Triviality of Higgs Sector

2) Second Higgs sector issue — suppose we tune the large cutoff dependence away — we still have the effective behavior of the coupling constant. Consider O(4) invariant scalar model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - V(\phi^2)$$

$$V(\phi^2) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}(\phi^2)^2.$$

The coupling constant has a β-function determined in the following.

Consider Higgs sector without gauge & fermion coupling

$$\mathcal{L} = \partial_\mu \phi_a^\dagger \partial^\mu \phi_a - m^2 \phi_a^\dagger \phi_a - \lambda (\phi_a^\dagger \phi_a)^2$$

$$\phi_a = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}_a \; ; \; \phi_a^\dagger = \phi^- \; \phi_0^\dagger \qquad \text{So}$$

$$\mathcal{L} = \partial_\mu \phi^+ \partial^\mu \phi^- + \partial_\mu \phi_0^\dagger \partial^\mu \phi_0 - m^2 \left[\phi^+ \phi^- + \phi_0^\dagger \phi_0 \right] - \lambda \left[\phi^+ \phi^- + \phi_0^\dagger \phi_0 \right]^2$$

Define

$$\phi^\pm \equiv \frac{1}{\sqrt{2}} \left[\phi_1 \mp i \phi_2 \right]$$

$$\phi_0 \equiv \frac{1}{\sqrt{2}} \left[\phi_3 \mp i \phi_4 \right]$$

$\Rightarrow$

$$\phi_0^\dagger \phi_0 = \frac{1}{2} \left[\phi_3^2 + \phi_4^2 \right] \; ; \; \phi^+ \phi^- = \frac{1}{2} \left[\phi_1^2 + \phi_2^2 \right]$$

So

$$\boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{2} \phi_i \phi_i - \frac{\lambda}{4} (\phi_i \phi_i)^2} \quad , \; i = 1,2,3,4.$$

So $\mathcal{L}$ is $SO(4) = SU(2) \times SU(2)$ invariant

Now consider the one-loop β-function for λ

As usual we convert from bare (above) to renormalized quantities

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_0 \cdot \partial^\mu \phi_0 - \frac{1}{2} m_0^2\, \phi_0 \cdot \phi_0 - \frac{\lambda_0}{4}(\phi_0 \cdot \phi_0)^2$$

let $\phi_0^i = Z^{1/2} \phi^i$

$$\mathcal{L} = \frac{1}{2} Z\, \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} Z m_0^2 (\phi\cdot\phi) - \frac{1}{4} Z^2 \lambda_0 (\phi\cdot\phi)^2$$

Next $Z m_0^2 \equiv m^2 + \delta m^2$

$Z^2 \lambda_0 \equiv Z_\lambda \lambda$

$\Rightarrow$

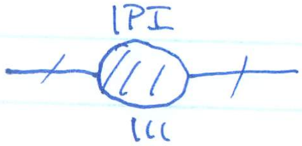
$$\boxed{\mathcal{L} = \frac{1}{2} Z\, \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2}(m^2 + \delta m^2)(\phi\cdot\phi) - \frac{Z_\lambda \lambda}{4}(\phi\cdot\phi)^2}$$

Normalization Conditions:

"Pole" 1) $\Gamma^{(2,0,0,0)}(p^2)\Big|_{p^2=0} \equiv -im$

$\parallel$

$$\left\{ i\left[Z p^2 - m^2 - \delta m^2\right] + \right.$$



$$\left.\right\}\Big|_{p^2=0}$$

1PI $= i\hat{\Pi}(p^2)$

$\Rightarrow$ $\boxed{\delta m^2 = \hat{\Pi}(p^2 = 0)}$

"residue" 2)

$$\frac{\partial}{\partial p^2} \Gamma^{(2,0,0,0)}(p^2)\Big|_{p^2=-\mu^2} \equiv i$$

$$\|$$

$$\frac{\partial}{\partial p^2}\left\{ i\left[Z p^2 - m^2 - \delta m^2\right] + i\,\hat{\Pi}(p^2)\right\}\Big|_{p^2=-\mu^2}$$

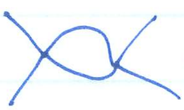
$$\Rightarrow \boxed{(Z-1) = \frac{\partial \hat{\Pi}(p^2)}{\partial p^2}\Big|_{p^2=-\mu^2}}$$

