3.) Consider the running of the gauge coupling constants for SU(3), SU(2) & U(1).
For each of the gauge coupling constants, g_1, g_2, g_3 we have the renormalization group running determined by $\beta_{1,2,3}$:

$$\mu \frac{d}{d\mu} \bar{g}_i(\mu) = \beta_i(\bar{g}_i(\mu)) .$$

As we saw for SU(3) $\beta_3(\bar{g}_3) = -\frac{\bar{g}_3^3}{(4\pi)^2}\left[\frac{11}{3}C_2(8) - \frac{4}{3}T_F\right]$

where $\delta_{ij} T_F = \frac{1}{2}\mathrm{Tr}[T^i T^j]$ where we sum over all fermi-quark-representations ⇒ # of Families

$N_F = 6 = \#$ of flavors

$T_F = \frac{1}{2}\#$ of flavours $= F$
$= \#$ of Families $= N_F \frac{1}{2}$

Now in general we find for a gauge theory

$$\beta = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}T_F - \frac{1}{3}T_S\right]$$

where $C_2(G)$ is the quadratic Casimir operator for gauge group G

$$C_2(SU(N)) = N \quad , \quad C_2(U(1)) = 0$$

and

$$T_F \delta_{ij} = \frac{1}{2}\mathrm{Tr}[T^i T^j] \quad \text{where we}$$

sum over all fermi representations [if Dirac fermions sum over both Left & Right handed representations, while if Weyl (Majorana) fermions only count ψ_L (ψ_α) or ψ_L^c ($\bar{\psi}_{\dot\alpha}$) once, not both. (these are Majorana fields)].

Likewise

$$T_S\,\delta_{ij} = \mathrm{Tr}\{T^i T^j\}$$ where we sum over all scalar field representations.

So for SU(3) we have as previously — The Higgs field is an SU(3) color singlet ($T^i=0$)

$$\beta_3 = -\frac{g_3^3}{(4\pi)^2}\left[\frac{11}{3}C_2(SU(3)) - \frac{4}{3}\cdot\frac{1}{2}N_F\right]$$ # of flavors

$\boxed{\frac{1}{2}N_F = F}$

$$\boxed{\beta_3 = -\frac{g_3^3}{(4\pi)^2}\left[11 - \frac{4}{3}F\right]}$$

F = # of Families

For SU(2) we also have $T_F = \frac{1}{2}N_F$ and one Higgs doublet $T_S = \frac{1}{2}$ i.e. $T_S\delta_{ij} = \mathrm{Tr}\left[\frac{\sigma^i}{2}\frac{\sigma^j}{2}\right] = \frac{1}{2}\delta^{ij}$ and

$$C_2(SU(2)) = 2 \quad \text{So}$$

$$\boxed{\beta_2 = -\frac{g_2^3}{(4\pi)^2}\left[\frac{22}{3} - \frac{4}{3}F - \frac{1}{6}H\right]}$$

H = # of Higgs doublets = 1 for SM

Finally the hypercharge coupling constant β_1 has

$$C_2(U(1)) = 0$$

$$T_F^{(U(1))} = \frac{1}{2}\sum_{\text{fields}}' y^2_{\text{fields}} = \frac{1}{2}\sum_L' y_L^2 + \frac{1}{2}\sum_R' y_R^2$$

$$= \frac{5}{3}F \cdot \quad = \frac{1}{2}\Big[\frac{1}{4} + \frac{1}{4} + 3\cdot\frac{1}{36} + 3\cdot\frac{1}{36} + 1 + \frac{4}{9}\cdot 3 + \frac{1}{9}\cdot 3\Big]$$

3 colours

and likewise $T_S = \frac{1}{2}\sum_{\substack{\text{real}\\ \text{scalars}}}' y_S^2$

$$= \frac{1}{2}\cdot 4 \times \frac{1}{4} = \frac{1}{2}$$

So

$$\boxed{\beta_1 = + \frac{g_1^3}{(4\pi)^2}\left[\frac{20}{9}F + \frac{1}{6}H\right]}$$

In general $\beta_i = g_i^3 b_i$ hence the running coupling constants obey the DE

$$\mu \frac{d}{d\mu}\bar{g}_i(\mu) = b_i \bar{g}_i^3(\mu)$$

$\Rightarrow$

$$\mu \frac{d}{d\mu}\frac{\bar{g}_i^2}{4\pi} = 8\pi b_i \left(\frac{\bar{g}_i^2}{4\pi}\right)^2$$

