SUSY Gauge Theories

Suppose we begin with consideration of a global symmetry group G with generators

$(T^i)_{ab}$ which obey the associated Lie Algebra

$[T^i, T^j] = i f_{ijk} T^k$ $(i = 1 \ldots, N$; $a, b = 1, \ldots, d$
N = dimension of group, d = dimension of representation)

Suppose we have we have chiral superfields in the d-dim. representation of G, then

$$\phi_a' = (e^{igw^i T^i})_{ab} \phi_b = U(\omega)_{ab} \phi_b$$

Since w^i is a constant in this case ϕ_a' is still a chiral superfield $\bar{D}_{\dot\alpha} \phi_a = 0 = \bar{D}_{\dot\alpha} \phi_a'$.

Likewise the anti-chiral field $\bar\Phi_a$ transforms as the complex conjugate transformation (ex. if ϕ_a is a 3 of $SU(3)$ then $\bar\Phi_a$ is a $\bar{3}$ of $SU(3)$)

$$\bar\Phi_a' = U^*(\omega)_{ab} \bar\Phi_b = \bar\Phi_b U^\dagger_{ba}(\omega) = \bar\Phi_b U^{-1}_{ba}(\omega) .$$

For infinitesimal w^i we have that

$$\phi_a' = \phi_a + i g w^i T^i_{ab} \phi_b$$
$$\bar\Phi_a' = \bar\Phi_a - i g \bar\Phi_b w^i T^i_{ba}$$

The SUSY and globally G invariant action is then given by the G singlet (invariant) monomials made from ϕ_a, & $\bar{\phi}_a$ integrated over the appropriate SUSY measure:

$$\Gamma = \int dV\, K(\phi_a, \bar{\phi}_a) + \int dS\, W(\phi_a) + \int d\bar{S}\, \bar{W}(\bar{\phi}_a)$$

where $K = \frac{1}{16}\bar{\phi}_a \phi_a$ while W & $\bar{W}$ are G invariant quadratic and cubic monomials.

ex. Consider a model with 2 fields in SU(3) 3 & $\bar{3}$ rep.'s $\phi_{3a}, X_{\bar{3}a}$, their anti-chiral counterparts are $\bar{\phi}_{\bar{3}a}, \bar{X}_{3a}$. The Kähler potential is

$$K = \frac{1}{16}\bar{\phi}_{\bar{3}a}\phi_{3a} + \frac{1}{16}X_{\bar{3}a}\bar{X}_{3a}$$

The possible superpotential terms are just mass terms

$$W(\phi, X) = \frac{m}{4} X_{\bar{3}a}\phi_{3a}$$

Now suppose we desire to make this symmetry local!

It is not sufficient to just let $\omega^i = \omega^i(x)$ as ϕ'_a will no longer be chiral! $\bar{D}_{\dot\alpha}\phi'_a \neq 0$. Hence we must let the gauge transformation parameter depend on θ & $\bar\theta$ as well. In addition it must preserve the chirality of the fields — hence the gauge parameter for a chiral field must be a chiral superfield, denoted $\Lambda^i(x,\theta,\bar\theta)$ and for an anti-chiral field an anti-chiral superfield parameter denoted $\bar\Lambda^i(x,\theta,\bar\theta)$ So

$\bar{D}_{\dot\alpha}\Lambda^i = 0 = D_\alpha \bar\Lambda^i$. Hence we define

the local gauge transformations as

$$\phi'_a = \left(e^{ig\Lambda^i T^i}\right)_{ab}\phi_b = U_{ab}(\Lambda)\phi_b$$

$$\bar\phi'_a = \bar\phi_b\left(e^{-ig\bar\Lambda^i T^i}\right)_{ba} = \bar\phi_b U^\dagger_{ba}(\bar\Lambda) = \bar\phi_b U^{-1}_{ba}(\bar\Lambda)$$

Now since the superpotentials are purely chiral, they remain invariant under these local gauge transformations as well. However the kinetic energy – Kähler potential – terms do not!

$$\bar\phi'_a \phi'_a = \bar\phi_b \left(e^{-ig\bar\lambda^i T^i}\right)_{ba} \left(e^{+ig\lambda^j T^j}\right)_{ac} \phi_c$$

$$\neq \bar\phi_a \phi_a \quad \text{(except for the } \lambda = \bar\lambda = \text{const. case!)}$$