"coupling constant" 3)

$$\Gamma^{(4,0,0,0)}\Big|_{s=t=u=-\frac{4}{3}\mu^2} \equiv -6i\lambda$$

$$\|$$

$$\left[-6i Z_\lambda \lambda + \hat{\Gamma}^{(4)}(\mu)\right]$$

=  +  + ...

$$\Rightarrow \boxed{(Z_\lambda - 1)\lambda = -\frac{i}{6}\hat{\Gamma}^{(4)}(\mu)}$$

Now if we change $\mu \to \mu'$ we have that

$$\Gamma(p, m', \lambda', \mu') = \hat{Z}\, \Gamma(p, m, \lambda, \mu)$$

$\Rightarrow$ RGE

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m \partial/\partial m - \gamma N\right] \Gamma^{(i_1, i_2, i_3, i_4)} = 0$$

$N = i_1 + i_2 + i_3 + i_4$

So let this act on $\Gamma^{(4,0,0,0)}_{(s,t,u)}$

$$\mu \frac{\partial}{\partial \mu} \Gamma^{(4,0,0,0)}_{(s,t,u)} + \beta \frac{\partial}{\partial \lambda} \Gamma^{(4,0,0,0)}_{(s,t,u)} + \gamma_m m \frac{\partial}{\partial m} \Gamma^{(4,0,0,0)}_{(s,t,u)} - 4\gamma \Gamma^{(4,0,0,0)}_{(s,t,u)} = 0$$

Let $s = t = u = -\frac{4}{3}\mu^2$

$\Rightarrow$

$$\left(\mu \frac{\partial}{\partial \mu} \Gamma^{(4,0,0,0)}_{(s,t,u)}\right)\Bigg|_{s=t=u=-\frac{4}{3}\mu^2} + \beta \frac{\partial}{\partial \lambda}\left[\Gamma^{(4,0,0,0)}_{(s,t,u)}\Big|_{-4/3\mu^2}\right] + \gamma_m m \frac{\partial}{\partial m}\left[\Gamma^{(4,0,0,0)}\Big|_{-4/3\mu^2}\right] - 4\gamma \Gamma^{(4,0,0,0)}\Big|_{-4/3\mu^2} = 0$$

So

$$\left(\mu \frac{\partial}{\partial \mu} \Gamma^{(4,0,0,0)}_{(s,t,u)}\right)\Bigg|_{\substack{s=t=u \\ =-\frac{4}{3}\mu^2}} + \beta \frac{\partial}{\partial \lambda}(-6i\lambda)$$

$$+ \gamma_\mu \, \mu \frac{\partial}{\partial \mu}(-6i\lambda) - 4\gamma(-6i\lambda) = 0$$

$\frac{\partial \lambda}{\partial \mu} = 0 \quad \Rightarrow$

$$\boxed{6i\beta = \left[\mu \frac{\partial}{\partial \mu} \Gamma^{(4,0,0,0)}_{(s,t,u)}\right]\Bigg|_{\substack{s=t=u \\ =-\frac{4}{3}\mu^2}} + 24i\lambda\gamma}$$

Now γ starts in 2-loops so

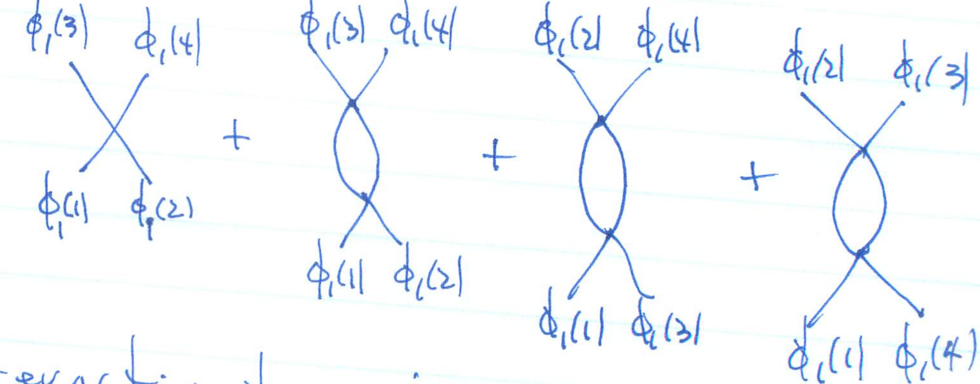
In one-loop

$$\boxed{\beta = -\frac{i}{6}\left[\mu \frac{\partial}{\partial \mu} \Gamma^{(4,0,0,0)}_{(s,t,u)}\right]\Bigg|_{\substack{s=t=u \\ =-\frac{4}{3}\mu^2}}}$$

with $\Gamma^{(4,0,0,0)}$ in one loop.

