I.B.| Also $\Rightarrow P_T^{\mu\nu}(p)$  $=0$

Renormalization Group: The Green's functions depend on the arbitrary scale μ at which the normalization conditions were given. Hence

$$\Gamma^{(m,n,l,c,\bar c)}(p,m,g_3,\alpha;\mu)$$

Now suppose we change $\mu \to \mu'$ as the normalization pt. and $m \to m'$, $g_3 \to g_3'$, $\alpha \to \alpha'$ at this norm. pt. The effects of such a change can be absorbed by a wave function re-scaling

$$\phi'(x,m',g_3',\alpha';\mu') = Z^{-1/2}(\mu',m,g_3,\alpha;\mu)\,\phi(x,m,g_3,\alpha;\mu)$$

hence

$$\Gamma^{(m,n,l,c,\bar c)}(p,m',g_3',\alpha';\mu') = Z_2^{\frac{m+n}{2}} Z_3^{l/2} Z_c^{\frac{c+\bar c}{2}} \Gamma^{(m,n,l,c,\bar c)}(p,m,g_3,\alpha;\mu)$$

[This applies to the bare $\to$ renormalized case as well

$$\Gamma^{(m,n,l,c,\bar c)}(p,m,g_3,\alpha;\mu) = Z_2^{\frac{m+n}{2}} Z_3^{l/2} Z_c^{c+\bar c/2} \Gamma^{(m,l,l,c,\bar c)}(p,m_0,g_3^0,\alpha_0;\underset{\text{cutoff}}{\Lambda})$$

$$Z_i = Z_i(\mu,m_0,g_3^0,\alpha_0;\Lambda)$$
]

I.B.) Now suppose $\mu' = \mu(1+\epsilon)$ with $\epsilon \ll 1$, small. For $\mu \to \mu'$ we can change $m, g_3, \alpha \to m', g_3', \alpha'$ and re-scale the fields with Z to yield the same Greens functions. So let

$$g_3' = g_3 + \epsilon\beta \quad ; \quad \alpha' = \alpha + \epsilon\beta_\alpha$$

$$m'_{(m)} = m_{(m)}(1 + \epsilon\gamma_m^{(m)})$$

$$\varphi' = (1 - \epsilon\gamma_\varphi)\varphi \text{ . that is } Z_\varphi = 1 + 2\epsilon\gamma_\varphi$$

$\Rightarrow$

$$\Gamma^{(m,n,\ell,c,\bar{c})}(p, m_{(m)}(1+\epsilon\gamma_m^{(m)}), g_3 + \epsilon\beta, \alpha + \epsilon\beta_\alpha; \mu(1+\epsilon))$$

$$= (1 + 2\epsilon\gamma_\varphi)^{\frac{m+n}{2}}(1 + 2\epsilon\gamma_G)^{\ell/2}(1 + 2\epsilon\gamma_c)^{c+\bar{c}/2} \times$$

$$\times \Gamma^{(m,n,\ell,c,\bar{c})}(p, m_{(m)}, g_3, \alpha; \mu)$$

$$\Rightarrow \left[1 + \epsilon\mu\frac{\partial}{\partial\mu} + \epsilon\beta\frac{\partial}{\partial g_3} + \epsilon\sum_m \gamma_m^{(m)} m_{(m)}\frac{\partial}{\partial m_{(m)}} + \epsilon\beta_\alpha\frac{\partial}{\partial\alpha}\right] \times$$

$$\times \Gamma^{(m,n,\ell,c,\bar{c})}(p, m_{(m)}, g_3, \alpha; \mu)$$

$$\doteq \left[1 + (m+n)\epsilon\gamma_\varphi + \epsilon\ell\gamma_G + \epsilon(c+\bar{c})\gamma_c\right]\Gamma^{(m,n,\ell,c,\bar{c})}(p, m_{(m)}, g_3, \alpha; \mu)$$

I.B.) Hence we find the Renormalization Group Eq.

