Physics 658: Elementary Particle Physics

I) The 1,2,3's of 3,2,1.

$SU_C(3)$: color gauge symmetry group of QCD } Local symmetry groups
$SU_L(2) \times U_Y(1)$: electroweak gauge symmetry group }

Fundamental degrees of freedom of the SM consist of

A) Matter Fields: 1) Spin 1/2

Leptons: e^-, μ^-, τ^- } No strong
ν_e, ν_μ, ν_τ } interactions
(consider neutrinos massless at this point)

Quarks: q^a_m $a = 1,2,3$ color index involved in QCD interactions
$m = 1,2,\ldots,6$ flavor index involved in electroweak interactions
$(m = u, d, c, s, t, b)$

2) Spin 0:

Higgs Bosons: ϕ^+, ϕ^0 Responsible for Spontaneous symmetry Breaking of $SU_L(2) \times U_Y(1) \to U_{EM}(1)$

3 give rise to $W^\pm, Z^0$ masses (eaten)

I) 2) Higgs particle $= \eta = \mathrm{Re}\,\phi^0$ left as particle
($\equiv h$ or H)

B) Gauge Bosons: spin 1

Photon γ : mediates E-M

IVB $W^{\pm}, Z^0$: mediates charged and neutral current weak interactions

Gluons G^i : mediates strong interactions, $i = 1, 2, \ldots, 8$ 8 types of gluons

---

Properties:

1) Mass & electric charge

	Leptons			Quarks		
	Flavor	Mass	Charge	Flavor	Mass (current)	Charge
1st Family	ν_e	($\lesssim .1$ eV)	0	u	~~5.6~~ 1.5–3.0 MeV	$+\frac{2}{3}e$
	e	0.51 MeV	$-e$	d	~~9.9~~ 3–7 MeV	$-\frac{1}{3}e$
2nd Family	ν_μ	($\lesssim .1$ eV)	0	c	1.25 GeV	$+\frac{2}{3}e$
	μ	106 MeV	$-e$	s	~~199~~ 95 MeV	$-\frac{1}{3}e$
3rd Family (or generation)	ν_τ	($\lesssim .1$ eV)	0	t	174 GeV	$+\frac{2}{3}e$
	τ	1.8 GeV	$-e$	b	4.7 GeV	$-\frac{1}{3}e$

($m_\nu < 2$ eV)

Boson Masses:

Boson	Mass	
γ	$< 10^{-36}$ GeV	($m < 6 \times 10^{-17}$ eV)
$W^{\pm}$	80.4 GeV	
Z^0	91.19 GeV	
$\eta = H^0$	$>$	114.4 GeV
G	0	

I.A.) Review of Gauge Theories

Symmetry Group (Lie) G with $\dim G = N$ dim of group

$$U_{\alpha\beta}(\omega) = \left(e^{ig\,\omega^i(x)\,T^i}\right)_{\alpha\beta} \text{ with } U \,\&\, T^i \text{ matrices}$$

$U \in G$

& Lie Algebra $[T^i, T^j] = i f_{ijk} T^k$

$T^i_{\alpha\beta}$ irreducible representations of G, $\alpha, \beta = 1, 2, \ldots, d$

d = dim of representation

Variations: $\psi^\alpha(x)$ carries d-dimen. rep. of G

$$\psi'^\alpha(x) \equiv \mathcal{U}^{-1}(\omega)\,\psi^\alpha(x)\,\mathcal{U}(\omega) = e^{-i\omega^i Q^i}\,\psi^\alpha\, e^{+i\omega^i Q^i_g}$$

$$= U_{\alpha\beta}(\omega)\,\psi^\beta = \left(e^{ig\,\omega^i T^i}\right)_{\alpha\beta}\psi^\beta$$

infinitesimal $\omega^i \Rightarrow$

I.A)

$$\psi'^{\alpha}_{(x)} = \psi^{\alpha}_{(x)} - i\,\omega^{i}_{(x)}\,g\,[Q^{i}, \psi^{\alpha}_{(x)}] = \psi^{\alpha} + \delta_{Q}(\omega)\,\psi^{\alpha}$$

$$= \psi^{\alpha}_{(x)} + i g\,\omega^{i}_{(x)}\,T^{i}_{\alpha\beta}\,\psi^{\beta}_{(x)}$$

$$\Rightarrow \quad [Q^{i}, \psi^{\alpha}_{(x)}] = -\,T^{i}_{\alpha\beta}\,\psi^{\beta}_{(x)}$$

$$= -i\left(-i\,T^{i}_{\alpha\beta}\,\psi^{\beta}\right)$$

$$\equiv +\frac{i}{g}\,\delta^{i}_{Q}\,\psi^{\alpha}_{(x)}$$

okay Q

i.e.