Fine Structure constant $\boxed{\alpha_i(\mu) \equiv \frac{\bar{g}_i^2(\mu)}{4\pi}}$

$\Rightarrow$

$$\mu \frac{d}{d\mu} \alpha_i = 8\pi b_i \alpha_i^2$$

$$\Rightarrow \int_{\alpha_i(\mu_1)}^{\alpha_i(\mu_2)} \frac{d\alpha_i}{\alpha_i^2} = 8\pi b_i \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu}$$

$$-\frac{1}{\alpha_i(\mu_2)} + \frac{1}{\alpha_i(\mu_1)} = 8\pi b_i \ln\left(\frac{\mu_2}{\mu_1}\right)$$

$$\boxed{\frac{1}{\alpha_i(\mu_2)} = \frac{1}{\alpha_i(\mu_1)} - 8\pi b_i \ln\left(\frac{\mu_2}{\mu_1}\right)}$$

&

$$\boxed{\alpha_i(\mu_2) = \frac{\alpha(\mu_1)}{1 - 8\pi b_i \alpha_i(\mu_1) \ln\left(\frac{\mu_2}{\mu_1}\right)}}$$

where we have

$$b_1 = \frac{1}{(4\pi)^2}\left[\frac{20}{9}F + \frac{1}{6}H\right] = \frac{41/6}{(4\pi)^2}$$

$$b_2 = \frac{-1}{(4\pi)^2}\left[\frac{22}{3} - \frac{4}{3}F - \frac{1}{6}H\right] = \frac{-19/6}{(4\pi)^2}$$

$$b_3 = \frac{-1}{(4\pi)}\left[11 - \frac{4}{3}F\right] = \frac{-7}{(4\pi)^2}$$

Further suppose $\mu_1 = M_Z$; $\mu_2 = e^t M_Z$

$$\ln\left(\frac{\mu_2}{\mu_1}\right) = t \qquad \left[\left(\mu\frac{d}{d\mu} = \frac{d}{dt}\right)\left(\mu = e^t M_Z\right)\right]$$

$g_1^2(M_Z) = 0.128$ $\quad g_3^2(M_Z) = 1.479$
$g_2^2(M_Z) = 0.423$

So we can plot

$$\boxed{\frac{1}{\alpha_i(t)} = \frac{1}{\alpha_i(M_Z)} - 8\pi b_i t}$$

Now $\alpha_3(M_Z) = 0.1176$ $\qquad M_Z = 91.1874$ GeV
$(g_3 = 1.216)$

$\alpha_2(M_Z) = 0.0336$ $\quad (g_2 = 0.65)$ $\quad (g_2^2 \approx 0.423)$

$\alpha_1(M_Z) = 0.0102$ $\quad (g_1 = 0.358)$

are the known initial conditions.

The running is displayed in a Mathcad program:

GUT Normalization: For a SU(5) or SO(10) GUT g_1 is normalized as

$$g_1(GUT) = \sqrt{\tfrac{5}{3}}\, g_1 \Rightarrow \alpha_1(t)(GUT) = \tfrac{5}{3}\alpha_1(t)$$

$$\Rightarrow \alpha_1(GUT)(M_Z) = \tfrac{5}{3}(0.0102)$$

and
$$\frac{1}{\alpha_1(GUT)(t)} = \frac{1}{\alpha_1(GUT)(M_Z)} - \left(\frac{3}{5}\right) 8\pi b_1 t$$

$$\Rightarrow b_1(GUT) = \frac{41/10}{(4\pi)^2} = \left(\frac{3}{5}\right) b_1$$

Running of the Standard Model Gauge Coupling Constants

Initial values of the fine structure constants are given at $M_Z := 91.1874\,\text{GeV}$

There the inverse fine structure constants for U(1), SU(2) and SU(3) gauge coupling constants are

$$\alpha\text{inverseM}_Z := \begin{pmatrix} \dfrac{1}{0.0102} \\ \dfrac{1}{0.0336} \\ \dfrac{1}{0.1176} \end{pmatrix}$$