$$\boxed{\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_3}+\sum_m \gamma_m^{(m)} m_{(m)}\frac{\partial}{\partial m_{(m)}}+\beta_\alpha\frac{\partial}{\partial\alpha} - (m+n)\gamma_q - l\gamma_G - (c+\bar{c})\gamma_c\right]\times \Gamma^{(m,n,l,c,\bar{c})}(p, m_{(m)}, g_3, \alpha; \mu) = 0}$$

Now note that

$$g_3' = g_3'(\mu', g_3, \alpha, m, \mu)$$
$$\alpha' = \alpha'(\quad " \quad)$$
$$m' = m'(\quad " \quad)$$

and $Z = Z(\mu', m, g_3, \alpha; \mu)$

Since g_3', α', Z are dimensionless we have

$$g_3' = g_3'(\mu'/\mu, g_3, \alpha, m/\mu)$$
$$\alpha' = \alpha'(\quad " \quad)$$
$$Z = Z(\mu'/\mu, m/\mu, g_3, \alpha).$$

So

$$\mu'\frac{dg_3'}{d\mu'}\bigg|_{\mu'\to\mu} = \boxed{\mu\frac{d}{d\mu}g_3 \equiv \beta(g_3, \alpha, \tfrac{m}{\mu})}$$

$$= \left[\frac{\partial g_3'}{\partial x}(x, g_3, \alpha, \tfrac{m}{\mu})\right]\bigg|_{x=1} \qquad (x = \mu'/\mu)$$

I.B.) For $\mu \gg m$, or in a scheme that is mass indep.
we can ignore m/μ $\beta(g_3, \alpha, \frac{m}{\mu}) \approx \beta(g_3, \alpha, 0) = \beta(g_3, \alpha)$

Similarly

$$\gamma_m^{(m)} = \mu' \frac{d}{d\mu'} \ln m'_{(m)} \Big|_{\mu' \to \mu} = \left[\mu' \frac{\partial m'_{(m)}(\mu', g_3, \alpha, m, \mu)}{\partial \mu'} \right] \Big|_{\mu' = \mu}$$

$$\beta_\alpha = \mu' \frac{d}{d\mu'} \alpha' \Big|_{\mu' \to \mu} = \left[\frac{\partial}{\partial x} \alpha'(x, g_3, \alpha; m/\mu) \right] \Big|_{x=1}$$

$$\gamma_q = \frac{1}{2} \mu' \frac{\partial \ln Z_2}{\partial \mu'} \Big|_{\mu' \to \mu} = \frac{1}{2} \left[\frac{\partial \ln Z_2(x, g_3, \alpha; m/\mu)}{\partial x} \right] \Big|_{x=1}$$

$$\gamma_G = \frac{1}{2} \mu' \frac{\partial \ln Z_3}{\partial \mu'} \Big|_{\mu' \to \mu} = \frac{1}{2} \left[\frac{\partial \ln Z_3(x, g_3, \alpha; m/\mu)}{\partial x} \right] \Big|_{x=1}$$

$$\gamma_c = \frac{1}{2} \mu' \frac{\partial \ln Z_c}{\partial \mu'} \Big|_{\mu' \to \mu} = \frac{1}{2} \left[\frac{\partial \ln Z_c(x, g_3, \alpha; m/\mu)}{\partial x} \right] \Big|_{x=1}$$

~~We~~ This is just what we found above

$$g_3' = g_3 + \epsilon \beta \Rightarrow \beta = \frac{g_3' - g_3}{\epsilon} = \mu' \frac{g_3' - g_3}{\mu' - \mu}$$

$$= \mu' \frac{dg_3'}{d\mu'} \Big|_{\mu' \to \mu} = \mu \frac{dg_3}{d\mu}.$$

I.B.) Essentially we are differentiating wrt μ' and setting $\mu' = 0$

$$\mu' \frac{d}{d\mu'} \left[Z_2^{-\frac{m+n}{2}} Z_3^{-\ell/2} Z_c^{-\frac{c+\bar{c}}{2}} \Gamma^{(m,n,\ell,c,\bar{c})}(p, m', g_3', \alpha'; \mu') \right]$$

$$= \mu' \frac{d}{d\mu'} \Gamma^{(m,n,\ell,c,\bar{c})}(p, m, g_3, \alpha; \mu) = 0$$