Consider the $\phi_1\phi_1\phi_1\phi_1$ vertex

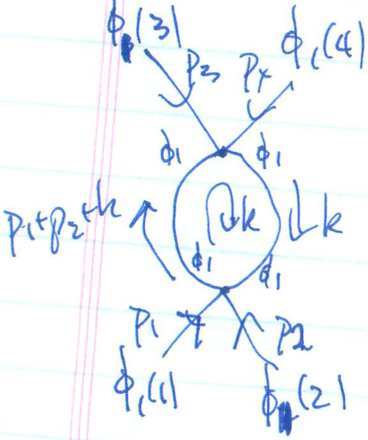
$$\Gamma^{(4,0,0,0)} =$$



Now the interaction term is

$$\mathcal{L}_{int} = -\frac{Z_\lambda \lambda}{4}\Big[\underbrace{\phi_1^4 + 2\phi_1^2\phi_2^2 + 2\phi_1^2\phi_3^2 + 2\phi_1^2\phi_4^2}_{\text{all terms involving } \phi_1} + \cdots\Big]$$

So we can have 4 possibilities for propagators in loop all ϕ_1's or all ϕ_2's etc.
The combinatorics will be different
First consider all ϕ_1's

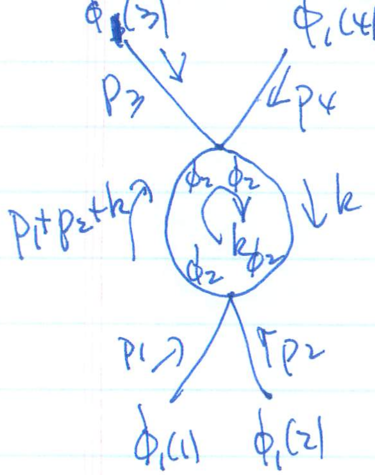


$$= \frac{(-i\lambda)^2}{4\cdot 4\cdot 2!}\Big[(2!)\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2\Big]\int \frac{d^4k}{(2\pi)^4}\left[\frac{i}{(p_1+p_2+k)^2-m^2}\right]\left[\frac{i}{k^2-m^2}\right]$$

$$= \frac{18\lambda^2}{16\pi^4}\int d^4k \frac{1}{[(p_1+p_2+k)^2-m^2][k^2-m^2]}$$

(this is one-loop so we have $(Z_\lambda \lambda)^2 \to (\lambda)^2$ here)

Next consider ϕ_2's in loop

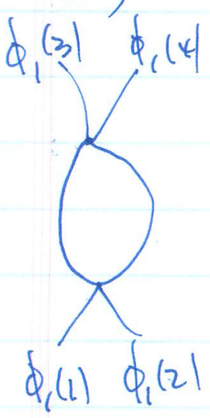


$$= \frac{(-i2\lambda)^2}{4\cdot 4\cdot 2!}\cdot\left[(2!)2\cdot 2\cdot 2\right]\int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{(p_1+p_2+k)^2-m^2}\right]\left[\frac{i}{k^2-m^2}\right]$$

$$= \frac{2\lambda^2}{16\pi^4}\int d^4k\,\frac{1}{[(p_1+p_2+k)^2-m^2][k^2-m^2]}$$

Now we can have ϕ_3 & ϕ_4 in loop so we get 3 times this result which we add to the ϕ_1's running in loop

$\Rightarrow$



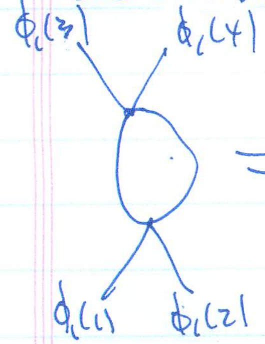
$$= \frac{24\lambda^2}{16\pi^4}\int d^4k\,\frac{1}{[(p_1+p_2+k)^2-m^2][k^2-m^2]}$$

Recall Mandelstam variables

$$s=(p_1+p_2)^2\ ;\ t=(p_1+p_3)^2$$

$$u=(p_1+p_4)^2$$

So we have



$$= \frac{3\lambda^2}{2\pi^4}\int d^4k \, \frac{1}{\left[(p_1+p_2+k)^2-m^2\right]\left[k^2-m^2\right]}$$

$$\equiv \Gamma(s)$$

So finally we have (Recall $X = -\frac{i\lambda Z_\lambda}{4}4! = -iZ_\lambda \lambda 6$)

$$\boxed{\Gamma^{(4,0,0,0)} = -6i\lambda Z_\lambda + \Gamma(s) + \Gamma(t) + \Gamma(u)}$$