Hence we must introduce the SUSY Yang-Mills field $V^i = V^i(x, \theta, \bar\theta)$ in the adjoint (global) representation of G and consider

$$\bar\phi'_a f_{ab}(V'^i T^i) \phi'_b$$

$$= \bar\phi\, e^{-ig\bar\lambda \cdot T} f(V' \cdot T)\, e^{+ig\lambda \cdot T} \phi$$

$$\equiv \bar\phi_a f_{ab}(V \cdot T) \phi_b$$

$\Rightarrow$

$$f(V \cdot T) = e^{-ig\bar\lambda \cdot T} f(V' \cdot T)\, e^{+ig\lambda \cdot T}$$

That is multiplying on the left & right

$$f(V' \cdot T) = e^{+ig\bar\lambda \cdot T} f(V \cdot T)\, e^{-ig\lambda \cdot T}$$

We define V^i to transform with an inhomogeneous piece - since we will evaluate these products by use of the Baker-Campbell-Hausdorff formula - the result will not depend on which representation matrix T^i we choose and we know that the exponential for f will yield the inhomogeneous term. So let

$$f = e^{gV\cdot T} = e^{gV^iT^i}$$

Thus $$\boxed{e^{gV'^iT^i} = e^{+ig\bar{\Lambda}\cdot T}\, e^{gV\cdot T}\, e^{-ig\Lambda\cdot T}}$$

So if we start expanding all the exponentials we find

$$1 + gV'^iT^i + \dots = 1 + ig\bar{\Lambda}\cdot T - ig\Lambda\cdot T + gV\cdot T + \dots$$

So $$\boxed{V'^i = V^i + i(\bar{\Lambda}^i - \Lambda^i) + \dots}$$ The inhomogeneous term is present! Thus we also have that the locally gauge invariant Kähler potential is given by

$$\boxed{K = K(\bar{\phi}\, e^{gV\cdot T}\phi) = \frac{1}{16}\bar{\phi}_a (e^{gV^iT^i})_{ab}\phi_b}$$

$$K' = K,$$

For infinitesimal gauge transformations we can further evaluate the Super Y-M's field's transformation. For infinitesimal A and finite B, we have the BCH formulae:

$$e^A e^B = e^{B - \mathcal{L}_{B/2}\cdot[A - \coth(\mathcal{L}_{B/2})\cdot A]}$$

$$e^B e^A = e^{B + \mathcal{L}_{B/2}\cdot[A + \coth(\mathcal{L}_{B/2})\cdot A]}$$

$$e^A e^B e^{-A} = e^{B + [A,B]}$$

and the Lie derivative is defined as the commutator

$$\mathcal{L}_{B/2}\cdot A \equiv \left[\frac{B}{2}, A\right] .$$

$$\Rightarrow \quad e^{gV'\cdot T} = e^{gV\cdot T - \mathcal{L}_{\frac{gV\cdot T}{2}}\left[+ig(\Lambda\cdot T + \bar{\Lambda}\cdot T) + \coth \mathcal{L}_{\frac{gV\cdot T}{2}}\cdot(ig(\Lambda\cdot T - \bar{\Lambda}\cdot T))\right]}$$

That is

$$\boxed{gV'\cdot T = gV\cdot T - ig\,\mathcal{L}_{\frac{gV\cdot T}{2}}\cdot\left[(\Lambda\cdot T + \bar{\Lambda}\cdot T) + \coth \mathcal{L}_{\frac{gV\cdot T}{2}}\cdot(\Lambda\cdot T - \bar{\Lambda}\cdot T)\right]}$$

factoring out the g's gives

$$\boxed{V'\cdot T = V\cdot T - i\mathcal{L}_{\frac{gV\cdot T}{2}}\cdot\left[(\Lambda+\bar\Lambda)\cdot T + \coth\mathcal{L}_{\frac{gV\cdot T}{2}}\cdot(\Lambda-\bar\Lambda)\cdot T\right]}$$

Defining $V_{ij} \equiv -if_{ijk}V^k = (T^k)_{ij}\,V^k$

The adjoint representation

we can express the above commutators of $[T^i, T^j] = if_{ijk}T^k$ as the formula

$$\boxed{V'^i = V^i - \frac{i}{2}g(\Lambda^j+\bar\Lambda^j)f_{ijk}V^k - i(\Lambda^j-\bar\Lambda^j)\left[\frac{g}{2}V\coth\left(\frac{g}{2}V\right)\right]_{ji}}$$

Recall $\coth x = \frac{1}{x} + \cdots$ so we have

$$V'^i = V^i - \frac{ig}{2}(\Lambda^j+\bar\Lambda^j)f_{ijk}V^k - i(\Lambda^i-\bar\Lambda^i) + \cdots$$

Hence we have the matter-YM part of the SUSY & gauge invariant action - next we need the pure super Yang-Mills part of the action. The anti-symmetric field strength tensor $F^i_{\mu\nu}$ is generalized by the SUSY covariant field strength chiral and anti-chiral spinors!