Equivalently we can determine β, γ, etc by applying the RGE to the normalization conditions. Recall the inverse 2-pt function ($\Gamma^{(2)} G^{(2)} \equiv -1$)

$\tilde{S}'^{-1ab}_{F\,mn}(p) = -\Gamma^{ab(1,1,0,0,0)}_{mn}(p)$; So the normalization condition that determines Z_2 is

$$\frac{\partial}{\partial \not{p}} \tilde{S}'^{-1}_F \Big|_{\substack{\not{p}=\mu \\ (p^2=-\mu^2)}} \equiv i \Rightarrow \boxed{\frac{\partial}{\partial \not{p}} \Gamma^{ab(1,1,0,0,0)}_{mn} \Big|_{\not{p}=\mu} \equiv i\delta^{ab}\delta_{mn}}$$

Now let the RGE act on $\Gamma^{(1,1,0,0,0)} \Rightarrow$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_3} + \gamma_m m\frac{\partial}{\partial m} + \beta_\alpha\frac{\partial}{\partial\alpha} - 2\gamma_q\right)\Gamma^{(1,1,0,0,0)} = 0$$

Now set $\not{p} = \mu$ an operation which commutes with $\frac{\partial}{\partial g_3}, \frac{\partial}{\partial m}, \frac{\partial}{\partial\alpha}$ so that ex. $\frac{\partial}{\partial g_3}\Gamma^{(1,1,0,0,0)}\Big|_{\not{p}=\mu} = 0$

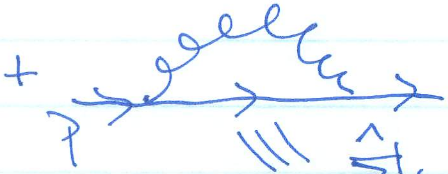
$\Rightarrow$

I.B.) $\Rightarrow$

$$\boxed{\left(\frac{\partial}{\partial \not p}\,\mu\frac{\partial}{\partial\mu}\Gamma^{ab\,(1,1,0,0,0)}_{mn}\right)\Bigg|_{\not p=\mu} - 2\gamma_q\, i\delta^{ab}\delta_{mn} = 0}$$

Now to one-loop order

$$\Gamma^{ab\,(1,1,0,0,0)}_{mn}(p) = iZ_2\not p\,\delta^{ab}\delta_{mn} - i(m_{(m)}+\delta m_{(m)})\delta_{mn}\delta^{ab}$$

$+$ 

$$\equiv -i\hat{\Sigma}_1(p)\delta_{mn}\delta^{ab}$$

(Recall Z_i counter-terms are from bare $\to$ renormalization conditions Not Z_i's of $\mu\to\mu'$) ↑ maybe should call these z_i

So

$$\frac{\partial}{\partial\not p}\Gamma^{ab\,(1,1,0,0,0)}_{mn}(p) = iZ_2\delta^{ab}\delta_{mn} - i\frac{\partial\hat{\Sigma}_1}{\partial\not p}\delta_{mn}\delta^{ab}$$

$\Rightarrow$

$$2\gamma_q\, i\delta^{ab}\delta_{mn} = \left(\mu\frac{\partial}{\partial\mu}\left[Z_2 - \frac{\partial\hat{\Sigma}_1}{\partial\not p}\right]\right)\Bigg|_{\not p=\mu} i\delta_{mn}\delta^{ab}$$