$$\delta^{i}_{Q}\,\psi^{\alpha} = +i g\,T^{i}_{\alpha\beta}\,\psi^{\beta}$$

$$\Rightarrow \quad [Q^{i}, Q^{j}] = i\,f_{ijk}\,Q^{k}$$

&

$$[\delta^{i}_{Q}, \delta^{j}_{Q}] = -g\,f_{ijk}\,\delta^{k}_{Q} \quad \left(\text{i.e. } \delta^{i}_{Q} = -i\,T^{i}\right)$$

So

$$\psi'^{\alpha}_{(x)} = \mathcal{U}^{-1}_{(\omega)}\,\psi^{\alpha}\,\mathcal{U}_{(\omega)} = U_{\alpha\beta}(\omega)\,\psi^{\beta}_{(x)}$$

$$= \psi^{\alpha}_{(x)} + i g\,\omega^{i}_{(x)}\,T^{i}_{\alpha\beta}\,\psi^{\beta}_{(x)}$$

$$\equiv \psi^{\alpha}_{(x)} + \delta_{Q}(\omega)\,\psi^{\alpha}_{(x)}$$

Covariant Derivative:

$$\boxed{D^{\alpha\beta}_{\mu}\,\psi^{\beta}_{(x)} \equiv \partial_{\mu}\,\psi^{\alpha}_{(x)} - i g\,T^{i}_{\alpha\beta}\,A^{i}_{\mu(x)}\,\psi^{\beta}_{(x)}}$$

$A^{i}_{\mu}(x)$ = gauge field = connection = Y-M field

IA.) A^i_μ is such that $(D^{\alpha\beta}_\mu \psi^\beta)$ transforms as does ψ^α

$$\boxed{(D^{\alpha\beta}_\mu \psi^\beta)' = \partial_\mu \psi'^\alpha - ig T^i_{\alpha\beta} A'^i_\mu(x)\, \psi'^\beta(x) \equiv U_{\alpha\beta}(\omega)\,(D^{\beta\gamma}_\mu \psi^\gamma)}$$

But $\boxed{\psi'^\alpha = U_{\alpha\beta}\psi^\beta} \Rightarrow \partial_\mu \psi'^\alpha = U_{\alpha\beta}\partial_\mu \psi^\beta + (\partial_\mu U_{\alpha\beta})\psi^\beta$

$\Rightarrow$

$$(D^{\alpha\beta}_\mu \psi^\beta)' = U_{\alpha\beta}(\partial_\mu \psi^\beta - ig T^i_{\beta\gamma} A^i_\mu \psi^\gamma)$$
$$+ U_{\alpha\beta}\, ig T^i_{\beta\gamma} A^i_\mu \psi^\gamma$$
$$+ (\partial_\mu U_{\alpha\beta})\psi^\beta$$
$$- ig T^i_{\alpha\beta} A'^i_\mu U_{\beta\gamma}\psi^\gamma$$

$$= U_{\alpha\beta}(D^{\beta\gamma}_\mu \psi^\gamma)$$
$$+ ig\left[U_{\alpha\beta} T^i_{\beta\gamma} A^i_\mu - T^i_{\alpha\beta} U_{\beta\gamma} A'^i_\mu\right]\psi^\gamma$$
$$+ (\partial_\mu U_{\alpha\beta})\psi^\beta$$

I.A.) The last terms must vanish $\Rightarrow$

$$U_{\alpha\beta} T^i_{\beta\gamma} A^i_\mu - \frac{i}{g}\partial_\mu U_{\alpha\gamma} = T^i_{\alpha\beta} A'^i_\mu U_{\beta\gamma}$$

multiply by U^{-1} $\Rightarrow$

$$\boxed{T^i_{\alpha\beta} A'^i_\mu = U_{\alpha\gamma}(T^i_{\gamma\delta} A^i_\mu) U^{-1}_{\delta\beta} - \frac{i}{g}\partial_\mu U_{\alpha\gamma} U^{-1}_{\gamma\beta}}$$

