Wess & Zumino found that the most general symmetry of the S-matrix involves charges which obey both commutation and anti-commutation relations. Such an algebra is called a graded Lie algebra. These algebras are generalizations of the Poincaré algebra. The simplest ($N=1$) supersymmetry (SUSY) algebra involves the generators of the Poincaré group P^μ, the generators of translations, $M^{\mu\nu}$, the generators of Lorentz transformations and two anti-commuting (Grassmann) spinor charges Q_α and $\bar{Q}_{\dot\alpha}$, the generators of supersymmetry transformations.

The $N=1$ SUSY graded Lie algebra consists of the Poincaré Algebra

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma})$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda})$$

$$\left(\begin{aligned} &= i\,[D_{\mu\nu}]_\lambda{}^\rho P_\rho \\ &= i(\delta_\mu{}^\rho g_{\nu\lambda} - \delta_\nu{}^\rho g_{\mu\lambda})P_\rho \end{aligned} \right)$$

Plus the anti-commutation relations

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = +2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

and the fact that the SUSY charges are spinors

$$[M^{\mu\nu}, Q_\alpha] = -\tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[M^{\mu\nu}, \bar{Q}_{\dot\alpha}] = +\tfrac{1}{2}(\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}$$

and finally the trivial zero commutators

$$[Q_\alpha, P^\mu] = 0 = [\bar{Q}_{\dot\alpha}, P^\mu] \quad .$$

The SUSY algebra is invariant under multiplication of Q_α by a phase that is

$$(Q'_\alpha =) \; e^{+i\alpha R} Q_\alpha e^{-i\alpha R} \equiv e^{+i\alpha} Q_\alpha$$

$$(\bar{Q}'_{\dot\alpha} =) \; e^{+i\alpha R} \bar{Q}_{\dot\alpha} e^{-i\alpha R} \equiv e^{-i\alpha} \bar{Q}_{\dot\alpha}$$

$$(P'_\mu =) \; e^{+i\alpha R} P_\mu e^{-i\alpha R} \equiv P_\mu$$

$$(M'_{\mu\nu} =) \; e^{-i\alpha R} M_{\mu\nu} e^{-i\alpha R} \equiv M_{\mu\nu}$$

This additional $U(1)$ automorphism group of the SUSY algebra is known as $U(1)_R$; the additional commutators are

$$[R, Q_\alpha] = +Q_\alpha$$
$$[R, \bar{Q}_{\dot\alpha}] = -\bar{Q}_{\dot\alpha}$$

$$[R, P^\mu] = 0 = [R, M^{\mu\nu}]$$

When studying representations of this algebra on single particle states (as we will do later) we note that $P^2 = P_\mu P^\mu$ still commutes with all the generators $P_\mu, M_{\mu\nu}, Q_\alpha, \bar{Q}_{\dot\alpha}$. Hence states (and fields) in a supermultiplet will have the same mass $P^2 = m^2$. However $W^2 = W_\mu W^\mu$ (where $W^\alpha = \frac{i}{4}\epsilon^{\alpha\nu\rho\sigma}P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski covariant spin operator) does not commute with the SUSY generators. Thus the particles in the same supermultiplet will have different spins. Fermions & bosons will be combined in the same supermultiplet and will have the same mass.

Represent the SUSY algebra by means of linear differential operators, as we did for the Poincaré generators P_μ & $M_{\mu\nu}$, acting on spinor & tensor fields. Since we now have anti-commuting charges we must extend space-time, x^μ, to include anti-commuting spinor parameters, $\theta_\alpha, \bar{\theta}^{\dot\alpha}$, to form superspace. A point in superspace is defined by

$$z^M \equiv (x^\mu, \theta_\alpha, \bar{\theta}^{\dot\alpha}) \quad \text{where}$$

the $\theta_\alpha, \bar{\theta}^{\dot\alpha}$ are (two component, complex) Weyl spinors which anti-commute, that is, are elements of a Grassmann algebra:

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha \quad \text{and since}$$

$\alpha = 1, 2$ we find

$$\theta^\alpha \theta^\beta \theta^\gamma = 0 \quad \text{and}$$

similarly for

$$\bar{\theta}_{\dot\alpha} \bar{\theta}_{\dot\beta} = -\bar{\theta}_{\dot\beta} \bar{\theta}_{\dot\alpha} \quad \text{with}$$

$$\bar{\theta}_{\dot\alpha} \bar{\theta}_{\dot\beta} \bar{\theta}_{\dot\gamma} = 0.$$