$\Rightarrow$

$$\boxed{2\gamma_q = \left(\mu\frac{\partial}{\partial\mu}\left[Z_2 - \frac{\partial\hat{\Sigma}_1}{\partial\not p}\right]\right)\Bigg|_{\not p=\mu}}$$

and

$$\frac{\partial}{\partial\not p}\Gamma^{ab\,(1,1,0,0,0)}_{mn}(p)\Bigg|_{\not p=\mu} \equiv i\delta^{ab}\delta_{mn} = iZ_2\delta^{ab}\delta_{mn} - i\frac{\partial\hat{\Sigma}_1}{\partial\not p}\Bigg|_{\not p=\mu}\delta_{mn}\delta^{ab}$$

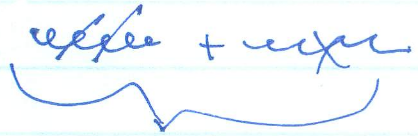
$$\Rightarrow \boxed{(Z_2-1) = \frac{\partial\hat{\Sigma}_1}{\partial\not p}\Bigg|_{\not p=\mu}}$$

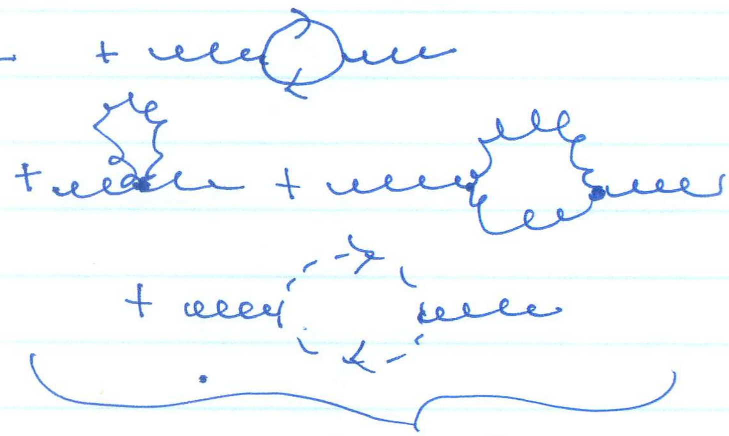
So

$$2\gamma_q = -\left(\mu\frac{\partial}{\partial\mu}\left[\frac{\partial\hat{\Sigma}_1}{\partial\not p}(p) - \frac{\partial\hat{\Sigma}_1}{\partial\not p}(\not p=\mu)\right]\right)\Bigg|_{\not p=\mu}$$

I.B.| Likewise

$$\Gamma^{(0,0,2,0,0)}_{\mu\nu}{}^{ij}(p) \equiv \Gamma_T(p^2)P^T_{\mu\nu}(p)\delta^{ij} + \Gamma_L(p^2)P^L_{\mu\nu}(p)\delta^{ij}$$

$=$ 



$$= -iZ_3 p^2 P^T_{\mu\nu}\delta^{ij} - i\frac{Z_\alpha}{\alpha}p^2 P^L_{\mu\nu}\delta^{ij}$$

$$\equiv \hat{\Pi}_{\mu\nu}(p)\delta^{ij}$$

$$\hat{\Pi}_T P^T_{\mu\nu} + \hat{\Pi}_L P^L_{\mu\nu}$$

$\Rightarrow$

$$P_T^{\mu\nu}(p)\,\Gamma^{(2)}_{\mu\nu}{}^{ij}(p) = 3\,\Gamma_T(p^2)\delta^{ij}$$

$$= 3\left(-iZ_3p^2 + \hat{\Pi}_T(p^2)\right)\delta^{ij}$$

$$\Rightarrow \boxed{\Gamma_T(p^2) = -iZ_3p^2 + \hat{\Pi}_T(p^2)}$$

Normalization: $\boxed{\left.\frac{\partial}{\partial p^2}\Gamma_T\right|_{p^2=\mu^2} \equiv -i}$

$$= -i\left[Z_3 + i\left.\frac{\partial\hat{\Pi}_T}{\partial p^2}(p^2)\right|_{p^2=\mu^2}\right]$$

$$\Rightarrow \boxed{Z_3 = 1 - i\frac{\partial}{\partial p^2}\hat{\Pi}_T(p^2)\Big|_{p^2=\mu^2}}$$