Denote $A_{\mu\,\alpha\beta} \equiv i T^i_{\alpha\beta} A^i_\mu \qquad = i \vec{T}_{\alpha\beta}\cdot\vec{A}$ $\Rightarrow$

$$\boxed{A'_{\mu\alpha\beta} = (U A_\mu U^{-1})_{\alpha\beta} + \frac{1}{g}(\partial_\mu U U^{-1})_{\alpha\beta}}$$

$$\boxed{{}'' \; \mathcal{U}^{-1}_{(\omega)} A_\mu \mathcal{U}_{(\omega)}}$$

Infinitesimal ω^i $\Rightarrow$

$$T^i_{\alpha\beta} A'^i_\mu = \left(\left[(1+ig\omega^j T^j) T^i A^i_\mu - \frac{i}{g} ig\partial_\mu \omega^j T^j\right] \times \times (1 - ig\omega^k T^k)\right)_{\alpha\beta}$$

$$= T^i_{\alpha\beta} A^i_\mu + ig\omega^j (T^j T^i - T^i T^j)\cdot A^i_\mu + \partial_\mu \omega^i T^i_{\alpha\beta}$$

$$= T^i_{\alpha\beta} A^i_\mu + T^i_{\alpha\beta}\partial_\mu \omega^i + \underbrace{g f_{ijk}\, \omega^j T^k_{\alpha\beta} A^i_\mu}_{= f_{kji}\,\omega^j A^k_\mu T^i_{\alpha\beta}}$$

$$f_{kji} = -f_{ijk}$$

I.A.)

$$T^i_{\alpha\beta} A'^i_\mu = T^i_{\alpha\beta} A^i_\mu + T^i_{\alpha\beta}\left[\partial_\mu \omega^i - g\, f_{ijk}\, \omega^j A^k_\mu\right]$$

$\Rightarrow$

$$\boxed{A'^i_\mu = A^i_\mu + \partial_\mu \omega^i + g\, f_{ijk} A^j_\mu \omega^k}$$

As a nemonic recall the adjoint (regular, real) representation of the group G is given by the structure constants

$$T^i_{jk} \equiv i f_{jik} = (T^i)_{jk}$$

So we have the covariant derivative for the adjoint representation

$$D_\mu^{ik} \equiv \partial_\mu \delta^{ik} - i g\, T^j_{ik} A^j_\mu$$

So

$$\boxed{A'^i_\mu = A^i_\mu + D^{ik}_\mu \omega^k}$$

(Notation: Rep: R, r, d $\left(T^i(R)\right)_{\alpha\beta}$ use if needed to denote which matrix T^i is considered)

(Recall $[T^i, T^j]_{lm} = i f_{ijk} T^k_{lm}$

follows from Jacobi identity
$$f_{ijl} f_{kml} + f_{jkl} f_{iml} + f_{kil} f_{jml} = 0\,)$$

I.A.) Hence we have if

$\mathcal{L}_{matter}(\psi, \partial_\mu \psi)$ is <u>globally</u> <u>invariant</u> then

$\mathcal{L}_{matter}(\psi, D_\mu \psi)$ is <u>locally gauge</u> <u>invariant</u>.

In addition we have covariant derivatives that yield the Y-M field strength

$$D_\mu = \mathbb{1}\partial_\mu - g A_\mu \qquad (A_\mu = i T^i_{\alpha\beta} A^i_\mu)$$

$$\Rightarrow \quad [D_\mu, D_\nu] = [\partial_\mu - g A_\mu, \partial_\nu - g A_\nu]$$

$$= -g(\partial_\mu A_\nu - \partial_\nu A_\mu) + g^2 [A_\mu, A_\nu]$$

Now $\qquad \equiv -g\, i T^i F^i_{\mu\nu} = -g F_{\mu\nu}$

$$i T^i F^i_{\mu\nu} = i T^i (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) + g [T^i, T^j] A^i_\mu A^j_\nu$$

$$= i T^i (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) + i g f_{ijk} T^k A^i_\mu A^j_\nu$$

$$\Rightarrow$$

$$\boxed{F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f_{ijk} A^j_\mu A^k_\nu}$$

I.A.) Now check the gauge transformation of $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]$$