Now
$$\hat{\Gamma}^{(4)} = \Gamma(s) + \Gamma(t) + \Gamma(u)$$

So
$$\hat{\Gamma}^{(4)}\Big|_{\substack{s=t=u\\ =-\frac{4}{3}\mu^2}} = 3\,\Gamma\!\left(-\tfrac{4}{3}\mu^2\right)$$

So $$\boxed{(Z_\lambda - 1)\lambda = -\frac{i}{2}\,\Gamma\!\left(-\tfrac{4}{3}\mu^2\right)}$$

So

$$\beta = -\frac{i}{6}\left[\mu\frac{\partial}{\partial\mu}\Gamma^{(4,0,0,0)}_{(s,t,u)}\right]\Bigg|_{-\frac{4}{3}\mu^2}$$

$$= -\frac{i}{6}\left[\mu\frac{\partial}{\partial\mu}\left(-6iZ_\lambda\lambda + \Gamma(s) + \Gamma(t) + \Gamma(u)\right)\right]\Bigg|_{-\frac{4}{3}\mu^2}$$

but $\Gamma(s), \Gamma(t), \Gamma(u)$ are indep. of μ in one-loop $\Rightarrow$

$$\boxed{\beta = -\lambda\,\mu\frac{\partial}{\partial\mu}Z_\lambda} \quad \text{in one-loop.}$$

$$\boxed{= +\frac{i}{2}\,\mu\frac{\partial}{\partial\mu}\Gamma\left(-\frac{4}{3}\mu^2\right)}$$

Recall

$$\Gamma(p) = \frac{3\lambda^2}{2\pi^4}\int d^4k \frac{1}{[(p+k)^2-m^2][k^2-m^2]}$$

Now Γ is dimensionless

$$\Gamma = \Gamma\left(\Lambda^2/s, m/\Lambda\right) \quad \text{where } \Lambda \text{ is the cut-off}$$

As usual we will only need the divergent term which comes from $\ln(\Lambda^2/s)$

A handwaving way to see this consider

$$\frac{1}{[(p+k)^2-m^2](k^2-m^2)} = \frac{1}{[p^2+k^2+2p\cdot k-m^2](k^2-m^2)}$$

$$= \frac{1}{[p^2+k^2][k^2]\left[1+\frac{2p\cdot k-m^2}{p^2+k^2}\right]\left[1-\frac{m^2}{k^2}\right]}$$

For large $k \to \Lambda$

$$= \frac{1}{[p^2+k^2][k^2]}\left[1 \underbrace{- \frac{2p\cdot k-m^2}{p^2+k^2} + \frac{m^2}{k^2} + \cdots}\right]$$

These are UV convergent so their terms are small compared to divergent term

$$\Gamma(s) = \frac{3\lambda^2}{2\pi^4}\int d4k\,\frac{1}{[p^2+k^2]k^2} + \text{finite terms}$$

Wick rotate to Euclidean space

$$= i\,\frac{3\lambda^2}{2\pi^4}\,\Omega_4\int_0^{\Lambda^2}\frac{\frac{1}{2}k^2\,dk^2}{[k^2-p^2]k^2}$$

$$= i\,\frac{3\lambda^2}{2\pi^4}\,2\pi^2\,\frac{1}{2}\int_0^{\Lambda^2}\frac{dx}{[x-p^2]}$$

$\Lambda \to \infty$

$$= i\,\frac{3\lambda^2}{2\pi^2}\ln\left(\frac{\Lambda^2-p^2}{-p^2}\right) = i\,\frac{3\lambda^2}{2\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right)$$

So
$$\boxed{\Gamma\left(-\tfrac{4}{3}\mu^2\right) = i\,\frac{3\lambda^2}{2\pi^2}\ln\left(\Lambda^2\big/\tfrac{4}{3}\mu^2\right) = i\,\frac{3\lambda^2}{2\pi^2}\,2\ln\left(\Lambda/\mu\right)} \qquad \left(\begin{matrix}\text{ignore}\\ \ln 4/3\end{matrix}\right)$$

So

(p.-9-)
$$\beta = \frac{i}{2}\;\mu\frac{\partial}{\partial\mu}\Gamma\left(-\tfrac{4}{3}\mu^2\right)$$
$$= \frac{i}{2}\;2i\,\frac{3\lambda^2}{2\pi^2}(-1)$$
$$\boxed{\boxed{\beta = +\frac{3\lambda^2}{2\pi^2}}}$$