The coefficients for their β functions are given by

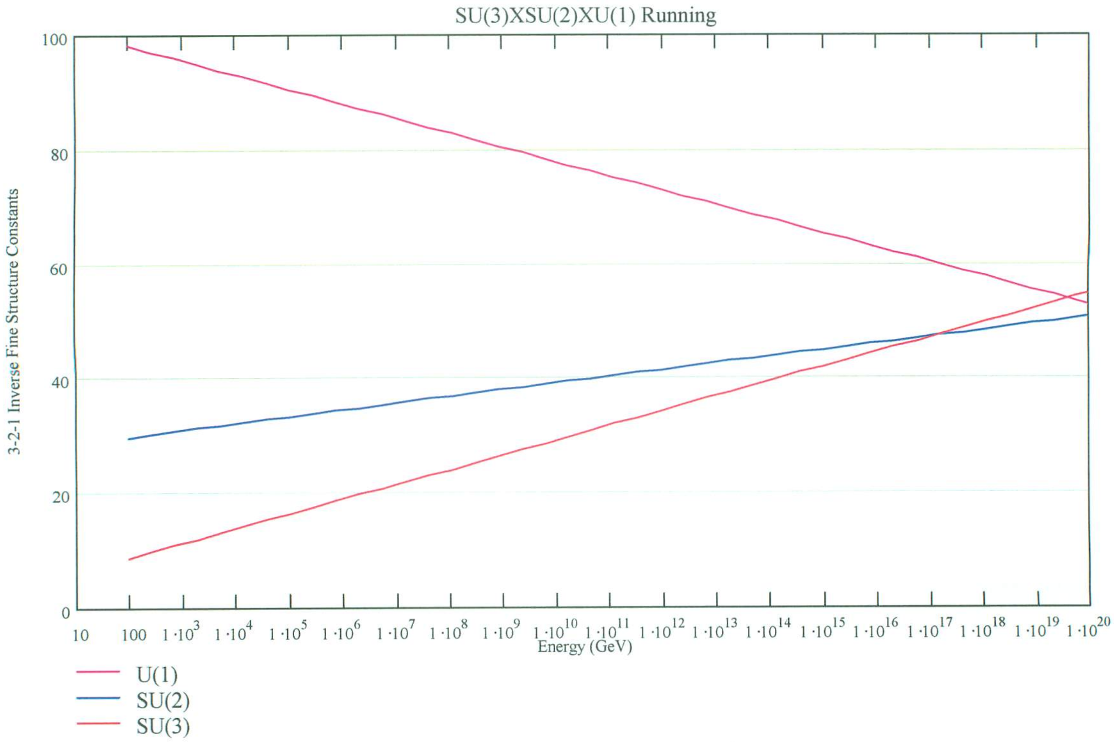
$$b := \begin{pmatrix} \dfrac{41}{6} \\ \dfrac{-19}{6} \\ -7 \end{pmatrix} \cdot \frac{1}{16 \cdot \pi^2}$$

The renormalization group running of the inverse fine structure constants is described by the RGE equation

$$\alpha\text{inverse}(t) := \overrightarrow{\left(\overrightarrow{\alpha\text{inverseM}_Z} - \overrightarrow{\left(8 \cdot \pi \cdot \vec{b} \cdot t \right)} \right)}$$

where the energy scale of the effective coupling constant is given by $Q(t) := M_Z \cdot e^t$

$t := 0, 1 .. 42$



Now if the SU(3)XSU(2)XU(1) groups are embedded in a SU(5) or SO(10) GUT it is conventional to normalize the U(1) coupling to be $g_1(GUT) = (5/3)^{(1/2)}\, g_1$. Hence the running for such a normalization is a bit different. The U(1) intial fine structure constant becomes

$$\alpha\text{inverseGUTM}_Z := \begin{pmatrix} \dfrac{\frac{3}{5}}{0.0102} \\ \dfrac{1}{0.0336} \\ \dfrac{1}{0.1176} \end{pmatrix}$$

and the normalization changes the β function to be

$$\text{bGUT} := \begin{pmatrix} \dfrac{41}{10} \\ \dfrac{-19}{6} \\ -7 \end{pmatrix} \cdot \frac{1}{16\cdot\pi^2}$$

The RGE running is now given by

$$\alpha\text{inverseGUT}(t) := \overrightarrow{\left(\overrightarrow{\alpha\text{inverseGUTM}_Z} - \overrightarrow{(8\cdot\pi\cdot\overrightarrow{\text{bGUT}\cdot t})}\right)}$$