So

$$F'_{\mu\nu} = \mathcal{U}^{-1}(\omega)\, F_{\mu\nu}\, \mathcal{U}(\omega)$$

$$= \partial_\mu A'_\nu - \partial_\nu A'_\mu - g[A'_\mu, A'_\nu]$$

$$= \partial_\mu\left[U A_\nu U^{-1} + \frac{1}{g}\partial_\nu U U^{-1}\right]$$

$$- \partial_\nu\left[U A_\mu U^{-1} + \frac{1}{g}\partial_\mu U U^{-1}\right]$$

$$- g\left[U A_\mu U^{-1} + \frac{1}{g}\partial_\mu U U^{-1},\ U A_\nu U^{-1} + \frac{1}{g}\partial_\nu U U^{-1}\right]$$

$$= U(\partial_\mu A_\nu - \partial_\nu A_\mu)U^{-1} + \cancel{\partial_\mu U A_\nu U^{-1}} + \cancel{U A_\nu \partial_\mu U^{-1}}$$

$$+ \frac{1}{g}\partial_\mu\partial_\nu U U^{-1} + \frac{1}{g}\partial_\nu U \partial_\mu U^{-1} - \partial_\nu U A_\mu U^{-1} - \cancel{U A_\mu \partial_\nu U^{-1}}$$

$$- \frac{1}{g}\partial_\nu\partial_\mu U U^{-1} - \frac{1}{g}\partial_\mu U \partial_\nu U^{-1}$$

$$- g[U A_\mu U^{-1}, U A_\nu U^{-1}] - \cancel{\frac{g}{g} U A_\mu U^{-1} \partial_\nu U U^{-1}}$$

$$+ \frac{g}{g}\partial_\nu U U^{-1} U A_\mu U^{-1} - \cancel{\frac{g}{g}\partial_\mu U U^{-1} U A_\nu U^{-1}}$$

$$+ \cancel{\frac{g}{g} U A_\nu U^{-1} \partial_\mu U U^{-1}} - \frac{g}{g^2}\left[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}\right]$$

I.A.) Recall $U^{-1}U = 1 \Rightarrow \partial_\mu U^{-1} U = -U^{-1}\partial_\mu U$ has been used.

$$\boxed{F'_{\mu\nu} = U\left[\partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]\right]U^{-1} = U F_{\mu\nu} U^{-1}}$$

$\underline{F_{\mu\nu}}$ is in the $\underline{\text{adjoint}}$ (real) representation.

So infinitesimal ω^i

$$iT^i_{\alpha\beta}F'^i_{\mu\nu} = i(U T^j U^{-1})_{\alpha\beta} F^j_{\mu\nu}$$

$$= i\left([1 + ig\omega^k T^k] T^j [1 - ig\omega^k T^l]\right)_{\alpha\beta} F^j_{\mu\nu}$$

$$= iT^i_{\alpha\beta}F^i_{\mu\nu} + i(ig)\omega^k i f_{kjl} T^l_{\alpha\beta} F^j_{\mu\nu}$$

$$= iT^i_{\alpha\beta}F^i_{\mu\nu} - igT^i_{\alpha\beta} f_{kji}\omega^k F^j_{\mu\nu}$$

$\Rightarrow$

$$\boxed{F'^i_{\mu\nu} = F^i_{\mu\nu} - g f_{ijk}\omega^j F^k_{\mu\nu}}$$

$$= F^i_{\mu\nu} + ig\,\omega^j T_i{}^j{}_k F^k_{\mu\nu} = F^i_{\mu\nu} + \delta_Q(\omega) F^i_{\mu\nu}$$

Hence exploiting $Tr[T^i T^j] = \frac{1}{2}\delta^{ij}$

(use $Tr\{AB\} = Tr\{BA\}$)

$$\Rightarrow \boxed{\begin{array}{c} Tr[F'_{\mu\nu}F'^{\mu\nu}] = Tr[F_{\mu\nu}F^{\mu\nu}] \\ F'^i_{\mu\nu}F'^{i\mu\nu} = F^i_{\mu\nu}F^{i\mu\nu} \end{array}}$$

or

They are locally gauge invariant.