Differentiation with respect to the anti-commuting parameters can be defined by the Taylor expansion formulae

$$\phi(\theta+\delta\theta) \equiv \phi(\theta) + \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\phi(\theta)$$

$$\phi(\bar\theta+\delta\bar\theta) \equiv \phi(\bar\theta) - \delta\bar\theta_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\phi(\bar\theta) .$$

Choosing $\phi(\theta) = \theta^\alpha$ or $\phi(\bar\theta) = \bar\theta^{\dot\alpha}$, we find

(i.e. $(\theta+\delta\theta)^\beta = \theta^\beta + \delta\theta^\beta \equiv \theta^\beta + \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\theta^\beta$

$\Rightarrow \delta\theta^\alpha \delta_\alpha{}^\beta = \delta\theta^\alpha \frac{\partial}{\partial\theta^\alpha}\theta^\beta \Rightarrow \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta$)

$$\boxed{\begin{array}{ll} \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta & \frac{\partial}{\partial\theta^\alpha}\theta_\beta = -\delta^\alpha{}_\beta \\ \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} & \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta_{\dot\beta} = -\delta^{\dot\alpha}{}_{\dot\beta} \end{array}}$$

with $\frac{\partial}{\partial\theta_\alpha} \equiv \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}$

$\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \equiv \epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}}$

Using these derivatives we can define linear differential operators that act on functions of $x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}$.

The SUSY algebra generators can then be represented as linear superspace differential operators acting on a superfield $\phi = \phi(x,\theta,\bar\theta)$. These are obtained by recalling the general transformation formula for operators now extended to include SUSY charges. Define the superfield $\phi(x,\theta,\bar\theta)$ so that

$$\phi(x,\theta,\bar\theta) = e^{ix^\mu P_\mu} e^{i(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})} \phi(0,0,0) \times e^{-i(\theta Q + \bar\theta \bar Q)} e^{-ix\cdot P}$$

where we have translated the field from the origin of superspace to the point $(x,\theta,\bar\theta)$.

Using the SUSY algebra we can transform the field by a further translation

$$e^{ia^\mu P_\mu} e^{ix^\nu P_\nu} = e^{i(x+a)^\mu P_\mu}$$

hence for an invariant field under translations

$$e^{ia^\mu P_\mu} \phi(x,\theta,\bar\theta) e^{-ia^\mu P_\mu} = \phi(x',\theta',\bar\theta')$$

$$= e^{ia\cdot P} e^{ix\cdot P} e^{i(\theta Q + \bar\theta \bar Q)} \phi(0,0,0) e^{-i(\theta Q + \bar\theta \bar Q)} e^{-ix\cdot P} \times e^{-ia\cdot P}$$

$$= e^{i(x+a)\cdot P} e^{i(\theta Q + \bar\theta \bar Q)} \phi(0,0,0) e^{-i(\theta Q + \bar\theta \bar Q)} e^{-i(x+a)\cdot P}$$

$$= \phi(x+a, \theta, \bar\theta) = e^{a^\mu \partial_\mu} \phi(x, \theta, \bar\theta)$$

$\Rightarrow$

$$\phi(x', \theta', \bar\theta') = \phi(x+a, \theta, \bar\theta).$$

So $$e^{ia^\mu P_\mu} \phi(x, \theta, \bar\theta) e^{-ia^\mu P_\mu} = \phi(x+a, \theta, \bar\theta)$$
$$= e^{a^\mu \partial_\mu} \phi(x, \theta, \bar\theta)$$

for infinitesimal a^μ

$$\Rightarrow \quad \phi(x, \theta, \bar\theta) + i a^\mu [P_\mu, \phi] = \phi(x, \theta, \bar\theta) + a^\mu \partial_\mu \phi$$

$$\Rightarrow \quad [P_\mu, \phi] = -i \partial_\mu \phi \quad \left(= -\text{``}P_\mu\text{''} \phi \right)$$

(Note an abuse of notation: The quantum symmetry generators should be written as $\mathcal{P}_\mu, \mathcal{M}_{\mu\nu}, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}^{\dot\alpha}$ with their representation as differential operators written as $P_\mu, M_{\mu\nu}, Q_\alpha, \bar{Q}^{\dot\alpha}$ as we did earlier for the Poincaré algebra. We will use the block letters for both, when the context is clear.)

