I.C) Electroweak Theory : $SU_W(2) \times U_Y(1)$

Electroweak theory is based on the $SU(2) \times U(1)$ gauge group with $SU(2)$ gauge fields A_μ^i ; $i = 1, 2, 3$ since $SU(2)$ has 3 generators

and a $U(1)$ gauge field B_μ only 1 generator.

What's "new" is that the gauge theory is "chiral". That means the left & right handed projections of the fermion fields are in different representations of $SU(2)$ and have different weak hypercharge.

Recall chiral (Weyl) projectors

$$a_\pm = \frac{1}{2}(1 \pm \gamma_5) = \gamma_\pm$$

$$\gamma_+^2 = \gamma_+ \qquad \gamma_+ \gamma_- = 0 = \gamma_- \gamma_+$$

$$\gamma_-^2 = \gamma_- \qquad \gamma_+ + \gamma_- = \mathbb{1}$$

Given a Dirac 4-component complex spinor ψ we defined Right handed spinors as

$$\psi_R \equiv \gamma_+ \psi$$

& left handed

$$\psi_L \equiv \gamma_- \psi$$

For $SU(2)_W$ <u>all</u> Right handed fields are $SU(2)$ singletts
<u>all</u> left handed fields are $SU(2)$ doublets (fundamental rep.)

1) Recall for $SU(2)$ fundamental rep. is given by Pauli Matrices

$$\sigma^1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^i_{ab} = \frac{1}{2} \sigma^i_{ab} \qquad a,b = 1,2$$
$$i = 1,2,3$$

& $[T^i, T^j] = i \epsilon^{ijk} T^k$. i.e. $f_{ijk} = \epsilon_{ijk}$

2) The adjoint rep. is given by structure constants

$$(T^i)_{jk} = i \epsilon_{jik}$$

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix}; \quad T^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[T^i, T^j] = i \epsilon^{ijk} T^k$$

I.C.) Hence all $\boxed{\psi'_R = \psi_R}$ for SU(2) transformations while

$$\boxed{\psi'_{L_a} = U(\omega)_{ab}\psi_{L_b}}$$

where $U(\omega)_{ab} = \left(e^{+ig_2 \omega^i T^i}\right)_{ab}$

$$= \left(e^{\frac{i}{2} g_2 \omega^i \sigma^i}\right)_{ab}.$$

So infinitesimally

$$\psi'_{La} = \psi_{La} + ig_2 \omega^i T^i_{ab} \psi_{Lb}$$

$$= \psi_{La} + ig_2 \omega^i \left(\frac{\sigma^i}{2}\right)_{ab} \psi_{Lb}$$

Now $U_Y(1)$ of weak hypercharge is just a phase symmetry whose value is given by the weak hypercharge of the field — that is a multiple of g_1 — call it y. So for left handed fields we have

$$\psi'_L = U(\theta)\psi_L = e^{ig_1 y_L \theta}\psi_L$$

& RH

$$\psi'_R = U(\theta)\psi_R = e^{ig_1 y_R \theta}\psi_R$$

or $\theta \ll 1$

$$\psi'_L = \psi_L + ig_1 y_L \theta \psi_L$$

$$\psi'_R = \psi_R + ig_1 y_R \theta \psi_R$$

The quantum numbers y_L, y_R are chosen so that $\mathcal{L}_{L,R}$ quantum numbers for T_3 & y add up to the electric charge Q of the field — which is diagonal — eigenvalues

$$Q \equiv T^3 + y.$$

(Some conventions are $Q = T^3 + y/2$ this is same as letting $2g_1 \to g_1$)

So how do the fermions of the SM fit into this scheme: 3 families of fermions. Each generation consists of a SU(2) doublet of left-handed quarks, a SU(2) doublet of left-handed leptons, 2 right-handed quark SU(2) singlets & one right handed lepton SU(2) singlet.