I.B.) Now apply RGE to $\Gamma_{\mu\nu}^{(0,0,2,0,0)}$ $\Rightarrow$

$$\left[\mu\partial_\mu + \beta\partial_{g_3} + \gamma_{m}m\partial_m + \beta_\alpha\partial_\alpha - 2\gamma_G\right]\Gamma_{\mu\nu}^{(0,0,2,0,0)}(p) = 0$$

apply $P_T^{\mu\nu}(p)$ $\Rightarrow$

$$\left(\mu\partial_\mu + \beta\partial_{g_3} + \cdots - 2\gamma_G\right)\Gamma_T(p^2) = 0$$

and normalization conditions:

$$\left(\mu\frac{\partial}{\partial\mu}\frac{\partial}{\partial p^2}\Gamma_T(p^2)\right)\Bigg|_{-p^2=\mu^2} - 2\gamma_G(-i) = 0$$

$\Rightarrow$

$$\boxed{2\gamma_G = +i\left(\mu\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial p^2}\Gamma_T(p^2)\right)\right)\Bigg|_{-p^2=\mu^2}}$$

$$= i\left(\mu\frac{\partial}{\partial\mu}\left[-iZ_3 + \frac{\partial\hat{\Pi}_T}{\partial p^2}(p^2)\right]\right)\Bigg|_{-p^2=\mu^2}$$

$$= i\left(\mu\frac{\partial}{\partial\mu}\left[\frac{\partial\hat{\Pi}_T}{\partial p^2}(p^2) - \frac{\partial}{\partial p^2}\hat{\Pi}(p^2)\Big|_{p^2=\mu^2}\right]\right)\Bigg|_{-p^2=\mu^2}$$

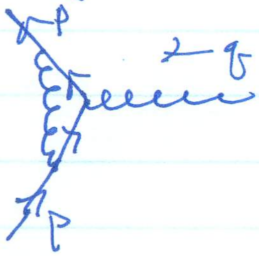
$$\langle 0| T\, \tilde{q}^a_m(p)\, \tilde{\bar{q}}^b_n(\bar{p})\, G^\mu_i(0)\,|0\rangle^{1PI} = \Gamma^{ab\,(1,1,1,0,0)\,\mu}_{mn\quad\ i}(p,\bar{p},q=\bar{p}-p)$$

$-ig=$

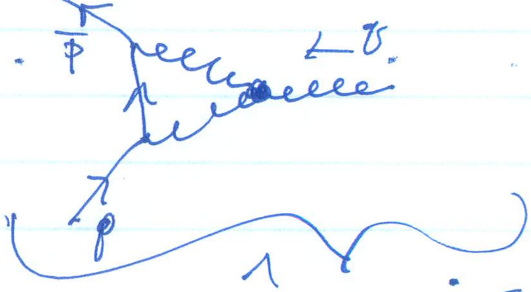
I.B.) and finally consider the quark-gluon vertex

$$\Gamma^{ab\,(1,1,1,0,0)\,\mu}_{mn\quad\ i}(p,\bar{p},q) =$$

$+$



$$= i Z_1^F g_3 T^i_{ba}\delta_{mn}\gamma^\mu +$$



$$\equiv i\hat{\Gamma}(p,\bar{p},q)\, g_3 T^i_{ba}\delta_{mn}\gamma^\mu$$

$$= i\Big[\overbrace{Z_1^F g_3 + g_3\hat{\Gamma}(p,\bar{p},q)}^{\equiv \Gamma_{g_3}}\Big] T^i_{ba}\delta_{mn}\gamma^\mu$$