So $$\text{``}P_\mu\text{''} = i\partial_\mu$$ as previously.

Likewise consider the scalar field under Rotations of superspace:

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\phi(0,0,0)\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \equiv \phi(0,0,0)$$

So using $e^A e^B e^{-A} = e^{B+[A,B]}$ for infinitesimal A

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{ix\cdot P}\,e^{i(\theta Q+\bar\theta\bar Q)}$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{ix\cdot P}\,e^{i(\theta Q+\bar\theta\bar Q)}\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$= e^{ix\cdot P+[\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},\,ix\cdot P]}\,e^{i(\theta Q+\bar\theta\bar Q)+[\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},\,i(\theta Q+\bar\theta\bar Q)]} \times e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$= e^{i[x^\rho-\frac{1}{2}\omega^{\mu\nu}x^\lambda[D_{\mu\nu}]_\lambda{}^\rho]P_\rho} \times e^{i[(\theta^\beta-\frac{i}{4}\omega^{\mu\nu}\theta^\alpha(\sigma_{\mu\nu})_\alpha{}^\beta)Q_\beta+(\bar\theta_{\dot\beta}-\frac{i}{4}\omega^{\mu\nu}\bar\theta_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta})\bar Q^{\dot\beta}]} \times e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$\boxed{\begin{aligned} & e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} e^{ix\cdot P} e^{i(\theta Q+\bar\theta\bar Q)} \\ &= e^{i[x^\mu-\omega^{\mu\nu}x_\nu]P_\mu} \times \\ &\times e^{i\left[\left(\theta^\beta-\frac{i}{4}\omega^{\mu\nu}(\theta\sigma_{\mu\nu})^\beta\right)Q_\beta+\left(\bar\theta_{\dot\beta}-\frac{i}{4}\omega^{\mu\nu}(\bar\theta\bar\sigma_{\mu\nu})_{\dot\beta}\right)\bar Q^{\dot\beta}\right]} \times \\ &\times e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \end{aligned}}$$

So

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\phi(x,\theta,\bar\theta)e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

$$\begin{aligned} &= \left(e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} e^{ix\cdot P} e^{i(\theta Q+\bar\theta\bar Q)} e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right)\times \\ &\times\left(e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\phi(0,0,0)\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right)\times \\ &\times\left(e^{+\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} e^{-i(\theta Q+\bar\theta\bar Q)} e^{-ix\cdot P} e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\right) \end{aligned}$$

Scalar field $e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\phi(0,0,0)e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$

$$= \text{"}\underset{\substack{\text{|||}\\ 1}}{D^{-1(\alpha)}{}_{\beta}\text{"}(S)}\,\phi(0,0,0)$$

$$= \phi(0,0,0)$$

So

$$\boxed{\begin{aligned} &e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\,\phi(x,\theta,\bar\theta)\,e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \\ &= \phi\left(x^\mu-\omega^{\mu\nu}x_\nu,\ \theta^\alpha-\tfrac{i}{4}\omega^{\mu\nu}(\theta\sigma_{\mu\nu})^\alpha,\ \bar\theta_{\dot\alpha}-\tfrac{i}{4}\omega^{\mu\nu}(\bar\theta\bar\sigma_{\mu\nu})_{\dot\alpha}\right) \\ &= \phi(x,\theta,\bar\theta) \\ &+\left[-\omega^{\mu\nu}x_\nu\partial_\mu-\tfrac{i}{4}\omega^{\mu\nu}\left(\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta}\right)+\tfrac{i}{4}\omega^{\mu\nu}\left(\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\right)\right] \\ &\qquad\times\phi(x,\theta,\bar\theta) \\ &= \phi(x,\theta,\bar\theta)+\left[\tfrac{i}{2}\omega^{\mu\nu}M_{\mu\nu},\ \phi(x,\theta,\bar\theta)\right] \end{aligned}}$$

$\Rightarrow$

$$\boxed{\begin{aligned} [M_{\mu\nu},\phi(x,\theta,\bar\theta)] &= -i\Big[x_\mu\partial_\nu-x_\nu\partial_\mu \\ &\qquad -\tfrac{i}{2}\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta}+\tfrac{i}{2}\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\Big]\phi \\ &\equiv -\mathcal{M}_{\mu\nu}\,\phi(x,\theta,\bar\theta) \end{aligned}}$$

