begin{aligned}\Gamma_W &= \frac{m}{4} \int dS\, \phi_+ \phi_- + \frac{m}{4} \int d\bar{S}\, \bar{\phi}_+ \bar{\phi}_- \\ &= -m \int d^4x \Big[ A_+ F_- + A_- F_+ - \frac{1}{2} \psi_+ \psi_- \\ &\qquad + \bar{A}_+ \bar{F}_- + \bar{A}_- \bar{F}_+ - \frac{1}{2} \bar{\psi}_+ \bar{\psi}_- \Big]\end{aligned}}$$

Thus we have the complete SQED action in the W-Z gauge

$$\Gamma^{WZ}_{SQED} = \Gamma^{WZ}_{sym} + \Gamma^{WZ}_{K} + \Gamma^{WZ}_{W}$$

$$\Gamma^{WZ}_{SQED} = \int d^4x \, \mathcal{L}^{WZ}_{SQED} \quad \text{with}$$

$$\mathcal{L}^{WZ}_{SQED} = \left(-2(16)^2 Z\right)\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{16}(\lambda \overleftrightarrow{\not{\partial}} \bar{\lambda}) + \frac{1}{16}D^2\right]$$

$$+16Z_+\Big(D_\mu \bar{A}_+ D^\mu A_+ + \frac{i}{4}\psi_+ \overleftrightarrow{\not{D}} \bar{\psi}_+ + \bar{F}_+ F_+ + D\frac{g}{4}\bar{A}_+ A_+ - \frac{g}{4}(\lambda \psi_+ \bar{A}_+ + \bar{\lambda}\bar{\psi}_+ A_+)\Big)$$

$$+16Z_-\Big(D_\mu \bar{A}_- D^\mu A_- + \frac{i}{4}\psi_- \overleftrightarrow{\not{D}} \bar{\psi}_- + \bar{F}_- F_- - D\frac{g}{4}A_- \bar{A}_- + \frac{g}{4}(\lambda \psi_- \bar{A}_- + \bar{\lambda}\bar{\psi}_- A_-)\Big)$$

$$-m\Big[A_+ F_- + A_- F_+ - \frac{1}{2}\psi_+ \psi_- + \bar{A}_+ \bar{F}_- + \bar{A}_- \bar{F}_+ - \frac{1}{2}\bar{\psi}_+ \bar{\psi}_-\Big]$$

The F and D fields are <u>auxiliary</u> fields since they have no derivatives acting on themselves - their equations of motion are algebraic and they can be eliminated from the Lagrangian.

We also note that the potential of the theory is given in terms of these <u>F-terms</u> and <u>D-terms</u>

$$V = \left(2(16)^2 Z \tfrac{1}{16}\right) D^2 \quad (16Z_+)\bar{F}_+ F_+$$

$$(16Z_-)\bar{F}_- F_-$$

$$+ m(F_- A_+ + F_+ A_-) + m(\bar{F}_- \bar{A}_+ + \bar{F}_+ \bar{A}_-)$$

$$+ (16Z_-)\frac{g}{4} A_- \bar{A}_- D - (16Z_+)\frac{g}{4} A_+ \bar{A}_+ D$$

---

The Euler-Lagrange field equations for the auxiliary fields are just

$$0 = \frac{\delta \Gamma^{WZ}_{SQED}}{\delta D(x)} = -\frac{\partial V(x)}{\partial D(x)}$$

$$0 = \frac{\delta \Gamma^{WZ}_{SQED}}{\delta F_+(x)} = -\frac{\partial V(x)}{\partial F_+(x)} \quad \text{and so on.}$$

Before carrying out all this algebra – let's choose a convenient normalization to cut down on the arbitrary factors floating around

let $Z_\pm \equiv \frac{1}{16}$ ; $Z = \frac{-1}{2(16)^2}$

Then re-scale the fields in the superfield expansion

$D \rightarrow \sqrt{8}\, D$ (i.e. $D^2 \rightarrow 8D^2$)

$\lambda, \bar\lambda \rightarrow \sqrt{8}\,\lambda, \bar\lambda$ (i.e $-\frac{i}{16}(\lambda \overleftrightarrow{\not\partial} \bar\lambda) \rightarrow -\frac{i}{2}\lambda \overleftrightarrow{\not\partial} \bar\lambda$)