Using the normalization condition $\boxed{\Gamma^{ab\,(1,1,1,0,0)\,\mu}_{mn\quad\ i}(p,\bar{p},q)\Big|_{NP} \equiv i g_3 T^i_{ba}\delta_{mn}\gamma^\mu}$

$$\Rightarrow \boxed{Z_1^F = 1 - \hat{\Gamma}(p,\bar{p},q)\Big|_{NP}}$$

applying the normal

I.B.) Finally consider the quark-gluon vertex RGE

$$\left[\mu\partial_\mu + \beta\partial_{g_3} \cdots - 2\gamma_q - \gamma_G\right]\Gamma^{(1,1,1,0,0)}_{(p,\bar{p};q)} = 0$$

apply the momentum point (note $q^\mu = \bar{p}^\mu - p^\mu$) so $q^2 = 0$

$-p^2 = \mu^2$
$-\bar{p}^2 = \mu^2$
$-p\cdot\bar{p} = \mu^2$
$p\cdot q = 0$
$\bar{p}\cdot q = 0$

$$\left(\mu\frac{\partial}{\partial\mu}\Gamma\right)\Big| + \beta - 2\gamma_q g_3 - \gamma_G g_3 = 0$$

$\Rightarrow$

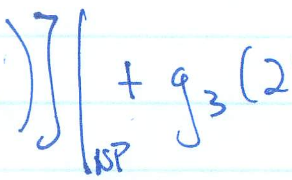
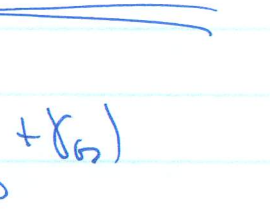
$$\boxed{\beta = g_3\left[-\mu\frac{\partial}{\partial\mu}\Gamma(p,\bar{p},q)\right]\Big| + (2\gamma_q + \gamma_G)g_3}$$

$$\beta = g_3\left[-\mu\frac{\partial}{\partial\mu}\left(Z_1^F + \hat{\Gamma}(p,\bar{p},q)\right)\right]\Big|_{NP} + g_3(2\gamma_q + \gamma_G)$$

$$\boxed{\beta = g_3\left\{-\mu\frac{\partial}{\partial\mu}\left(\hat{\Gamma}(p,\bar{p},q) - \hat{\Gamma}\Big|_{NP}\right) + 2\gamma_q + \gamma_G\right\}}\Bigg|_{NP}$$

Now let's recall what is happening —

for example
in 1-loop order

$$\hat{\Gamma}(p,\bar{p},q) \sim$$  $+$ 

There is no explicit μ dependence at this point, so $\frac{\partial}{\partial\mu}\hat{\Gamma}(p,\bar{p},q) = 0$

I.B) So in fact

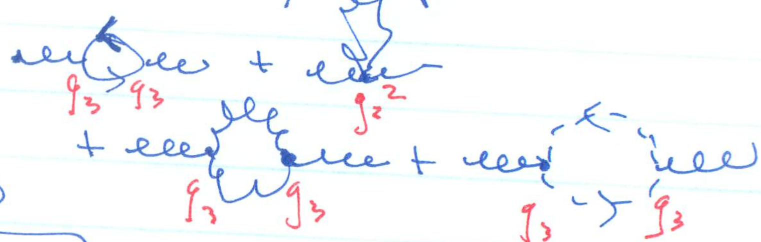
$$\boxed{\beta = g_3\, \mu \frac{\partial}{\partial \mu}\left[-Z_1^F\right] + g_3\left(2\gamma_q + \gamma_G\right)}$$ 1-loop order

Now in similar manner

$$2\gamma_G = i\left[\mu \frac{\partial}{\partial \mu}\left[-iZ_3 + \frac{\partial \hat{\Pi}_T}{\partial p^2}(p^2)\right]\right]\Big|_{p^2=\mu^2}$$

also 1-loop: $\hat{\Pi}_T(p^2)$ has no explicit μ dependence in

$\hat{\Pi}_T(p^2) \sim$



$$\frac{\partial}{\partial \mu}\hat{\Pi}_T(p^2) = 0$$

So

$$\boxed{2\gamma_G = \mu \frac{\partial}{\partial \mu} Z_3}$$ 1-loop order

likewise $\frac{\partial \hat{\Sigma}_i}{\partial \not p}(p)$ is indep. of explicit μ so $\frac{\partial}{\partial \mu}\hat{\Sigma}_i(p) = 0$ in 1-loop order