1) Electron Family

$$\begin{bmatrix} \nu_e \\ e \end{bmatrix}_L , \begin{bmatrix} u \\ d \end{bmatrix}_L , e_R , u_R , d_R$$

2) Muon Family

$$\begin{bmatrix} \nu_\mu \\ \mu \end{bmatrix}_L , \begin{bmatrix} c \\ s \end{bmatrix}_L , \mu_R , c_R , s_R$$

3) Tau Family

$$\begin{bmatrix} \nu_\tau \\ \tau \end{bmatrix}_L , \begin{bmatrix} t \\ b \end{bmatrix}_L , \tau_R , t_R , b_R$$

$T^3 \leftrightarrow \frac{1}{2}\sigma^3$ So isospin quantum numbers of the doublet are

$+\frac{1}{2}$ for upper field

$-\frac{1}{2}$ for lower field

ex. $T^3 \begin{bmatrix} \nu_e \\ e \end{bmatrix}_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} \frac{1}{2}\nu_e \\ -\frac{1}{2} e \end{pmatrix}_L$

So we can make a table of T^3, y, Q quantum #'s

		T^3	y	$Q = T^3 + y$
$l_L =$	$(\nu_e, \nu_\mu, \nu_\tau)_L$	$+\frac{1}{2}$	$-\frac{1}{2}$	0
	e_L, μ_L, τ_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$q_L =$	u_L, c_L, t_L	$+\frac{1}{2}$	$+\frac{1}{6}$	$+2/3$
	d_L, s_L, b_L	$-\frac{1}{2}$	$+\frac{1}{6}$	$-1/3$
	e_R, μ_R, τ_R	0	-1	-1
	u_R, c_R, t_R	0	$+2/3$	$+2/3$
	d_R, s_R, b_R	0	$-1/3$	$-1/3$

(Also the y assignments are fixed from anomaly cancellation)

Note the SU(3) color indices on the quarks have been surpressed each u, d, c, s, t, b has a color index also $u^a, d^a, c^a, s^a, t^a, b^a$; $a = 1,2,3$.

In general we will also use the notation

l_{mL} for each lepton doublet & q_{mL} for each quark doublet, $m = e, \mu, \tau$ or $1, 2, 3$ family index

(note earlier we let $m = 1, \ldots, 6$ now $m = 1, 2, 3$)

and we use e_{mR}, u_{mR}, d_{mR} for right-handed fields ~~where~~

$e_{1R} = e_{eR} = e_R$	$u_{1R} = u_{eR} = u_R$	$d_{1R} = d_{eR} = d_R$
$e_{2R} = e_{\mu R} = \mu_R$	$u_{2R} = u_{\mu R} = c_R$	$d_{2R} = d_{\mu R} = s_R$
$e_{3R} = e_{\tau R} = \tau_R$	$u_{3R} = u_{\tau R} = t_R$	$d_{3R} = d_{\tau R} = b_R$

So we can write the $SU(2) \times U(1)$ transformations of the fermion fields

$$l'_L = U(\omega, \theta)\, l_L = \left(e^{ig_2 \omega^i t^i}\right)\left(e^{ig_1 (-\frac{1}{2})\theta}\right) l_L$$

$$= l_L + \frac{i}{2} g_2 \omega^i \sigma^i l_L - \frac{i}{2} g_1 \theta\, l_L$$

(in detail
$$l'_{La} = l_{La} + \frac{i}{2} g_2 \omega^i (\sigma^i)_{ab}\, l_{Lb} - \frac{i}{2} g_1 \theta\, l_{La}$$
)

Likewise

$$q'_L = \left(e^{ig_2 \omega^i T^i}\right)\left(e^{ig_1 (\frac{1}{6})\theta}\right) q_L$$

$$= q_L + \frac{i}{2} g_2 \omega^i \sigma^i q_L + \frac{i}{6} g_1 \theta\, q_L$$

I.C.) Next the SU(2) Righthanded singlets

$$e'_R = \left(e^{ig_1(-1)\theta}\right) e_R$$

$$= e_R - ig_1\theta e_R$$

$$u'_R = \left(e^{ig_1\left(\frac{2}{3}\right)\theta}\right) u_R$$

$$= u_R + i\frac{2}{3} g_1 \theta u_R$$

$$d'_R = \left(e^{ig_1\left(-\frac{1}{3}\right)\theta}\right) d_R$$

$$= d_R - \frac{i}{3} g_1 \theta d_R$$

These transformations apply to each family

Note each flavor of quark, left- or right-handed, still transforms as a $\underline{3}$ under SU(3)

