Spinor Interlude:

In what was done we separated Dirac fermions into a left & Right handed spinors

$$\psi_D = (\gamma_+ + \gamma_-)\psi_D = \underbrace{\gamma_+\psi_D}_{\equiv \psi_R} + \underbrace{\gamma_-\psi_D}_{\equiv \psi_L}$$

And they interacted differently under $SU(2)\times U(1)$. The 3-2-1 representations of the matter field is ($a = 1,2,3$ = color = R,G,B)

Field	Families ($m=1,2,3$)	$(SU(3), SU(2), U(1))$	Family Electro-Weak Multiplets
$l_{mL} = \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix}$	3	$(1, 2, -\frac{1}{2})$	$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$
$q^a_{mL} = \begin{pmatrix} u_{mL} \\ d_{mL} \end{pmatrix}$	3	$(3, 2, +\frac{1}{6})$	$\begin{pmatrix} u^a_L \\ d^a_L \end{pmatrix}, \begin{pmatrix} c^a_L \\ s^a_L \end{pmatrix}, \begin{pmatrix} t^a_L \\ b^a_L \end{pmatrix}$
e_{mR}	3	$(1, 1, -1)$	e_R, μ_R, τ_R
u^a_{mR}	3	$(3, 1, +\frac{2}{3})$	u^a_R, c^a_R, t^a_R
d^a_{mR}	3	$(3, 1, -\frac{1}{3})$	d^a_R, s^a_R, b^a_R
ϕ		$(1, 2, +\frac{1}{2})$	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$
$(\tilde{\phi} = i\sigma^2\phi^*)$		$(1, 2, -\frac{1}{2})$	$\tilde{\phi} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$

It is useful in GUTS & SUSY to deal with fields with the same chirality. For instance in a SU(5) GUT the RH down quarks and the left handed lepton e, ν_e are put in the same $\bar{5}$ representation — they must have the same chirality. To accomplish this we can use the <u>charge conjugate</u> fields instead of the RH fields as they will be LH. <u>ie.</u> $e^{c}_{R}, u^{a}_{R}, d^{a}_{R}$ will become LH under charge conjugation.

Recall Charge Conjugation:

$$\psi \to \mathcal{C}\psi\mathcal{C}^{-1} \equiv \psi^c = C\bar{\psi}^T$$

$$\bar{\psi} \to \mathcal{C}\bar{\psi}\mathcal{C}^{-1} \equiv \overline{\psi^c} = -\psi^T C^{-1} \quad (= \psi^{c\dagger}\gamma^0)$$

where $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ & for our representation

$$C = -C^{-1} = -C^\dagger = -C^T = i\gamma^2\gamma^0$$

So

$$\mathcal{C}\psi_{\substack{L\\R}}\mathcal{C}^{-1} = \gamma_{\mp}\mathcal{C}\psi\mathcal{C}^{-1}$$

$$= \gamma_{\mp}\psi^c = \gamma_{\mp}C\bar{\psi}^T$$

$$= C\gamma_{\mp}\bar{\psi}^T = C(\bar{\psi}\gamma_{\mp})^T \qquad (\gamma_5^T = \gamma_5)$$

$$= C\,\overline{\psi_{\substack{R\\L}}}^{\,T} \equiv \psi^c_{\substack{L\\R}}$$

So $\boxed{\psi^c_{L \atop R} = C\,\overline{\psi}_{R \atop L}{}^T} \Rightarrow \overline{\psi}_{R \atop L} C^T = (\psi^c_{L \atop R})^T$

Similarly $\Rightarrow \overline{\psi}_{R \atop L} = (\psi^c_{L \atop R})^T C$

$$\mathcal{C}\,\overline{\psi}_{L \atop R}\,\mathcal{C}^{-1} = \mathcal{C}\,\overline{\psi}\,\mathcal{C}^{-1}\gamma_\pm = -\psi^T C^{-1}\gamma_\pm$$

$$= -\psi^T \gamma_\pm C^{-1} = -(\gamma_\pm \psi)^T C^{-1}$$

$$\boxed{= -\psi_{R \atop L}{}^T C^{-1} = \overline{\psi^c_{L \atop R}}} \quad \left(= \psi^{c\,\dagger}_{L \atop R}\gamma^0\right)$$