And finally consider a SUSY transformation

$$e^{i(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})}\phi(x,\theta,\bar\theta)e^{-i(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})}$$

$$= e^{ix^\mu P_\mu}e^{i(\xi\cdot Q+\bar\xi\bar Q)}e^{i(\theta Q+\bar\theta\bar Q)}\phi(0,0,0)\times$$
$$\times e^{-i(\theta Q+\bar\theta\bar Q)}e^{-i(\xi Q+\bar\xi\bar Q)}e^{-ix\cdot P}$$

But $e^{i(\xi Q+\bar\xi\bar Q)}e^{i(\theta Q+\bar\theta\bar Q)}$

$$= e^{i[(\theta+\xi)Q+(\bar\theta+\bar\xi)\bar Q]+\frac{1}{2}[i(\xi Q+\bar\xi\bar Q),\, i(\theta Q+\bar\theta\bar Q)]}$$

$$= e^{i[(\theta+\xi)Q+(\bar\theta+\bar\xi)\bar Q]+\frac{1}{2}i^2([\xi Q,\bar\theta\bar Q]+[\bar\xi\bar Q,\theta Q])}$$

Now

$$[\xi Q,\bar\theta\bar Q] = -\xi^\alpha\bar\theta_{\dot\alpha}\{Q_\alpha,\bar Q^{\dot\alpha}\}$$

$$= +\xi^\alpha\bar\theta^{\dot\alpha}\{Q_\alpha,\bar Q_{\dot\alpha}\} = \xi^\alpha\bar\theta^{\dot\alpha}\,2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

$$= (2\xi\sigma^\mu\bar\theta)P_\mu$$

Similarly

$$[\bar\xi\bar Q,\theta Q] = +\bar\xi^{\dot\alpha}\theta^\alpha\{Q_\alpha,\bar Q_{\dot\alpha}\}$$

$$= 2\bar\xi^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha P_\mu = +(2\bar\xi\bar\sigma^\mu\theta)P_\mu$$

$$\boxed{e^{i(\xi Q + \bar\xi \bar Q)} e^{i(\theta Q + \bar\theta \bar Q)} = e^{+i[\,i(\xi\sigma^\mu\bar\theta + \bar\xi\bar\sigma^\mu\theta)P_\mu]} \times e^{i[(\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q]}}$$

For $\xi, \bar\xi$ finite or infinitesimal

So

$$e^{i(\xi Q + \bar\xi \bar Q)} \phi(x,\theta,\bar\theta) e^{-i(\xi Q + \bar\xi \bar Q)}$$

$$= e^{i[x^\mu + i(\xi\sigma^\mu\bar\theta + \bar\xi\bar\sigma^\mu\theta)]P_\mu} \times e^{i[(\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q]} \phi(0,0,0) e^{-i[(\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q]} \times e^{-ix\cdot P}$$

$$= \phi(x^\mu + i\xi\sigma^\mu\bar\theta + i\bar\xi\bar\sigma^\mu\theta, \theta+\xi, \bar\theta+\bar\xi)$$

$$= \phi(x^\mu + i(\xi\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\xi), \theta+\xi, \bar\theta+\bar\xi)$$

$$= \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right)\phi(x,\theta,\bar\theta) + \bar\xi_{\dot\alpha}\left(-\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\right)\phi(x,\theta,\bar\theta) + \phi(x,\theta,\bar\theta)$$

$\Rightarrow$

$$i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}, \phi(x,\theta,\bar\theta)]$$
$$= \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right)\phi(x,\theta,\bar\theta)$$
$$+ \bar\xi_{\dot\alpha}\left(-\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\right)\phi(x,\theta,\bar\theta)$$

$\Rightarrow$

$$\boxed{\begin{aligned}[Q_\alpha, \phi(x,\theta,\bar\theta)] &= -i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not\partial\bar\theta)_\alpha\right]\phi(x,\theta,\bar\theta)\\ &\equiv -\mathcal{Q}_\alpha\phi(x,\theta,\bar\theta)\\ [\bar{Q}_{\dot\alpha}, \phi(x,\theta,\bar\theta)] &= -i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not\partial)_{\dot\alpha}\right]\phi(x,\theta,\bar\theta)\\ &\equiv -\bar{\mathcal{Q}}_{\dot\alpha}\phi(x,\theta,\bar\theta)\end{aligned}}$$