So we can make an SU(2)×U(1) globally invariant action by recalling

$$\psi_L = \gamma_- \psi; \quad (\psi_L)^\dagger = \psi^\dagger \gamma_-^\dagger = \psi^\dagger \gamma_-$$

$$\overline{(\psi_L)} \equiv \psi_L^\dagger \gamma^0 = \psi^\dagger \gamma_- \gamma^0 = \psi^\dagger \gamma^0 \gamma_+ = (\overline{\psi})_R$$

$$= \overline{\psi} \gamma_+$$

Further

$$\overline{(\psi_L)}\gamma^\mu\psi_L = \bar\psi\gamma_+\gamma^\mu\gamma_-\psi = \bar\psi\gamma^\mu\gamma_-\psi$$

But

$$\overline{(\psi_R)}\gamma^\mu\psi_L = \bar\psi\gamma_-\gamma^\mu\gamma_-\psi = \bar\psi\gamma^\mu\cancelto{0}{\gamma_+\gamma_-}\psi = 0$$

So kinetic terms are of the form

$$\overline{(\psi_L)}\,i\not\partial\,\psi_L \quad \text{and} \quad \overline{(\psi_R)}\,i\not\partial\,\psi_R$$

So if $\psi_L' = U\psi_L$ then

$$\psi_L'^{\dagger} = \psi_L^\dagger U^\dagger = \psi_L^\dagger U^{-1}$$

& $\overline{(\psi_L')} = \overline{(\psi_L)}\,U^{-1}$

likewise $\overline{(\psi_R')} = \overline{(\psi_R)}\,U^{-1}$

Notation Convention:

$$\bar\psi_L \equiv \overline{(\psi_L)}$$
$$\bar\psi_R \equiv \overline{(\psi_R)}$$

(Not $(\bar\psi)_L$, $(\bar\psi)_R$)

Hence we have globally invariant kinetic terms

$$\mathcal{L}_{inv} = \bar l_L\, i\not\partial\, l_L + \bar q_L\, i\not\partial\, q_L + \bar e_R\, i\not\partial\, e_R + \bar u_R\, i\not\partial\, u_R + \bar d_R\, i\not\partial\, d_R$$

$\mathcal{L}_{inv}' = \mathcal{L}_{inv}$ for w^i & θ space-time indep.

In order to make a globally invariant action locally invariant we replace $\partial_\mu \psi$ with $D_\mu \psi$ — covariant derivatives! Introducing the covariant derivatives

$$D_\mu \equiv \partial_\mu - i g_2 T^i A^i_\mu - i g_1 y B_\mu$$

so that

$$D_\mu l_L = \left(\partial_\mu - \frac{i g_2}{2} \vec{\sigma} \cdot \vec{A}_\mu + \frac{i g_1}{2} B_\mu\right) l_L$$

$$D_\mu q_L = \left(\partial_\mu - \frac{i g_2}{2} \vec{\sigma} \cdot \vec{A}_\mu - \frac{i g_1}{6} B_\mu\right) q_L$$

$$D_\mu e_R = \left(\partial_\mu + i g_1 B_\mu\right) e_R$$

$$D_\mu u_R = \left(\partial_\mu - \frac{2i}{3} g_1 B_\mu\right) u_R$$

$$D_\mu d_R = \left(\partial_\mu + \frac{i}{3} g_1 B_\mu\right) d_R$$

Recall that if $U(\omega) = e^{i g_2 T^i \omega^i}$; $U(\theta) = e^{+i g_1 y \theta}$
$U(\omega, \theta) = U(\omega) U(\theta)$ then

$$T^i A'^i_\mu = U(\omega) T^i A^i_\mu U^{-1}(\omega) - \frac{i}{g_2} \left(\partial_\mu U(\omega)\right) U^{-1}(\omega)$$

or $A_\mu \equiv i T^i A^i_\mu \Rightarrow$

$$A'_\mu = U(\omega) A_\mu U^{-1}(\omega) + \frac{1}{g_2} \left(\partial_\mu U(\omega)\right) U^{-1}(\omega)$$

So infinitesimally

$$A'^i_\mu = A^i_\mu + \partial_\mu \omega^i + g_2 \epsilon_{ijk} A^j_\mu \omega^k$$

$$= A^i_\mu + D^{ij}_\mu \omega^j \qquad \text{with } D^{ij}_\mu = \partial_\mu \delta^{ij} - i g_2 T^k_{ij} A^k_\mu$$

adj. rep.