$\psi_\pm, \bar\psi_\pm \rightarrow \sqrt{2}\,\psi_\pm, \bar\psi_\pm$

So the $\mathcal{L}^{WZ}_{SQED}$ lagrangian becomes

$$
\begin{aligned}
\mathcal{L}^{WZ}_{SQED} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\lambda\overleftrightarrow{\not\partial}\bar\lambda + \tfrac{1}{2}D^2 \\
&+ D_\mu \bar A_+ D^\mu A_+ + \tfrac{i}{2}\psi_+ \overleftrightarrow{\not D}\bar\psi_+ + \bar F_+ F_+ \\
&+ D_\mu \bar A_- D^\mu A_- + \tfrac{i}{2}\psi_- \overleftrightarrow{\not D}\bar\psi_- + \bar F_- F_- \\
&- m\big[ A_+ F_- + A_- F_+ + \bar A_+ \bar F_- + \bar A_- \bar F_+ \\
&\qquad - \psi_+\psi_- - \bar\psi_+\bar\psi_- \big] \\
&- \tfrac{g}{2}\left(\lambda\psi_+\bar A_+ + \bar\lambda\bar\psi_+ A_+ - \lambda\psi_-\bar A_- - \bar\lambda\bar\psi_- A_-\right) \\
&+ \tfrac{g}{\sqrt{2}} D\left(\bar A_+ A_+ - A_-\bar A_-\right)
\end{aligned}
$$

And likewise the Potential becomes

$$\boxed{\begin{aligned} V = & -\tfrac{1}{2}D^2 - \bar{F}_+ F_+ - \bar{F}_- F_- \\ & + m\left(A_+ F_- + A_- F_+ + \bar{A}_+ \bar{F}_- + \bar{A}_- \bar{F}_+\right) \\ & + \tfrac{g}{\sqrt{2}} D\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right) \end{aligned}}$$

So the auxiliary fields' equations of motion are simply algebraic

0) $$-\frac{\partial V}{\partial D} = 0 = D - \frac{g}{\sqrt{2}}\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right)$$

$$\Rightarrow \boxed{D = \frac{g}{\sqrt{2}}\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right)}$$

1) $$-\frac{\partial V}{\partial F_+} = 0 = \bar{F}_+ - m A_- \Rightarrow \boxed{\bar{F}_+ = m A_-}$$

2) $$-\frac{\partial V}{\partial \bar{F}_+} = 0 = F_+ - m \bar{A}_- \Rightarrow \boxed{F_+ = m \bar{A}_-}$$

3) $$-\frac{\partial V}{\partial F_-} = 0 = \bar{F}_- - m A_+ \Rightarrow \boxed{\bar{F}_- = m A_+}$$

4) $$-\frac{\partial V}{\partial \bar{F}_-} = 0 = F_- - m \bar{A}_+ \Rightarrow \boxed{F_- = m \bar{A}_+}$$

Plugging this into the potential we find

$$V = -\frac{1}{2}D^2 - \bar{F}_+ F_+ - \bar{F}_- F_- + D^2$$

$$+ \left( \bar{F}_- F_- + \bar{F}_+ F_+ + F_- \bar{F}_- + F_+ \bar{F}_+ \right)$$

$$\boxed{V = +\frac{1}{2}D^2 + \bar{F}_+ F_+ + \bar{F}_- F_-}$$

We note an important property of the potential energy in SUSY theories it is <u>non-negative</u>. The ground state (vacuum) must have the $V=0$ energy $\Rightarrow$

$D=0$ ; $F_+ = 0$ , $F_- = 0$, $\bar{F}_+ = 0$, $\bar{F}_- = 0$
for the SUSY vacuum

In general for any # of vector fields $V^i \to D^i$ & chiral fields $\phi^a, \bar{\phi}^a \to F^a, \bar{F}^a$ the potential is

$$V = \frac{1}{2}\sum_i (D^i)^2 + \sum_a \left( \bar{F}^a F^a \right) \geq 0$$

Substituting the field equations we have

$$\bar{F}_+ F_+ = m^2 \bar{A}_- A_- \;, \quad \bar{F}_- F_- = m^2 \bar{A}_+ A_+$$

$$\tfrac{1}{2} D^2 = \frac{g^2}{4}\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right)^2$$

