I.B.) Quantum Chromodynamics – QCD

QCD is based on the gauge group SU(3) with gauge fields (Y-M) $G^i_\mu(x)$; $i = 1, 2, \ldots, 8$
$= \dim SU(3)$

and Dirac fermion quark fields carrying the color charge

$$q^a_m(x) \text{ where}$$

$m = 1, 2, \ldots, 6 = u, d, c, s, t, b$ lists the flavor of quark and $a = 1, 2, 3 =$ color.
(We will often suppress the flavor index m in this section.)

SU(3) is the group of 3×3 unitary matrices with determinant = 1. There are $\dim SU(3) = 3^2 - 1 = 8$ generators or charges

$$[T^i, T^j] = i f_{ijk} T^k \qquad i, j, k = 1, 2, \ldots, 8$$

f_{ijk} are completely anti-symmetric and

$$f_{123} = +1$$
$$f_{147} = f_{246} = f_{257} = f_{345} = +\tfrac{1}{2}$$
$$f_{156} = f_{367} = -\tfrac{1}{2}$$
$$f_{458} = \tfrac{1}{2}\sqrt{3} = f_{678}$$

all others (not related by a permutation) are zero.

I.B.) SU(3) has an infinite number of irreducible representations labelled by their dimension $d = 1, 3, \underset{\bar{3}}{\underset{\shortparallel}{3^*}}, 6, \underset{\bar{6}}{\underset{\shortparallel}{6^*}}, 8, 10, \underset{\bar{10}}{\underset{\shortparallel}{10^*}}, \ldots$

Quarks are in 3 rep. (fundamental rep)
antiQuarks " " 3^* rep. $= \bar{3}$ (anti " ")
Gluons " " 8 rep. (adjoint rep.)

1) 3 is given by the 8 Gell-Mann λ^i matrices (3×3) (analogous to Pauli matrices of SU(2))

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{bmatrix}; \quad \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix}; \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then the generators in the 3 are

$$(T^i)_{\substack{\cancel{\alpha\beta} \\ ab}} = \frac{1}{2} (\lambda^i)_{\substack{\cancel{\alpha\beta} \\ ab}}$$

and $[T^i, T^j]_{\substack{\cancel{\alpha\beta} \\ ab}} = i f_{ijk} T^k_{\substack{\cancel{\alpha\beta} \\ ab}}$ as can be checked

I.B.)1) That is $$\boxed{[\lambda^i, \lambda^j] = 2i f_{ijk} \lambda^k}$$

Properties & identities

1) Completeness:

$$\lambda^i_{\underset{ab}{\cancel{\alpha\beta}}} \lambda^i_{\underset{cd}{\cancel{\gamma\delta}}} = -\frac{2}{3} \delta_{\underset{ab}{\cancel{\alpha\beta}}} \delta_{\underset{cd}{\cancel{\gamma\delta}}} + 2 \delta_{\underset{ad}{\cancel{\alpha\delta}}} \delta_{\underset{bc}{\cancel{\beta\gamma}}}$$

2) $\{\lambda^i, \lambda^j\} = \frac{4}{3} \delta_{ij} \mathbb{1} + 2 d_{ijk} \lambda k$

where d_{ijk} are completely symmetric

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}$$

$$d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = +\frac{1}{2}$$

$$d_{247} = d_{366} = d_{377} = -\frac{1}{2}$$

$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}$$

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3) $\mathrm{Tr}[\lambda^i] = 0$

4) $\mathrm{Tr}[\lambda^i \lambda^j] = 2\delta_{ij}$

I.B.) 5) Casimir invariants for representation d

a) Quadratic Casimir $C_2(d)$

$$C_2(d)\,\mathbb{1} \equiv \sum_{i=1}^{8} T^i(d)\,T^i(d)$$

b) Third order Casimir $C_3(d)$

$$C_3(d)\,\mathbb{1} \equiv \sum_{i,j,k=1}^{8} d_{ijk}\,T^i(d)\,T^j(d)\,T^k(d)$$

Quark and antiquark are in smallest irred. rep. (fundamental & anti fundamental) Build up higher dim. rep. by products of 3 & 3^* of quarks & anti-quarks. So each rep. d corresponds to a pair of numbers p & q : (p,q) with p = factors of quarks, q = factors of anti-quark

$1 \sim (0,0)$; $3 \sim (1,0)$, $3^* \sim (0,1)$; $8 \sim (1,1)$

$10 \sim (3,0)$ The dimension of rep. (p,q) is

$$\dim(p,q) = \tfrac{1}{2}(p+1)(q+1)(p+q+2)$$

&

$$C_2(p,q) = \tfrac{1}{3}(3p+3q+p^2+pq+q^2)$$

$$C_3(p,q) = \tfrac{1}{18}(p-q)(2p+q+3)(2q+p+3)$$

$$\Rightarrow \quad C_2(3) = \tfrac{4}{3} = C_2(3^*)$$

I.B.5.) Hence

$$\lambda^i_{ab}\,\lambda^i_{bc} = \frac{16}{3}\,\delta_{ac} = 4\,C_2(3)\,\delta_{ac}$$

Quarks transform as 3

$$\boxed{q'^{a}_{m}(x) = U_{ab}(\omega)\, q^{b}_{m}(x)}$$

for each flavor $= u, d, c, s, t, b$.