$$(T^k)_{ij} = i\,\epsilon_{ikj}$$

Likewise

$$y B'_\mu = U(\theta)\, y B_\mu U^{-1}(\theta) - \frac{i}{g_1}[\partial_\mu U(\theta)]U^{-1}(\theta)$$

$$= y B_\mu + y\,\partial_\mu \theta \qquad \text{with } \theta \text{ finite or infinitesimal}$$

i.e. $B'_\mu = B_\mu + \partial_\mu \theta$

So we have that if $l'_L = U(\omega,\theta)\, l_L$ then

$$(D_\mu l_L)' = D'_\mu l'_L = U(\omega,\theta)(D_\mu l_L)$$

likewise for all other terms

$$q'_L = U(\omega,\theta)\, q_L \;\&\; (D_\mu q_L)' = U(\omega,\theta)(D_\mu q_L)$$

and since $\bar{q}'_L = \bar{q}_L U^{-1}(\omega,\theta)$ etc.

we have the invariant kinetic term

$$(\bar{q}_L i \not{D} q_L)' = \bar{q}_L i \not{D} q_L \quad \text{, etc.}$$

So
$$\mathcal{L}_F = \bar{l}_L i\not{D} l_L + \bar{q}_L i\not{D} q_L + \bar{e}_R i\not{D} e_R + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R$$

Recall that we are implicitly summing over each family e, μ, τ.

$$\mathcal{L}_F' = \mathcal{L}_F \quad SU(2)\times U(1) \text{ invariant}$$

Note we cannot make $SU(2)\times U(1)$ invariant fermion mass terms!

$$\bar{\psi}\psi = \bar{\psi}(\gamma_+ + \gamma_-)\psi = \bar{\psi}\gamma_+\psi + \bar{\psi}\gamma_-\psi$$

$$= \bar{\psi}\gamma_+^2\psi + \bar{\psi}\gamma_-^2\psi$$

$$= \psi^\dagger\gamma^0\gamma_+\psi_R + \psi^\dagger\gamma^0\gamma_-\psi_L$$

$$= \psi^\dagger\gamma_-\gamma^0\psi_R + \psi^\dagger\gamma_+\gamma^0\psi_L$$

$$= \psi_L^\dagger\gamma^0\psi_R + \psi_R^\dagger\gamma^0\psi_L$$

$$= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$$

But for the $SU(2)\times U(1)$ all left handed fields are $SU(2)$ doublets while all RH fields are

SU(2) singlet — we cannot make an SU(2) invariant mass term!

Introduce Higgs multiplet to spontaneously break $SU(2) \times U(1) \longrightarrow U_{em}(1)$

<u>Higgs Field</u>: complex scalar field ϕ in the doublet representation of SU(2) (i.e. 4 hermitean fields)

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \qquad \begin{matrix} \phi^+ \text{ has } Q = +1 \\ \phi^0 \text{ has } Q = 0 \end{matrix}$$

Since $Q = T^3 + y \Rightarrow \phi$ has $y = +\frac{1}{2}$.

So again for $SU(2) \times U(1)$ transformations we have

$$\phi' = U(\omega, \theta)\phi$$

$$= \left(e^{ig_2 \omega^i T^i}\right)\left(e^{ig_1\left(\frac{1}{2}\right)\theta}\right)\phi$$

$$= \phi + \frac{ig_2}{2}\vec{\omega}\cdot\vec{\sigma}\phi + \frac{ig_1}{2}\theta\phi$$

$\Rightarrow$ covariant derivative of ϕ

$$\boxed{D_\mu \phi = \partial_\mu \phi - \frac{ig_2}{2} A^i_\mu \sigma^i \phi - \frac{ig_1}{2} B_\mu \phi}$$

$$D_\mu \phi = \left[\partial_\mu - i\frac{g_2}{2}\vec{A}_\mu \cdot \vec{\tau} - \frac{ig_1}{2}B_\mu\right]\phi$$

and as usual for $\phi' = U(\omega,\theta)\phi$ so does

$$(D_\mu\phi)' = U(\omega,\theta)(D_\mu\phi)$$

Now we must list possible invariant terms: first $SU(2)\times U(1)$ invariant Yukawa interaction terms:

$$\bar{q}_L \phi d_R \quad , \quad \bar{l}_L \phi e_R$$

Note

$$\bar{q}_L \phi \text{ is } SU(2) \text{ invariant}$$

but q_L has $y = +\frac{1}{6} \Rightarrow \bar{q}_L$ has $y = -\frac{1}{6}$

and ϕ has $y = +\frac{1}{2}$ so

$$\bar{q}_L \phi \text{ has } y = +\frac{1}{2} - \frac{1}{6} = +\frac{1}{3}$$

since d_R has $y = -\frac{1}{3}$

$\bar{q}_L \phi d_R$ is also $U(1)$ invariant.