(That is $\overline{\psi^c_{L \atop R}} = \psi^{c\,\dagger}_{L \atop R}\gamma^0 = \overline{\psi}^*_{R \atop L} C^\dagger \gamma^0 = \psi^T_{R \atop L}\gamma^{0*} C^{-1}\gamma^0$

$$= -\psi^T_{R \atop L} C^{-1} \checkmark \text{)}$$

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hence $\left(\overline{\psi^c_{L \atop R}}\right)^T = -C^{-1\,T}\psi_{R \atop L}$

$$= -C\,\psi_{R \atop L}$$

and so $\boxed{\psi_{R \atop L} = C\left(\overline{\psi^c_{L \atop R}}\right)^T}$

& from above $\boxed{\overline{\psi}_{R \atop L} = \psi^{c\,T}_{L \atop R}\,C}$

So instead of (ψ_L, ψ_R) as fundamental fields, we can use the equivalent pairs

$$(\psi_L, \psi_L^c) \text{ or } (\psi_R, \psi_R^c) \text{ or } (\psi_R^c, \psi_L^c)$$

we will replace $\underline{\psi_R \text{ with } \psi_L^c}$ in the SM

So (ψ_L, ψ_L^c) will become the fundamental fields. So for example we will replace e_R^-, u_R^a, d_R^a with the fields

$$e_L^+, u_L^{ca}, d_L^{ca}$$

where $e_L^+ = C\bar{e}_R^{-T} = e_L^{-c}$ (making electric charge explicit)
or leaving off $\pm$ $e_L^c, u_L^{ca}, d_L^{ca}$

Now the charge conjugation takes the fields to the complex conjugation group representation and opposite U(1) charge

So for example d_R^a is a $(3, 1, -\frac{1}{3})$

but $d_L^c = C\bar{d}_R^T$ transforms as

a $(\bar{3}, 1, +\frac{1}{3})$ under $SU(3) \times SU(2) \times U(1)$.

since the $\bar{d}_R$ transforms according to

U^{-1} taking a 3 to a $\bar{3}$ & $-\frac{1}{3}$ to $+\frac{1}{3}$.

Likewise $e^c_L = C(\overline{e_R})^T$ the $\overline{e}_R$ flips the hypercharge of e_R from -1 to $+1$.
So we have the 3-2-1 table

Field	$(SU(3), SU(2), U(1))$	Family Multiplets
$l_{mL} = \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix}$	$(1, 2, -\frac{1}{2})$	$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$
$q^a_{mL} = \begin{pmatrix} u_{mL} \\ d_{mL} \end{pmatrix}$	$(3, 2, +\frac{1}{6})$	$\begin{pmatrix} u^a_L \\ d^a_L \end{pmatrix}, \begin{pmatrix} c^a_L \\ s^a_L \end{pmatrix}, \begin{pmatrix} t^a_L \\ b^a_L \end{pmatrix}$
$e^+_{mL} = e^c_{mL}$	$(1, 1, +1)$	e^c_L, μ^c_L, τ^c_L
u^{ca}_{mL}	$(\bar{3}, 1, -\frac{2}{3})$	$u^{ca}_L, c^{ca}_L, t^{ca}_L$
d^{ca}_{mL}.	$(\bar{3}, 1, +\frac{1}{3})$	$d^{ca}_L, s^{ca}_L, b^{ca}_L$
ϕ	$(1, 2, +\frac{1}{2})$	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$
$(\tilde{\phi} = i\sigma^2\phi^*)$	$(1, 2, -\frac{1}{2})$	$\tilde{\phi} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$

Note:

$i\sigma^2 l_L$	$(1, \bar{2}, -\frac{1}{2})$	$i\sigma^2 l_L = \begin{pmatrix} e^-_L \\ -\nu_{eL} \end{pmatrix}$