&

$$U_{ab}(\omega) = \left(e^{+i g_3 \omega^i T^i}\right)_{ab} = \left(e^{+\frac{i}{2} g_3 \omega^i \lambda^i}\right)_{ab}$$

So for infinitesimal $\omega^i \Rightarrow$

$$\boxed{q'^{a}_{m}(x) = q^{a}_{m}(x) + \frac{i}{2} g_3\, \omega^i \lambda^i_{ab}\, q^{b}_{m}(x)}$$

Adjoint transforms as a 3^*

$$\bar{q}^{a}_{m}(x) \equiv q^{\dagger a}_{m}(x)\,\gamma^0$$

$$\boxed{\begin{aligned}\bar{q}'^{a}_{m}(x) &= U^*_{ab}(\omega)\,\bar{q}^{b}_{m}(x) = \bar{q}^{b}_{m}(x)\,U^{\dagger}_{ba}(\omega)\\ &= \bar{q}^{b}_{m}(x)\,U^{-1}_{ba}(\omega)\end{aligned}}$$

I.B.) $\Rightarrow$ $$\boxed{\bar{q}'^{a}_{m}(x) = \bar{q}^{a}_{m}(x) - \frac{i}{2} g_3 \bar{q}^{b}_{m}(x) \lambda^{i}_{ba} \omega^{i}(x)}.$$

Now for a global SU(3) symmetry $\omega^i = \text{const.}$ we have that

$$\left(\partial^\mu q^a_m(x)\right)' = \underbrace{U_{ab}(\omega)}_{\text{space-time const.}} \left(\partial^\mu q^b_m(x)\right)$$

it also is a 3.

Hence we can make an SU(3) singlet by contracting a 3 with a 3* so the globally SU(3) invariant $\mathcal{L}$ is

$$\mathcal{L}(q, \partial_\mu q) = \bar{q}^a_m i \overrightarrow{\not\partial} q^a_m - \sum_{m=1}^{b} \bar{q}^a_m m_{(m)} q^a_m$$

$\mathcal{L}' = \mathcal{L}$ with $m_{(m)} = (m_u, m_d, m_c, \ldots, m_b)$ the mass terms for each flavor.

(Recall the most general $\mathcal{L}_{gen} = \bar{q}^a_m [Z_{mn} i\not\partial - M_{mn}] q^a_n$

with Z_{mn}, M_{mn} hermitean flavor matrices

Let $q'^a_m = A_{mn} q^a_n$ with $AZA^\dagger = Z_{diagonal}$

$q''^a_m = Z^{1/2}_{diagonal\, mn} q'^a_n$

I.B.) So that

$$\mathcal{L}_{gen} = \bar{q}''^{a}_{m}\left[\delta_{mn} i\not{\partial} - \underbrace{\left(Z^{-1/2}_{diag.} A m A^{\dagger} Z^{-1/2}_{diag}\right)_{mn}}_{= \mathcal{M}_{mn} \text{ hermitean}}\right] q''^{a}_{n}$$

Let

$$q'''^{a}_{m} = B_{mn} q''^{a}_{n} \quad \text{with } B\mathcal{M}B^{\dagger} = \mathcal{M}_{diag.}$$

$$\Rightarrow \quad \mathcal{L}_{gen} = \bar{q}'''^{a}_{m}\left[i\not{\partial}\delta_{mn} \; \mathcal{M}_{diag.\,mn}\right] q'''^{a}_{n}$$

which is just our globally invariant $\mathcal{L}$)

To find the locally invariant $\mathcal{L}$ we introduce the covariant derivative of q^{a}

$$q'^{a}_{m}(x) = q^{a}_{m}(x) + \frac{i}{2} g_3 \omega^{i}(x) \lambda^{i}_{ab} q^{b}_{m}(x)$$

So

$$D^{ab}_{\mu} q^{b}_{m}(x) = \left[\partial_\mu \delta^{ab} - \frac{i}{2} g_3 \; G^{i}_{\mu}(x) \lambda^{i}_{ab}\right] q^{b}_{m}(x)$$

and

$$\left(D_\mu q_m\right)^{a\prime} = U_{ab}(\omega) \left(D_\mu q_m\right)^{b} \quad \text{still a } \underline{3}$$