$$W_\alpha \equiv \bar{D}\bar{D}\left[e^{-gV\cdot T} D_\alpha e^{+gV\cdot T}\right]$$

where $(T^i)_{jk} = i f_{jik}$ are the adjoint representation generators. and $\bar{D}_{\dot\alpha} W_\alpha = 0$

Using $e^{gV'\cdot T} e^{-gV'\cdot T} = 1$, we have that

$$e^{ig\bar{\Lambda}\cdot T} e^{gV\cdot T} e^{-ig\Lambda\cdot T} e^{-gV'\cdot T} = 1$$

hence $e^{-gV'\cdot T} = e^{+ig\Lambda\cdot T} e^{-gV\cdot T} e^{-ig\bar{\Lambda}\cdot T}$.

Using this we can calculate the variation of the field strength spinor

$$W'_\alpha = \bar{D}\bar{D}\left[e^{-gV'\cdot T} D_\alpha e^{+gV'\cdot T}\right]$$

$$= \bar{D}\bar{D}\left[e^{+ig\bar{\Lambda}\cdot T} e^{-gV\cdot T} e^{-ig\bar{\Lambda}\cdot T} D_\alpha \left(e^{+ig\bar{\Lambda}\cdot T} e^{gV\cdot T} e^{-ig\Lambda\cdot T}\right)\right]$$

Since $D_\alpha \bar{\Lambda} = 0$ & $\bar{D}_{\dot\alpha} \Lambda = 0$ this simplifies to

$$W'_\alpha = e^{+ig\Lambda\cdot T} \bar{D}\bar{D}\left[e^{-gV\cdot T} D_\alpha \left(e^{gV\cdot T} e^{-ig\Lambda\cdot T}\right)\right]$$

$$= e^{+ig\Lambda\cdot T}\left(\bar{D}\bar{D}\left[e^{-gV\cdot T} D_\alpha e^{+gV\cdot T}\right)e^{-ig\Lambda\cdot T}\right.$$

$$+ e^{ig\Lambda\cdot T} \bar{D}\bar{D}\left[D_\alpha e^{-ig\Lambda\cdot T}\right]$$

But recall $[\bar{D}\bar{D}, D_\alpha] = \bar{D}_{\dot\alpha}\{\bar{D}^{\dot\alpha}, D_\alpha\} - \{D_\alpha, \bar{D}_{\dot\alpha}\}\bar{D}^{\dot\alpha}$

$$= -4i\,\not\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} = -4i(\not\partial\bar{D})_\alpha$$

So $\bar{D}\bar{D}D_\alpha e^{-ig\Lambda\cdot T} = D_\alpha \bar{D}\bar{D} e^{-ig\Lambda\cdot T} - 4i(\not\partial\bar{D})_\alpha e^{-ig\Lambda\cdot T}$

$$= 0 \text{ since } \bar{D}_{\dot\alpha}\Lambda = 0$$

Hence we find the homogeneous chiral gauge transformation for the field strength

$$\boxed{W'_\alpha = e^{+ig\Lambda\cdot T} W_\alpha e^{-ig\Lambda\cdot T}}$$

The field strength spinor is in the <u>adjoint</u> representation of the gauge group.

Hence we can make a gauge (& Lorentz) invariant quantity that starts bilinear in the gauge field by

$$\mathrm{Tr}[W^\alpha W_\alpha]$$ where the trace is over the <u>adjoint Rep.</u> $(T^i)_{jk}$ matrices —

$$\mathrm{Tr}[W'^\alpha W'_\alpha] = \mathrm{Tr}\{e^{+ig\Lambda\cdot T} W^\alpha e^{-ig\Lambda\cdot T} e^{+ig\Lambda\cdot T} \times W_\alpha e^{-ig\Lambda\cdot T}\}$$

$$= \mathrm{Tr}[e^{ig\Lambda\cdot T} W^\alpha W_\alpha e^{-ig\Lambda\cdot T}]$$

(cyclicity of trace) $$= \mathrm{Tr}\{e^{-ig\Lambda\cdot T} e^{+ig\Lambda\cdot T} W^\alpha W_\alpha\}$$

$$= \mathrm{Tr}\{W^\alpha W_\alpha\}$$ gauge invariant.

Since W_α is a chiral superfield — the SUSY invariant action is made by integrating over the chiral measure

$$\frac{1}{g^2}\int dS\, \mathrm{Tr}\{W^\alpha W_\alpha\}$$ where the $\frac{1}{g^2}$ factor cancels the g^2 from the bilinear term from the W's.