Similarly for $\bar{l}_L \phi e_R$.

Now we must sum these invariants over the families with inter-family couplings

$$\Gamma^d_{mn} \bar{q}_{mL} \phi d_{nR} + h.c.$$

$$= \Gamma^d_{mn} \bar{q}_{mL} \phi d_{nR} + \Gamma^{d*}_{mn} d^\dagger_{nR} \phi^\dagger \gamma^0 q_{mL}$$

$$= \Gamma^d_{mn} \bar{q}_{mL} \phi d_{nR} + \Gamma^{d*}_{mn} \bar{d}_{nR} \phi^\dagger q_{mL}$$

and

$$\Gamma^e_{mn} \bar{l}_{mL} \phi e_{nR} + h.c.$$

$$= \Gamma^e_{mn} \bar{l}_{mL} \phi e_{nR} + \Gamma^{e*}_{mn} \bar{e}_{nR} \phi^\dagger l_{mL}.$$

Note looks like no u quark mass! since

$\bar{q}_L \phi u_R$ has $y = -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = 1$ (ξ $Q = +1 \neq 0$!)

But we can also couple to $\phi^\dagger$ the hermitean conjugate of ϕ.

$$\phi' = U(\omega, \theta) \phi \Rightarrow \phi'^\dagger = \phi^\dagger U^{-1}(\omega, \theta)$$

$$\Rightarrow \phi^{\dagger\prime} = \phi^\dagger - \frac{ig_2}{2} \phi^\dagger \vec{\omega} \cdot \vec{\sigma} - \frac{ig_1}{2} \theta \phi^\dagger$$

So if ϕ transforms as $(2, +\frac{1}{2})$ under $(SU(2), U(1))$

then $\phi^\dagger$ transforms as $(2^*, -\frac{1}{2})$ under $(SU(2), U(1))$

But for $SU(2)$ 2 & 2^* are equivalent:

$$\phi^{\dagger\prime} = \phi^\dagger - \frac{ig_2}{2}\phi^\dagger \vec{\omega}\cdot\vec{\sigma} \quad \text{if } \phi \text{ is a } 2 \text{ of } SU(2)$$

So

$$\phi_a^{*\prime} = \phi_a^* - \frac{ig_2}{2}(\vec{\omega}\cdot\vec{\sigma})_{ba}\phi_b^*$$

$$= \phi_a^* - \frac{ig_2}{2}(\vec{\omega}\cdot\vec{\sigma}^T)_{ab}\phi_b^*$$

But $\quad -\sigma^{iT} = (i\sigma^2)^\dagger \sigma^i (i\sigma^2)$

So

$$\phi^{*\prime} = \phi^* + \frac{ig_2}{2}(i\sigma^2)^\dagger(\vec{\omega}\cdot\vec{\sigma})(i\sigma^2)\phi^*$$

$\Rightarrow$

$$(i\sigma^2\phi^*)' = (i\sigma^2\phi^*) + \frac{ig_2}{2}(\vec{\omega}\cdot\vec{\sigma})(i\sigma^2\phi^*)$$

Define $\quad \tilde{\phi} \equiv i\sigma^2\phi^*$

Then $\quad \boxed{\tilde{\phi}' = \tilde{\phi} + \frac{ig_2}{2}\vec{\omega}\cdot\vec{\sigma}\,\tilde{\phi}}$

Thus $\tilde{\phi}$ is a 2 of $SU(2)$ just like ϕ

$$\boxed{\tilde{\phi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \phi^- \\ \phi^{0\dagger} \end{bmatrix} = \begin{bmatrix} \phi^{0\dagger} \\ -\phi^- \end{bmatrix} \quad \text{as } (2, -\tfrac{1}{2})}$$