The invariant fermion kinetic energy terms become for example

$$\overline{u}_R i \not{D} u_R = \overline{u}_R^a i \left[(\partial_\mu - \tfrac{2i}{3} g_1 B_\mu)\delta^{ab} - \tfrac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)_{ab} \right] \gamma^\mu u_R^b$$

chain Rule Differentiation throw away total derivative since action $i\int d^4x$

sign flip

$3 \to \overline{3}$

$$= -i\left[(\partial_\mu + \tfrac{2i}{3} g_1 B_\mu)\delta^{ba} + \tfrac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T \right] \overline{u}_R^a \cdot \gamma^\mu u_R^b$$

$$= -i\left[(\partial_\mu + \tfrac{2i}{3} g_1 B_\mu)\delta^{ba} + \tfrac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T_{ba} \right] u_L^{ca\,T} C \cdot \gamma^\mu C \left(\overline{u}_L^{cb}\right)^T$$

$$= + \overline{u}_L^{cb} C^T \gamma^{\mu T} C^T i\left[(\partial_\mu + \tfrac{2i}{3} g_1 B_\mu)\delta^{ba} + \tfrac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T_{ba} \right] u_L^{ca}$$

(Now $C^T \gamma^{\mu T} C^T = \gamma^\mu$)

$$= \overline{u}_L^{c\,b} \gamma^\mu i \left[(\partial_\mu + \tfrac{2i}{3} g_1 B_\mu)\delta^{ba} + \tfrac{ig_3}{2}(\vec{\lambda}\cdot\vec{G}_\mu)^T_{ba} \right] u_L^{c\,a}$$

covariant derivative for $(\overline{3}, 1, -\tfrac{2}{3})$

$$\Rightarrow \boxed{\overline{u}_R i \not{D} u_R = \overline{u}_L^c \, i \not{D} u_L^c}$$

Likewise

$$\bar{e}_R i \not{D} e_R = \bar{e}^c_L i \not{D} e^c_L$$

&

$$\bar{d}_R i \not{D} d_R = \bar{d}^c_L i \not{D} d^c_L$$

with

$$D_\mu e^c_L = (\partial_\mu - i g_1 B_\mu) e^c_L$$

$$D^{ab}_\mu u^{cb}_L = \left[\left(\partial_\mu + \frac{2i}{3} g_1 B_\mu\right) \delta^{ab} + \frac{i g_3}{2} (\vec{\lambda} \cdot \vec{G}_\mu)^T_{ab} \right] u^{cb}_L$$

$$D^{ab}_\mu d^{cb}_L = \left[\left(\partial_\mu - \frac{i}{3} g_1 B_\mu\right) \delta^{ab} + \frac{i g_3}{2} (\vec{\lambda} \cdot \vec{G}_\mu)^T_{ab} \right] d^{cb}_L$$

and

$$\mathcal{L}_F = \bar{l}_L i \not{D} l_L + \bar{q}_L i \not{D} q_L + \bar{e}^c_L i \not{D} e^c_L + \bar{u}^c_L i \not{D} u^c_L + \bar{d}^c_L i \not{D} d^c_L$$

Finally the Yukawa terms must be re-expressed in terms of the left handed charge conjugate fields.

These are Fermion bilinears of the form

$$\bar{l}_L e_R = \bar{l}_L C (\overline{e^c_L})^T$$

$$\bar{e}_R l_L = e^{c\,T}_L C l_L$$

$$\bar{q}_L d_R = \bar{q}_L C (\overline{d^c_L})^T \qquad \bar{q}_L u_R = \bar{q}_L C \overline{u^c_L}^T$$

$$\bar{d}_R q_L = d^{c\,T}_L C q_L \qquad \bar{u}_R q_L = u^{c\,T}_L C q_L$$

So

$$\boxed{\begin{aligned} \mathcal{L}_{yuk} = {} & \Gamma^e_{mn} \bar{l}_{mL} \phi C \overline{e^c_{nL}}^T + \Gamma^{e\dagger}_{mn} e^{c\,T}_{mL} \phi^\dagger C l_{nL} \\ & + \Gamma^d_{mn} \bar{q}_{mL} \phi C \overline{d^c_{nL}}^T + \Gamma^{d\dagger}_{mn} d^{c\,T}_{mL} \phi^\dagger C q_{nL} \\ & + \Gamma^u_{mn} \bar{q}_{mL} \tilde{\phi} C \overline{u^c_{nL}}^T + \Gamma^{u\dagger}_{mn} u^{c\,T}_{mL} \tilde{\phi}^\dagger C q_{nL} \end{aligned}}$$

As usual the mass & Higgs couplings are L-R types of coupling

$$\overline{\psi^c_R} \phi_L = \psi^T_L C \phi_L$$

& $\bar{\phi}_L \psi^c_R = \bar{\phi}_L C \overline{\psi_R}^T$ as above.