Analogously we derive the expressions involving the complex conjugate anti-chiral field strength spinor $\bar{W}_{\dot{\alpha}}$

$$\boxed{\bar{W}_{\dot{\alpha}} \equiv DD\left[e^{+gV\cdot T}\,\bar{D}_{\dot{\alpha}}\,e^{-gV\cdot T}\right]}$$

So $D_\alpha \bar{W}_{\dot{\alpha}} = 0$ since $D_\alpha D_\beta D_\gamma = 0$.

Likewise $\bar{W}_{\dot{\alpha}}$ is in the anti-chiral adjoint representation of the gauge group

$$\bar{W}'_{\dot{\alpha}} = e^{+ig\bar{\Lambda}\cdot T}\,\bar{W}_{\dot{\alpha}}\,e^{-ig\bar{\Lambda}\cdot T}$$

where we used $[DD, \bar{D}_{\dot{\alpha}}] = +4i(D\not{\partial})_{\dot{\alpha}}$.

Hence the gauge invariant (& Lorentz inv.) action term is made from

$$Tr[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}]$$

$\left(Tr[\bar{W}'_{\dot{\alpha}}\bar{W}'^{\dot{\alpha}}] = Tr[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}]\right.$ so it is G invariant$\left.\right)$

The SUSY and gauge invariant action term is made by integrating over the anti-chiral measure

$$\frac{1}{g^2}\int d\bar{S}\, \mathrm{Tr}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}] \ .$$

Actually, there is now reason the coefficient of these terms needs to be real — we just ask for the pure SUSY Y-M action to be real — hence we have

$$\Gamma_{sym} = \frac{-1}{32}\int dS\left(\frac{1}{g^2} - i\frac{\theta}{8\pi^2}\right)\mathrm{Tr}[W^\alpha W_\alpha]$$

$$-\frac{1}{32}\int d\bar{S}\left(\frac{1}{g^2} + i\frac{\theta}{8\pi^2}\right)\mathrm{Tr}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}]$$

The holomorphic gauge coupling $\frac{\tau}{4\pi i} = \frac{1}{g^2} - i\frac{\theta}{8\pi^2}$

The θ terms as usual will lead to the $F_{\mu\nu}\tilde{F}^{\mu\nu}$ surface term in the component action.

So we have the SUSY & gauge invariant ~~acti~~ pure Y-M action

$$\boxed{\Gamma_{sym} = \frac{Z}{g^2}\int dS\, \mathrm{Tr}\{W^\alpha W_\alpha\} + \frac{Z}{g^2}\int d\bar{S}\, \mathrm{Tr}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}]}$$

where we have set the θ-term to zero for our perturbative work and choose Z for

The normalization of the fields to be specified later.

So we have the generic form of a SUSY and gauge invariant action

$$\Gamma = \Gamma_{SYM} + \Gamma_K + \Gamma_W$$

SUSY Y-M: $\Gamma_{SYM} = \frac{Z}{g^2}\int dS\, Tr\{W^\alpha W_\alpha\} + \frac{Z}{g^2}\int d\bar{S}\, Tr\{\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}\}$

The gauge invariant Kähler potential term is

$$\Gamma_K = \int dV\, K(\bar\phi e^{gV\cdot T} \phi)$$

with $K = \frac{1}{16}\bar\phi_a (e^{gV^i T^i})_{ab} \phi_b$ where $(T^i)_{ab}$ are the matter field representation matrices

and finally the gauge invariant superpotential terms with the same form as in the global symmetry case

$$\Gamma_W = \int dS\, W(\phi) + \int d\bar{S}\, \bar{W}(\bar\phi)$$

where $W(\phi') = W(\phi)$ is gauge invariant.

for example let ϕ_N be an N of $SU(N)$ and $\Sigma_{\bar{N}}$ be in the $\bar{N}$ of $SU(N)$ then

$$\Gamma_W = m\int dS\, \Sigma_{\bar{N}}\, \phi_N + m\int d\bar{S}\, \bar{\phi}_{\bar{N}}\, \bar{\Sigma}_N$$

or if ϕ^i is in the adjoint rep. of $SU(N)$

$$\Gamma_W = m\int dS\, \mathrm{Tr}[\phi^2] + m\int d\bar{S}\, \mathrm{Tr}[\bar{\phi}^2]$$

$$+ g_Y\int dS\, \mathrm{Tr}[\phi^3] + g_Y^*\int d\bar{S}\, \mathrm{Tr}[\bar{\phi}^3]$$

where here $(\phi)_{jk} = i(T^i)_{jk}\, \phi^i$

As a specific example let's study <u>SUSY QED</u> an $U(1)$ abelian gauge theory.