So

$$\mathcal{L}_{matter} = \bar{q}^{a}_{m} i \not{D}^{ab} q^{b}_{m} - \sum_{m}{}' \bar{q}^{a}_{m} m_{(m)} q^{a}_{m}$$

$$\mathcal{L}'_{matter} = \mathcal{L}_{matter}$$

I.B.) The matrix notation $G^{\mu}_{ab} \equiv i \frac{\lambda^{i}_{ab}}{2} G^{i}_{\mu}$ let's us write

$$(D_\mu q_m)^a = \left([\partial_\mu - g_3 G_\mu] q_m \right)^a$$

and

$$\mathcal{L}_{matter} = \bar{q}\, i(\not\partial - g_3 \not G) q - \bar{q} m q$$

suppressing all indices.

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* insert p. -22'-

Hence the covariant field strength tensor for the gluons becomes

$$G^{i}_{\mu\nu} = \partial_\mu G^{i}_{\nu} - \partial_\nu G^{i}_{\mu} + g_3 f_{ijk} G^{j}_{\mu} G^{k}_{\nu}$$

and we have the invariant Y-M & θ Lag.

$$\mathcal{L}_{YM} = -\frac{1}{4} G^{i}_{\mu\nu} G^{i\mu\nu} = -\frac{1}{2} \mathrm{Tr}[G_{\mu\nu} G^{\mu\nu}]$$

$$\mathcal{L}_{\theta} = \frac{g_3}{16\pi^2} \theta\, \mathrm{Tr}[G_{\mu\nu} \tilde{G}^{\mu\nu}]$$

So the QCD invariant Lag. is

$$\boxed{\mathcal{L}_{QCD} = \bar{q}[i\not\partial - i g_3 \not G - m] q - \frac{1}{2} \mathrm{Tr}[G_{\mu\nu} G^{\mu\nu}] + \frac{g_3}{16\pi^2} \theta\, \mathrm{Tr}[G_{\mu\nu} \tilde{G}^{\mu\nu}]}$$

I.B) 2) The adjoint representation is the 8 of $SU(3)$

$$(T^i)_{jk} \equiv i f_{jik} \qquad 8\times 8 \text{ matrices}$$

which obey $([T^i, T^j])_{lm} = i f_{ijk} (T^k)_{lm}$

Properties:

1) $C_2(8) = 3$

2) $f_{ikl} f_{jkl} = C_2(8)\delta_{ij} = 3\delta_{ij}$

3) $f_{ijk}\lambda^j\lambda^k = \frac{1}{2} f_{ijk}[\lambda^j, \lambda^k]$

$$= i f_{ijk} f_{jkl}\lambda^l = i C_2(8)\lambda^i$$

4) $\lambda^j\lambda^i\lambda^j = \frac{1}{2}\Big(\lambda^j[\lambda^i, \lambda^j] - [\lambda^i, \lambda^j]\lambda^j$

$$+ \lambda^j\lambda^j\lambda^i + \lambda^i\lambda^j\lambda^j\Big)$$

$$= 4 C_2(3)\lambda^i + i f_{ijk}[\lambda^j, \lambda^k]$$

$$= 4\left[C_2(3) - \tfrac{1}{2} C_2(8)\right]\lambda^i$$

I.B) 2) The gluons G^i_μ transform in the 8 the homogeneous part that is G^j_μ

$$G'^i_\mu = G^i_\mu + \partial_\mu \omega^i + g_3 f_{ijk} G^j_\mu \omega^k$$
$$= G^i_\mu + D^{ij}_\mu \omega^j$$

Introducing $G_{\mu\,ab} \equiv i T^i_{ab} G^i_\mu$

$$G'_\mu = U G_\mu U^{-1} + \frac{1}{g_3}(\partial_\mu U\, U^{-1})$$
$$(= \mathcal{U}^{-1}(\omega)\, G_\mu\, \mathcal{U}(\omega) \quad \text{quantum field})$$

Now recall $\mathcal{U}(\omega) = e^{i g_3 \omega^i Q^i}$

So infinitesimal

$$G'_\mu = G_\mu - i g_3 [\omega^i Q^i, G_\mu]$$

$\Rightarrow$

$$\boxed{\begin{aligned} G'^i_\mu &= G^i_\mu - i g_3 [\omega^j Q^j, G^i_\mu] \equiv G^i_\mu + \delta_{SU(3)}(\omega) G^i_\mu \\ &= G^i_\mu + D^{ij}_\mu \omega^j \end{aligned}}$$