So we have the additional $SU(2) \times U(1)$ invariant Yukawa term

$$\bar{q}_L \tilde{\phi}\, u_R \qquad \text{where}$$

$\bar{q}_L \tilde{\phi}$ is ~~an~~ $SU(2)$ singlet with $y = -\frac{1}{6} - \frac{1}{2} = -\frac{2}{3}$

u_R is a $SU(2)$ singlet with $y = +\frac{2}{3}$

So

$$\Gamma^u_{mn}\, \bar{q}_{mL} \tilde{\phi}\, u_{nR} + h.c. \text{ is an } SU(2) \times U(1) \text{ invariant}$$

So we have the Yukawa interaction Terms

$$\boxed{\mathcal{L}_{yuk} = \Gamma^e_{mn}\, \bar{l}_{mL} \phi\, e_{nR} + \Gamma^d_{mn}\, \bar{q}_{mL} \phi\, d_{nR} + \Gamma^u_{mn}\, \bar{q}_{mL} \tilde{\phi}\, u_{nR} + h.c.}$$

We will show later not all $\Gamma^{e,d,u}_{mn}$ are observable — we will be able to redefine some couplings away.

Now we also have the Higgs' self-interaction that are invariant given by the potential

$$V = V(\phi^\dagger \phi) \equiv m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

and the globally invariant KE $\partial_\mu \phi^\dagger \partial^\mu \phi$

which becomes $(D_\mu \phi)^\dagger (D^\mu \phi)$ to give the gauge interactions of the Higgs field

So the locally $SU(2) \times U(1)$ Higgs Lagrangian is

$$\mathcal{L}_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger \phi)$$

with $V(\phi^\dagger \phi) = m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$

$$D_\mu \phi = \left(\partial_\mu - \frac{i g_2}{2} \vec{\tau} \cdot \vec{A}_\mu - \frac{i g_1}{2} B_\mu\right) \phi$$

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \qquad \left((\phi^+)^\dagger = \phi^-\right)$$

Finally we have the gauge fields' kinetic energy

$$D_\mu = \partial_\mu - i g_2 \vec{T}\cdot\vec{A}_\mu$$

$$[D_\mu, D_\nu] = -i g_2 T^i F^i_{\mu\nu}$$

$$\Rightarrow \quad F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_2 \epsilon_{ijk} A^j_\mu A^k_\nu$$

or $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]$$

$$\left(\begin{matrix} \parallel \\ i T^i F^i_{\mu\nu} \end{matrix}\right)$$

with the transformation

$$F'_{\mu\nu} = U(\omega) F_{\mu\nu} U^{-1}(\omega) \quad \text{that is infinitesimally}$$

$$F'^i_{\mu\nu} = F^i_{\mu\nu} + i g_2 \vec{\omega}\cdot\vec{T}_{ij} F^j_{\mu\nu} \quad \text{adjoint rep}$$

Hence $$(F^i_{\mu\nu} F^{i\mu\nu})' = (F^i_{\mu\nu} F^{i\mu\nu}) \text{ is } SU(2)\times U(1) \text{ invariant}$$

Similarly

$$D_\mu = \partial_\mu - i g_1 y B_\mu$$

$$[D_\mu, D_\nu] = -i g_1 y (\partial_\mu B_\nu - \partial_\nu B_\mu) \equiv -i g_1 y B_{\mu\nu}$$

$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ the Abelian field strength tensor.

$$B'_{\mu\nu} = \partial_\mu B'_\nu - \partial_\nu B'_\mu$$

$$= \partial_\mu(B_\nu + \partial_\nu \theta) - \partial_\nu(B_\mu + \partial_\mu \theta)$$

$$= B_{\mu\nu} + \cancel{\partial_\mu \partial_\nu \theta} - \cancel{\partial_\nu \partial_\mu \theta}$$

$$B'_{\mu\nu} = B_{\mu\nu}$$

$\Rightarrow \quad (B_{\mu\nu} B^{\mu\nu})' = (B_{\mu\nu} B^{\mu\nu})$ is $SU(2) \times U(1)$ invariant

Hence we have the Yang-Mills action

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}}$$

$$= -\frac{1}{2} \mathrm{Tr}\left[F_{\mu\nu} F^{\mu\nu} \right] - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$