The negative β implies asymptotic freedom

Recall Asymptotic Behavior of Greens functions & RGE

Consider $\Gamma^{(m,n,l,c,\bar{c})}(p, m, g_3, \alpha; \mu)$, we are interested in scaling the momenta to large values — scale with factor ρ & use engineering dimensional analysis

$$\Gamma^{(m,n,l,c,\bar{c})}(\rho p, m, g_3, \alpha; \mu) = \rho^{4-\frac{3}{2}(m+n)-l-c-\bar{c}}\,\Gamma^{(m,n,l,c,\bar{c})}(p, \tfrac{m}{\rho}, g_3, \alpha; \mu/\rho)$$

$\Rightarrow$

$$\rho\frac{\partial}{\partial\rho}\Gamma^{(m,n,l,c,\bar{c})}(\rho p, m, g_3, \alpha; \mu) = \left[4-\tfrac{3}{2}(m+n)-l-c-\bar{c}-\mu\frac{\partial}{\partial\mu}-m\frac{\partial}{\partial m}\right]\times$$
$$\times\,\rho^{4-\frac{3}{2}(m+n)-l-c-\bar{c}}\,\Gamma^{(m,n,l,c,\bar{c})}(p, m/\rho, g_3, \alpha; \mu/\rho)$$

$\Rightarrow$

engineering dimensional analysis

$$\boxed{\left[\rho\frac{\partial}{\partial\rho}+\mu\frac{\partial}{\partial\mu}+\sum_{m}m_{(m)}\frac{\partial}{\partial m_{(m)}}-\left(4-\tfrac{3}{2}(m+n)-l-c-\bar{c}\right)\right]\Gamma^{(m,n,l,c,\bar{c})}(\rho p, m, g_3, \alpha; \mu)=0}$$

Renormalization group

$$\Big[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_3}+\sum_{m}\gamma_m^{(m)}m_{(m)}\frac{\partial}{\partial m_{(m)}}+\beta_\alpha\frac{\partial}{\partial\alpha}$$
$$-(m+n)\gamma_f-l\gamma_G-(c+\bar{c})\gamma_c\Big]\Gamma^{(m,n,l,c,\bar{c})}(\rho p, m, g_3, \alpha; \mu)=0$$

arb. momentum so use ρp

This yields

$$\boxed{\begin{aligned}\Big[\rho\frac{\partial}{\partial\rho} - \beta\frac{\partial}{\partial g_3} + \sum_m{}' \left(1-\gamma_m^{(m)}\right) m_{(m)}\frac{\partial}{\partial m_m} - \beta_\alpha\frac{\partial}{\partial\alpha} \\ - \left(4-\left(\tfrac{3}{2}+\gamma_q\right)(m+n) - l(1+\gamma_G) - (c+\bar{c})(1+\gamma_c)\right)\Big] \times \\ \times\, \Gamma^{(m,n,l,c,\bar{c})}(\rho p, m, g_3, \alpha; \mu) = 0\end{aligned}}$$

Let $t \equiv \ln\rho$ and define the

Running or effective mass $\bar{m}_{(m)}(t)$ &

coupling constant $\bar{g}_3(t)$ & gauge parameter $\bar{\alpha}(t)$

by the DE

$$\frac{d\bar{g}_3(t)}{dt} = \beta(\bar{g}_3)$$

(for simplicity assume a scheme (minimal) s.t. γ, β only depend on g_3 (1-loop is okay))

$$\frac{d\bar{m}_{(m)}(t)}{dt} = -\left(1-\gamma_m^{(m)}\right) m_{(m)}$$

$$\frac{d\bar{\alpha}(t)}{dt} = \beta(\bar{g}_3)$$

with i.c.

$$\bar{g}_3(t=0) = g_3$$

$$\bar{m}_{(m)}(t=0) = m_{(m)}$$

$$\bar{\alpha}(t=0) = \alpha$$

Then the solution to the RGE above is

$$\boxed{\begin{aligned}\Gamma^{(m,n,l,c,\bar c)}(\rho p, m, g_3, \alpha; \mu) &= e^{\left(4-\frac{3}{2}(m+n)-l-c-\bar c\right)t} \cdot e^{-(m+n)\int_0^t \gamma_q(\bar g_3(t'))dt'} \cdot \\ &\cdot e^{-l\int_0^t dt'\,\gamma_G(\bar g_3(t'))}\, e^{-(c+\bar c)\int_0^t dt'\,\gamma_c(\bar g_3(t'))} \times \\ &\times \Gamma^{(m,n,l,c,\bar c)}(p, \bar m(t), \bar g_3(t), \bar\alpha(t); \mu)\end{aligned}}$$

So $\rho \frac{\partial}{\partial \rho} = \rho \frac{\partial t}{\partial \rho}\frac{\partial}{\partial t} = \rho \frac{1}{\rho}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}$

$$\rho = e^t$$

So

$$\left[\frac{\partial}{\partial t} - \beta\frac{\partial}{\partial g_3} + \dots\right]\Gamma(e^t p, m, g_3, \alpha; \mu) = 0$$