Thus the effective or running coupling constant for the SM Higgs self-interactions is given by

$$\boxed{\mu\frac{d\lambda(\mu)}{d\mu} = \beta(\lambda(\mu)) = \frac{3\,\lambda(\mu)^2}{2\pi^2}}$$

So

$$\frac{d\lambda}{\lambda^2} = \frac{3}{2\pi^2}\frac{d\mu}{\mu}$$

$$\boxed{-\frac{1}{\lambda(\mu_2)} + \frac{1}{\lambda(\mu_1)} = \frac{3}{2\pi^2}\ln\left(\frac{\mu_2}{\mu_1}\right)}$$

Solve for $\lambda(\mu_2)$

$$\frac{1}{\lambda(\mu_2)} = \frac{1}{\lambda(\mu_1)} - \frac{3}{2\pi^2}\ln(\mu_2/\mu_1)$$

$$= \frac{1 - \frac{3}{2\pi^2}\lambda(\mu_1)\ln(\mu_2/\mu_1)}{\lambda(\mu_1)}$$

$\Rightarrow$

$$\boxed{\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \frac{3}{2\pi^2}\lambda(\mu_1)\ln(\mu_2/\mu_1)}}$$

So we note the Landau singularity the coupling $\lambda(\mu_2)$ blows up at some value of $\mu_2 \gg \mu_1$

This appears as some cutoff energy — even though we renormalized the model

Let $\mu_2 = \Lambda$ where the denominator vanishes

$$\boxed{1 - \frac{3}{2\pi^2}\lambda(\mu_1)\ln\left(\frac{\Lambda}{\mu_1}\right) = 0 \quad \Lambda \text{ is where } \lambda(\Lambda) \to \infty.}$$

The theory is defined below this scale then.

$$\lambda(\mu_1) = \frac{2\pi^2/3}{\ln(\Lambda/\mu_1)}$$

So the larger Λ the smaller $\lambda(\mu_1)$ at low energy. In the limit $\Lambda \to \infty$, $\lambda(\mu_1) \to 0$ the theory is free! Hence we cannot remove the cut off. The Higgs model is an effective theory - good to use below Λ.

$\lambda(\mu_1) < \lambda(\text{bound})$ or else $\lambda(t)$ diverges before Λ.

Now since the Higgs mass is given by $\lambda(\mu_1)$ we can obtain a bound for it depending ~~on~~ the cutoff Λ. Suppose we fine tune the mass parameter so that m^2 is negative for the broken phase the Higgs mass is given by