$\Rightarrow$

$$\boxed{V = \frac{g^2}{4}\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right)^2 + m^2\left(\bar{A}_+ A_+ + \bar{A}_- A_-\right)}$$

& The SQED Lagrangian becomes

$$\boxed{\begin{aligned}\mathcal{L}^{WZ}_{SQED} = & -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{i}{2} \lambda \overleftrightarrow{\not{\partial}} \bar{\lambda} \\ & + D_\mu \bar{A}_+ D^\mu A_+ + \tfrac{i}{2} \psi_+ \overleftrightarrow{\not{D}} \bar{\psi}_+ + m \psi_+ \psi_- \\ & + D_\mu \bar{A}_- D^\mu A_- + \tfrac{i}{2} \psi_- \overleftrightarrow{\not{D}} \bar{\psi}_- + m \bar{\psi}_+ \bar{\psi}_- \\ & - m^2\left(\bar{A}_+ A_+ + \bar{A}_- A_-\right) - \frac{g^2}{4}\left(\bar{A}_+ A_+ - A_- \bar{A}_-\right)^2 \\ & - \frac{g}{2}\left(\lambda \psi_+ \bar{A}_+ + \bar{\lambda} \bar{\psi}_+ A_+ - \lambda \psi_- \bar{A}_- - \bar{\lambda} \bar{\psi}_- A_-\right)\end{aligned}}$$

Recall in the covariant derivatives we had $\frac{g}{2}$, so we can also rescale $\frac{g}{2} \to g$ (i.e. $g \to 2g$) So only $g$ appears everywhere.

So we see this looks like ordinary QED with a ~~photon~~ field $A_\mu$ & ~~and~~ electron (massive Dirac electron) field

$$\psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix}.$$

In addition the SQED theory contains a massless Majorana fermion $\lambda$, and massive charged scalar fields $A_\pm$. The Yukawa and quartic couplings are required to have the same coupling constant g ~~a~~ by SUSY — In addition this is the gauge coupling constant. Further the SUSY requires the mass of the scalars and electron to be the same — m while the photon and photino $\lambda, \bar\lambda$ are required to have the same mass 0.

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The photinos can be put in the 4-component Majorana field

$$\Lambda = \begin{pmatrix} \lambda \\ \bar\lambda \end{pmatrix}$$ and the

whole Lagrangian written in terms of Dirac $\gamma$-matrices $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$ and 4-component spinors to look more like QED.

For example the Dirac mass term for the electron is

$$m\bar\psi\psi = m\psi^\dagger\gamma^0\psi = m\,(\bar\psi_+\ \psi_-)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\psi_+\\ \bar\psi_-\end{pmatrix}$$

$$= m(\bar\psi_+\bar\psi_- + \psi_-\psi_+)$$ as it appears in the Lagrangian.

Also $$\bar\psi\gamma\psi = (\bar\psi_+\ \psi_-)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\gamma\\ \bar\gamma&0\end{pmatrix}\begin{pmatrix}\psi_+\\ \bar\psi_-\end{pmatrix}$$

$$= (\psi_-\ \bar\psi_+)\begin{pmatrix}\gamma\bar\psi_-\\ \bar\gamma\psi_+\end{pmatrix} = \psi_-\gamma\bar\psi_- + \bar\psi_+\bar\gamma\psi_+$$

$$= \psi_-\gamma\bar\psi_- - \psi_+\overleftarrow{\gamma}\bar\psi_+$$

and

$$-\bar\psi\overleftarrow{\gamma}\psi = -\psi_-\overleftarrow{\gamma}\bar\psi_- - \bar\psi_+\overleftarrow{\bar\gamma}\psi_+$$

$$= -\psi_-\overleftarrow{\gamma}\bar\psi_- + \psi_+\overrightarrow{\gamma}\bar\psi_+$$

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So together we have

$$\boxed{\bar\psi\overleftrightarrow{\gamma}\psi = \psi_-\overleftrightarrow{\gamma}\bar\psi_- + \psi_+\overleftrightarrow{\gamma}\bar\psi_+}$$

and so on.

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