$\Rightarrow$

$$\left[\frac{\partial}{\partial t} - \beta\frac{\partial}{\partial g_3} + \sum_m (1-\gamma_m^{(m)})m_{(m)}\frac{\partial}{\partial m_{(m)}} - \beta_\alpha\frac{\partial}{\partial \alpha}\right]\times$$

$$\times\Gamma^{(m,n,l,c,\bar c)}(p, \bar m(t), \bar g_3(t), \bar\alpha(t); \mu) = 0$$

$$\Rightarrow \left[\left(\frac{d\bar g_3(t)}{dt} - \beta\right)\frac{\partial}{\partial g_3} + \sum_m\left(\frac{d\bar m_{(m)}(t)}{dt} + (1-\gamma_m^{(m)})m_{(m)}\right)\frac{\partial}{\partial \bar m_{(m)}} + \left(\frac{d\bar\alpha(t)}{dt} - \beta_\alpha\right)\frac{\partial}{\partial \alpha}\right]\Gamma = 0$$

which is satisfied by the running parameters.

(So we include $\frac{m}{\mu}$ effects for general normalization scheme let $\mu >> m$ so we can neglect it in β, γ).

So in the deep Euclidean region of momentum the Green's functions are governed by the value of the parameters at that scale $\bar{g}_3(t)$ etc.

In particular we found that

$$\beta(\bar{g}_3) = -\frac{\bar{g}_3^3}{(4\pi)^2}\left[\frac{11}{3}C_2(8) - \frac{2}{3}N_F\right]$$

$$= -\frac{\bar{g}_3^3}{(4\pi)^2}[7]$$

So let

$$\beta = -\bar{g}_3^3\, b \quad ; \quad b = \frac{7}{(4\pi)^2}$$

Then $\displaystyle \frac{d\bar{g}_3}{dt} = -b\bar{g}_3^3$

$$\Rightarrow \int_{g_3}^{\bar{g}_3(t)} \frac{d\bar{g}_3}{\bar{g}_3^3} = -b\int_0^t dt$$

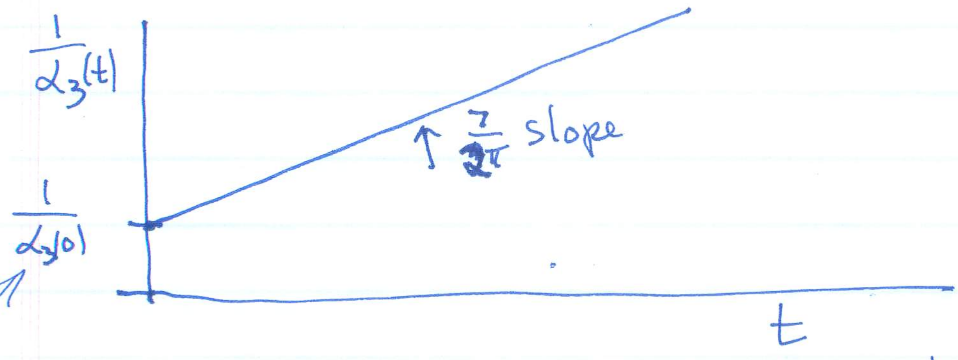
$$-\frac{1}{2\bar{g}_3^2(t)} + \frac{1}{2\bar{g}_3^2(0)} = -bt$$

So introduce the fine structure constant

$$\alpha_3(t) \equiv \frac{\bar{g}_3^{\,2}(t)}{4\pi}$$

$$+ \quad \frac{4\pi}{\bar{g}_3^{\,2}(t)} - \frac{4\pi}{\bar{g}_3^{\,2}(0)} = \frac{2\cdot 7t}{4\pi} = \frac{2}{4\pi}\left[\frac{11}{3}C_2(8) - \frac{2}{3}N_F\right]t$$

$$\boxed{\frac{1}{\alpha_3(t)} - \frac{1}{\alpha_3(0)} = \frac{7}{2\pi}t}$$



Say $\alpha_3(M_Z) = .1176$

$\Rightarrow \quad \alpha_3(t) \to 0$ as $t \to \infty$ perturbation theory applies.

Or use
$$-\frac{1}{2\bar{g}_3^{\,2}(t)} + \frac{1}{2\bar{g}_3^{\,2}(t_0)} = -b(t-t_0)$$

$$\Rightarrow \quad \bar{g}_3^{\,2}(t) = \frac{\bar{g}_3^{\,2}(t_0)}{1 + \bar{g}_3^{\,2}(t_0)\, 2b(t-t_0)}$$

Often useful to consider scale at which $\bar{g}_3^2(t_0)$ diverges — call it Λ. ($\simeq 200\,\mathrm{MeV}$)

Then consider scaled momentum $q^2 = e^{2(t-t_0)}\Lambda^2$

So

$$\frac{1}{\bar{g}_3^2(q^2)} - \cancelto{0}{\frac{1}{\bar{g}_3^2(\Lambda^2)}} = b\ln(q^2/\Lambda^2)$$

$\Rightarrow$

$$\alpha_3(q^2) = \frac{1}{4\pi b\ln(q^2/\Lambda^2)}$$

$$= \frac{4\pi}{\left[\frac{11}{3}C_2(8) - \frac{2}{3}N_F\right]\ln(q^2/\Lambda^2)}$$

$$= \frac{4\pi}{7\ln(q^2/\Lambda^2)}$$

Dimensional Transmutation: trade dimensionless coupling constant g_3 with dimensionful scale Λ as defining the theory