$$m_H^2 = 2\lambda(m_H)v^2$$

The $W^\pm$ mass is $M_W^2 = \frac{g_2^2 v^2}{4}$

So $$\frac{m_H^2}{M_W^2} = 8\frac{\lambda(m_H)}{g_2^2(M_W)}$$

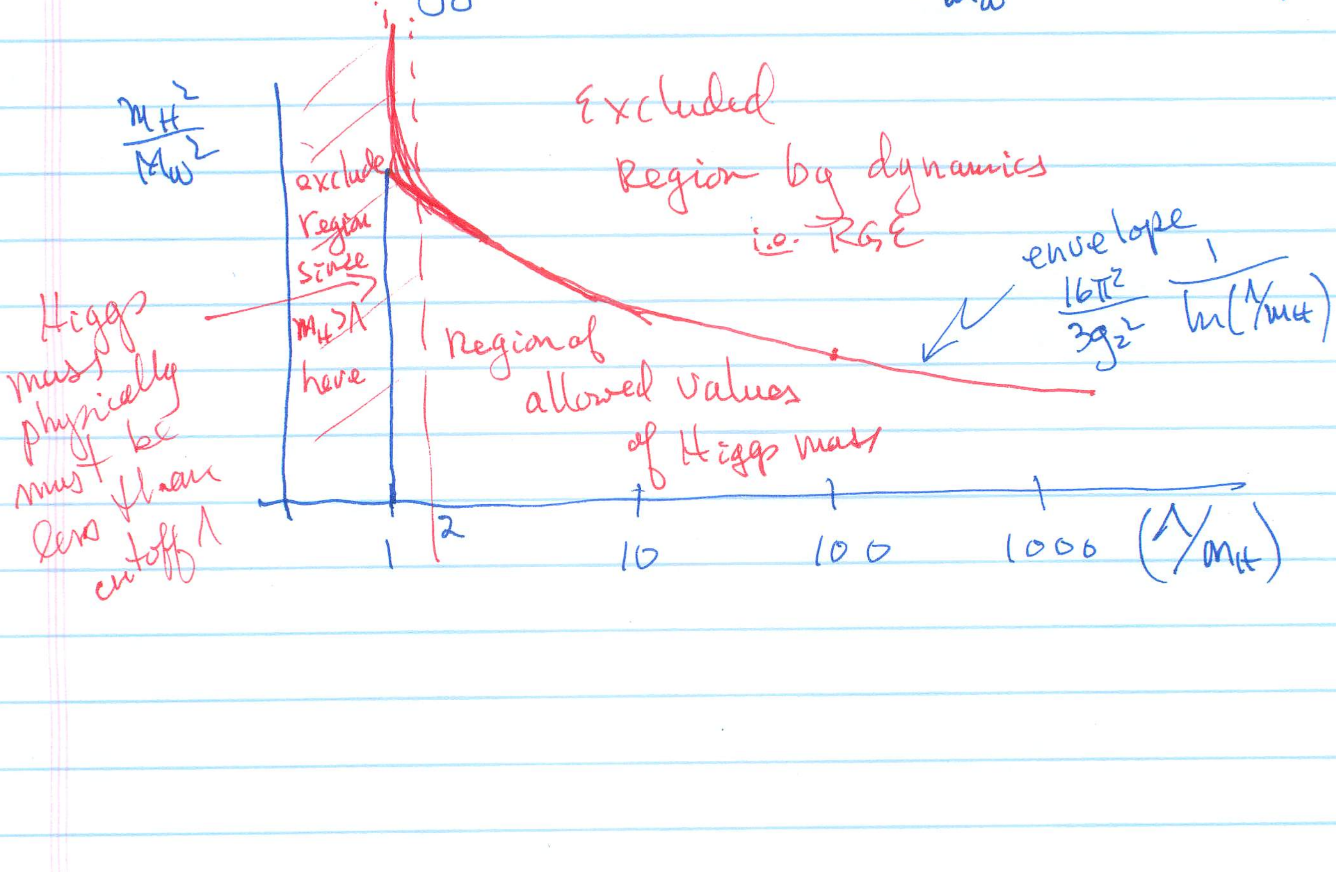
Now assume the gauge couplings run slowly so that $g_2^2 \approx 0.4$ fixed we have

$$\frac{m_H^2}{M_W^2} = \frac{8}{g_2^2}\,\frac{2\pi^2/3}{\ln(\Lambda/m_H)}$$

So

$$\boxed{\frac{m_H^2}{M_W^2} = \frac{16\pi^2}{3g_2^2}\,\frac{1}{\ln(\Lambda/m_H)}}$$

This an envelope of allowed values for the Higgs mass in the $\left(\frac{M_H^2}{M_W^2} - \Lambda/m_H\right)$ plane



So let's put in some numbers

$$\frac{m_H^2}{M_W^2} = \frac{4\pi}{3}\frac{1}{\alpha_2}\frac{1}{\ln(\Lambda/m_H)} \quad ; \quad \alpha_2 = \frac{g_2^2}{4\pi}$$

$$g_2 = \frac{e}{\sin\theta_W}$$

$$= \frac{4\pi}{3}\frac{\sin^2\theta_W}{\alpha}\frac{1}{\ln(\Lambda/m_H)} \qquad \alpha_2 = \frac{\alpha}{\sin^2\theta_W} = \frac{1/137}{0.231}$$

$$= \frac{132.6}{\ln(\Lambda/m_H)} \qquad M_W = 80.4\ \text{GeV}$$

Λ/m_H	$(m_H/m_W)^2$	(M_H/m_W)	m_H (GeV)	Λ (GeV)
2	191	13.8	1,100	2,200
5	82.4	9.1	732	3.7×10^3
10	57.6	7.56	608	6.1×10^3
10^2	28.8	5.37	432	43×10^3
10^3	19.2	4.38	352	350×10^3
10^4	14.4	3.8	306	3×10^6
10^6	9.60	3.1	249	2.5×10^8
10^{15}	3.84	2.00	161	1.6×10^{17}
10^{17}	3.39	1.84	148	1.5×10^{19}
10^{20}	2.88	1.70	137	1.4×10^{22}

$\Lambda/m_H \sim 5$–10 expect ϕ^4 as a reasonable effective theory $\Rightarrow m_H \lesssim 700$ GeV