So one can check explicitly that the group multiplication laws ($e^{ia\cdot P}e^{ix\cdot P} = e^{i(x+a)\cdot P}$ etc.) ~~has~~ given us a representation of the commutation relations by superspace linear differential operators acting on superfields:

$$\boxed{\begin{aligned} P_\mu \phi &= i\partial_\mu \phi \\ M_{\mu\nu}\phi &= i\left[x_\mu \partial_\nu - x_\nu \partial_\mu - \tfrac{i}{2}\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta} + \tfrac{i}{2}\bar\theta\bar\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\right]\phi \\ Q_\alpha \phi &= i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right]\phi \\ \bar Q_{\dot\alpha}\phi &= i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right]\phi \end{aligned}}$$

For example the SUSY charge differential operators yield:

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right][i]\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right] + \bar Q_{\dot\alpha} Q_\alpha$$

$$= +2i\not{\partial}_{\alpha\dot\alpha} = +2\sigma^\mu_{\alpha\dot\alpha} P_\mu \quad .$$

The other commutators can be similarly checked.

This is called the <u>real representation</u> of the SUSY algebra. There are 2 other representations of the algebra that are quite useful.

Real Representation:

Group Element $\Omega(x,\theta,\bar\theta) \equiv e^{ix^\mu P_\mu} e^{i(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})}$

$\Rightarrow$

$$e^{i(\xi Q + \bar\xi \bar Q)} \Omega(x,\theta,\bar\theta) = \Omega(x + i(\xi\sigma\bar\theta - \theta\sigma\bar\xi), \theta+\xi, \bar\theta+\bar\xi)$$

1) Chiral Representation:

Group Element $\Omega_1(x,\theta,\bar\theta) \equiv e^{ix\cdot P} e^{i\theta Q} e^{i\bar\theta\bar Q}$

$$e^{i(\xi Q + \bar\xi\bar Q)} \Omega_1(x,\theta,\bar\theta) = \Omega_1(x - 2i\theta\sigma\bar\xi, \theta+\xi, \bar\theta+\bar\xi)$$

2) Anti-Chiral Representation:

Group Element $\Omega_2(x,\theta,\bar\theta) \equiv e^{ix\cdot P} e^{i\bar\theta\bar Q} e^{i\theta Q}$

$$e^{i(\xi Q + \bar\xi\bar Q)} \Omega_2(x,\theta,\bar\theta) = \Omega_2(x + 2i\xi\sigma\bar\theta, \theta+\xi, \bar\theta+\bar\xi)$$

In each case we can define a superfield in that representation

Real representation:

$$\phi(x,\theta,\bar\theta) \equiv \Omega(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega^{-1}(x,\theta,\bar\theta)$$

Chiral representation:

$$\phi_1(x,\theta,\bar\theta) \equiv \Omega_1(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega_1^{-1}(x,\theta,\bar\theta)$$

Anti-Chiral representation:

$$\phi_2(x,\theta,\bar\theta) \equiv \Omega_2(x,\theta,\bar\theta)\,\phi(0,0,0)\,\Omega_2^{-1}(x,\theta,\bar\theta)$$

The SUSY charges in each representation becomes

0) Real:
$$Q_\alpha\phi = i\left[\frac{\partial}{\partial\theta^\alpha} + i(\not{\partial}\bar\theta)_\alpha\right]\phi$$
$$\bar Q_{\dot\alpha}\phi = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\not{\partial})_{\dot\alpha}\right]\phi$$

1) Chiral:
$$Q_{1\alpha}\phi_1 = i\left[\frac{\partial}{\partial\theta^\alpha}\right]\phi_1$$
$$\bar Q_{1\dot\alpha}\phi_1 = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i(\theta\not{\partial})_{\dot\alpha}\right]\phi_1$$

2) Anti-Chiral:
$$Q_{2\alpha}\phi_2 = i\left[\frac{\partial}{\partial\theta^\alpha} + 2i(\not{\partial}\bar\theta)_\alpha\right]\phi_2$$
$$\bar Q_{2\dot\alpha}\phi_2 = i\left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right]\phi_2$$