$\Rightarrow$

$$\boxed{2\gamma_q = \mu \frac{\partial}{\partial \mu} Z_2}$$ 1-loop order

Hence we finally obtain

$$\boxed{\boxed{\beta = g_3\, \mu \frac{\partial}{\partial \mu}\left[-Z_1^F + Z_2 + \tfrac{1}{2} Z_3\right]}}$$ in 1-loop order

IB.) So we need only to calculate the counter-terms.

Recall $g_3 = Z_1^{F-1} \cdot Z_2 Z_3^{1/2} g_3^0$ (this is the case of $\phi = Z^{1/2}\phi_0$ Bare theory)

$$\boxed{\beta = \mu \frac{\partial}{\partial\mu} g_3 = g_3\left[-\mu \frac{\partial \ln Z_1^F}{\partial\mu} + \mu \frac{\partial \ln Z_2}{\partial\mu} + \frac{1}{2}\mu \frac{\partial \ln Z_3}{\partial\mu}\right] = g_3\left[2\gamma_q + \gamma_G - \mu\frac{\partial}{\partial\mu}\ln Z_1^F\right]}$$

where we relate to bare theory

above is the $\mu \to \mu'$ theory.

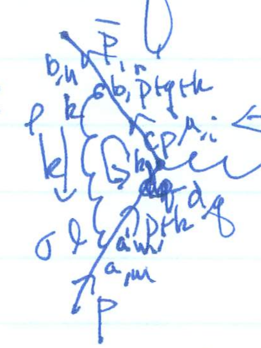
So now to calculate the counter-terms:

Now the Z's are divergent — so we only need these parts — we might as well put $m_{cur} = 0$ ($\mu \gg m$) in the quark masses and choose a convenient gauge — say Feynman gauge $\alpha = 1$.

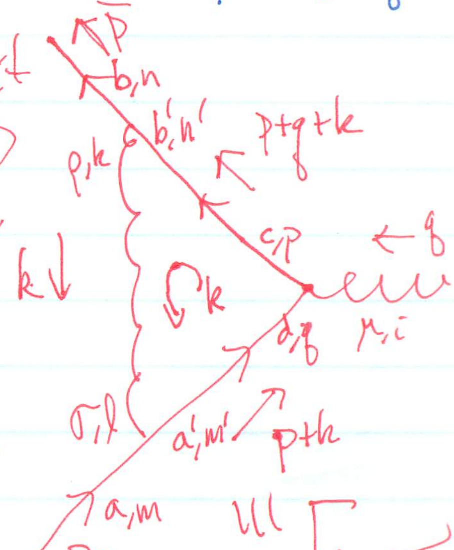
(all can be shown rigorously)

Consider the vertex first: E-Minkowski. $\bar{p}^\mu = p^\mu + q^\mu$

$\hat{\Gamma}^{ab\,\mu}_{mn\;i}(p,\bar{p},q) =$



blow it up bigger



Γ_1

Feynman diagram

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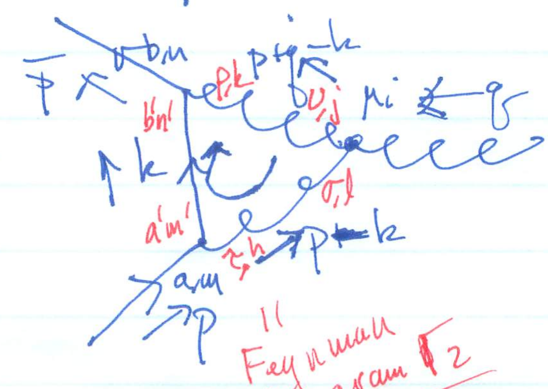

Γ_2

Feynman diagram Γ_2