$\Rightarrow$

$$\boxed{\delta_{SU(3)}(\omega)\, G^i_\mu = D^{ij}_\mu \omega^j}$$

I.B.) Less cryptically

$$\boxed{\mathcal{L}_{QCD} = \sum_{m=1}^{6} \bar{q}^a_m (i \not{D}^{ab} - m_{(m)} \delta^{ab}) q^b_m - \frac{1}{4} G^i_{\mu\nu} G^{i\mu\nu} + \frac{g_3}{32\pi^2} \theta G^i_{\mu\nu} \tilde{G}^{i\mu\nu}}$$

The gauge fixing function will be chosen as the Stückelberg type

$$f_i = \frac{1}{\alpha} \partial^\mu G^i_\mu \qquad \alpha = \text{arb. Real parameter}$$

The gauge variation of f_i is

$$\delta_{SU(3)}(\omega) f_i = \frac{1}{\alpha} \partial^\mu \left[\delta_{SU(3)}(\omega) G^i_\mu \right]$$

recall

$$G'^i_\mu = G^i_\mu + \partial_\mu \omega^i + g_3 f_{ijk} G^j_\mu \omega^k$$

$$= G^i_\mu + D^{ij}_\mu \omega^j$$

$$D^{ij}_\mu = \partial_\mu \delta^{ij} - i g_3 T^k_{ij} G^k_\mu$$

adjoint rep.

$$T^k_{ij} = i f_{ikj}$$

I.B.) So $\left(\delta_{SU(3)}(\omega) \psi^\alpha = -i T^i_{\alpha\beta} \omega^i \psi^\beta \quad (\text{p. -4-}) \right)$

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$$\boxed{\begin{aligned} \delta_{SU(3)}(\omega) G^i_\mu &\equiv G'^i_\mu - G^i_\mu \\ &= (D_\mu \omega)^i \end{aligned}}$$

So
$$\delta_{SU(3)}(\omega) f_i = \frac{1}{\alpha} \partial^\mu \left(\partial_\mu \omega^i + g_3 f_{ikj} G^k_\mu \omega^j \right)$$

$$\Rightarrow M^{ij}_f(x,y) = \frac{\delta\, \delta_{SU(3)}(\omega) f_i(x)}{\delta \omega^j(y)}$$

$$= \frac{1}{\alpha} \partial^\mu_x \left[\partial^x_\mu \delta^{ij} + g_3 f_{ikj} G^k_\mu(x) \right] \delta^4(x-y)$$

So
$$\boxed{\mathcal{L}_f = -\frac{\alpha}{2} f^2 = -\frac{1}{2\alpha} \left(\partial^\mu G^i_\mu \right)^2}$$

$$\mathcal{L}_{\phi\text{-}\pi} = \int d^4y\, \bar{c}_i(x) M^{ij}_f(x,y)\, c_j(y)$$

$$= -\frac{1}{\alpha} \partial^\mu \bar{c}_i \partial_\mu c_i - \frac{1}{\alpha} \partial^\mu \bar{c}_i \left[g_3 f_{ikj} G^k_\mu c_j \right]$$

$$+ \underbrace{\frac{1}{\alpha} \partial^\mu \left[\bar{c}_i \partial_\mu c_i \right]}_{\text{drop total divergence}}$$

I.B) So

$$\mathcal{L}_{\phi\pi} = -\frac{1}{\alpha}\partial^\mu \bar{c}_i\left[\partial_\mu \delta^{ij} + g_3 f_{ikj} G^k_\mu\right] c_j$$

$$\boxed{\mathcal{L}_{\phi\pi} = -\frac{1}{\alpha}\partial^\mu \bar{c}_i D^{ij}_\mu c_j}$$

Thus the action for quantized QCD is

$$\mathcal{L} = \sum_{m=1}^{6} \bar{q}^a_m \left(i \not{D}^{ab} - m_{(m)}\delta^{ab}\right) q^b_m - \frac{1}{4} G^i_{\mu\nu} G^{i\mu\nu} + \frac{g_3}{32\pi^2}\theta G^i_{\mu\nu}\tilde{G}^{i\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu G^{i\mu}\right)^2 - \frac{1}{\alpha}\partial^\mu \bar{c}_i D^{ij}_\mu c_j$$

and

$$Z[\eta,\bar{\eta},J_\mu,\xi,\bar{\xi}] = \int [dq^a][d\bar{q}^a][dG^i_\mu][dc_i][d\bar{c}_i] \times e^{i\int d^4x \left[\mathcal{L} + \bar{\eta}^a_m q^a_m + \bar{q}^a_m \eta^a_m + J^i_\mu G^{i\mu} + \bar{\xi}_i c_i + \bar{c}_i \xi_i\right]}$$