For each representation

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = +2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

Since the group elements are related according to

$$\Omega_1(x,\theta,\bar\theta) = \Omega(x+i\theta\sigma\bar\theta, \theta, \bar\theta)$$

$$\Omega_2(x,\theta,\bar\theta) = \Omega(x-i\theta\sigma\bar\theta, \theta, \bar\theta)$$

The fields are related as

$$\boxed{\begin{aligned}\phi(x,\theta,\bar\theta) &= \phi_1(x-i\theta\sigma\bar\theta, \theta, \bar\theta) \\ &= \phi_2(x+i\theta\sigma\bar\theta, \theta, \bar\theta)\end{aligned}}$$

Hence $\phi(x,\theta,\bar\theta) = e^{-i\theta\not{\partial}\bar\theta} \phi_1(x,\theta,\bar\theta)$

$$= e^{+i\theta\not{\partial}\bar\theta} \phi_2(x,\theta,\bar\theta)$$

Likewise $Q_\alpha = e^{-i\theta\not{\partial}\bar\theta} Q_{1\alpha} e^{+i\theta\not{\partial}\bar\theta}$

$$\bar{Q}_{\dot\alpha} = e^{-i\theta\not{\partial}\bar\theta} \bar{Q}_{1\dot\alpha} e^{+i\theta\not{\partial}\bar\theta}$$

and $Q_\alpha = e^{+i\theta\not{\partial}\bar\theta} Q_{2\alpha} e^{-i\theta\not{\partial}\bar\theta}$

$$\bar{Q}_{\dot\alpha} = e^{+i\theta\not{\partial}\bar\theta} \bar{Q}_{2\dot\alpha} e^{-i\theta\not{\partial}\bar\theta}$$

So $e^{-i\theta\not{\partial}\bar\theta} Q_{1\alpha}\phi_1 = Q_\alpha \phi$, etc., as can be checked explicitly using the differential operators.

From the transformation property of the superfield,

$$U(\xi,\bar\xi)\,\Phi(x,\theta,\bar\theta)\,U^{-1}(\xi,\bar\xi) = \Phi(x',\theta',\bar\theta')$$

(p. 237)

$$= \Phi(x+i(\xi\sigma\bar\theta-\theta\sigma\bar\xi),\ \theta+\xi,\ \bar\theta+\bar\xi),$$

we see that SUSY transformations correspond to <u>translations</u> in superspace

$$\boxed{\begin{aligned} x'^{\mu} &= x^{\mu} + i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi) \\ \theta'^{\alpha} &= \theta^{\alpha} + \xi^{\alpha} \\ \bar\theta'_{\dot\alpha} &= \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha} \end{aligned}}$$

Note that

$$[i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi)]^{*}$$

$$= [i(\xi\sigma^{\mu}\bar\theta - \theta\sigma^{\mu}\bar\xi)]$$

is <u>real</u>; thus $\Phi(x,\theta,\bar\theta)$ can be taken as a real superfield $\phi = \phi^*$ and is called a vector superfield it forms a real representation of SUSY.

In the chiral representation x^μ is translated by a pure imaginary vector

$$\phi_1(x,\theta,\bar\theta) = \phi(x + i\theta\sigma\bar\theta, \theta, \bar\theta)$$

with $[i\theta\sigma\bar\theta]^* = -[i\theta\sigma\bar\theta]$

and likewise $\phi_2(x,\theta,\bar\theta) = \phi(x - i\theta\sigma\bar\theta, \theta, \bar\theta)$

Hence ϕ_1 & ϕ_2 transform as complex representations of SUSY

Chiral:
$$x'^\mu = x^\mu - 2i\theta\sigma\bar\xi$$
$$\theta'^\alpha = \theta^\alpha + \xi^\alpha$$
$$\bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

Anti-Chiral:
$$x'^\mu = x^\mu + 2i\xi\sigma^\mu\bar\theta$$
$$\theta'^\alpha = \theta^\alpha + \xi^\alpha$$
$$\bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

[Notation: Complex conjugation changes $\theta_\alpha \to \bar\theta_{\dot\alpha}$ and also interchanges the order of Grassmann spinors e.g.

$$[\theta^\alpha \chi_\alpha]^* = \bar\chi_{\dot\alpha}\bar\theta^{\dot\alpha} = \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}.$$
)