Note $F'^{i}_{\mu\nu} F'^{i\mu\nu} = F^{i}_{\mu\nu} F^{i\mu\nu} - 2g F^{i}_{\mu\nu} f_{ijk} \omega_j F^{k\mu\nu}$

but $F^{i}_{\mu\nu} F^{k\mu\nu}$ is (i,k) symmetric while f_{ijk} is (i,k) anti-symmetric — hence

$$F'^{i}_{\mu\nu} F'^{i\mu\nu} = F^{i}_{\mu\nu} F^{i\mu\nu} .$$

Thus we have a gauge invariant kinetic energy term for the Y-M field

$$\boxed{\mathcal{L}_{Y\text{-}M} = -\frac{1}{4} F^{i}_{\mu\nu} F^{i\mu\nu} = -\frac{1}{2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}$$

with $\mathcal{L}'_{Y\text{-}M} = \mathcal{L}_{Y\text{-}M}$.

In addition another gauge invariant term (but CP odd) is made with the dual $\tilde{F}^{i}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{i\rho\sigma}$

So $\left(F^{i}_{\mu\nu} \tilde{F}^{i\mu\nu}\right)' = F^{i}_{\mu\nu} \tilde{F}^{i\mu\nu}$

or $\left(\mathrm{Tr}[F_{\mu\nu} \tilde{F}^{\mu\nu}]\right)' = \mathrm{Tr}[F_{\mu\nu} \tilde{F}^{\mu\nu}]$

$$\boxed{\mathcal{L}_{\theta} \equiv \frac{g^2}{32\pi^2} \theta F^{i}_{\mu\nu} \tilde{F}^{i\mu\nu} = \frac{g^2}{16\pi^2} \theta \mathrm{Tr}[F_{\mu\nu} \tilde{F}^{\mu\nu}]}$$

with $\mathcal{L}'_{\theta} = \mathcal{L}_{\theta}$.

Now $F^{i}_{\mu\nu} \tilde{F}^{i\mu\nu} = \partial_\mu K^\mu$

$$K^\mu = \epsilon^{\mu\nu\rho\sigma} A^{i}_{\nu} \left[F^{i}_{\rho\sigma} - \frac{g}{6} f_{ijk} A^{j}_{\rho} A^{k}_{\sigma} \right]$$

I.A.) So

$$\boxed{\mathcal{L}_\theta = \frac{g^2}{32\pi^2}\theta\,\partial_\mu\left[\epsilon^{\mu\nu\rho\sigma}A^i_\nu\left(F^i_{\rho\sigma} - \frac{g}{6}f_{ijk}A^j_\rho A^k_\sigma\right)\right]}$$

It is possible that K^μ is a singular operator and its divergence cannot be thrown out in the action. This will generate CP violating terms in QCD given in terms of the parameter θ. Exp. $\Rightarrow$ $\theta < 10^{-10}$ very small (Strong CP problem)

So in general the locally G-invariant Lagrangian is given by

$$\boxed{\mathcal{L}_{inv} = \mathcal{L}_{matter} + \mathcal{L}_{Y\text{-}M} + \mathcal{L}_\theta}$$

& $\mathcal{L}'_{inv} = \mathcal{L}_{inv}$

In order to quantize a gauge must be chosen and a $\phi\Pi$-term added

We will use the gauge fixing function ($\alpha \in \mathbb{R}_{\alpha \geq 0}$)

$$f_i(A,\psi) = \frac{1}{\alpha}\partial_\mu A^{i\mu} - a^i(\psi, A)$$

I.A.) for example $f_i = \frac{1}{\alpha}\partial_\mu A^{i\mu} + \xi\, v^{\alpha} (T^i)_{\alpha\beta}\, \phi^\beta$ in the case of a Higgs-Kibble type model.

So the gauge fixing Lagrangian will be given by

$$\mathcal{L}_f = -\frac{\alpha}{2} f_i(A,\psi) f_i(A,\psi)$$

The gauge variation of the gauge fixing function f_i will give us the ϕ-π Lagrangian

$$M_f^{ij}(x,y) \equiv \frac{\delta\, \delta_Q(\omega) f_i(A,\psi)(x)}{\delta \omega^j(y)}$$

The ϕ-π Lagrangian is then

$$\mathcal{L}_{\phi\pi} = \int d^4y\, \bar{C}_i(x)\, M_f^{ij}(x,y)\, C_j(y)$$

Thus the generating functional for $\{\psi, A, C, \bar{C}\}$ green's functions is given by the path integral

I.A.)

$$Z[J, J_\mu, \xi, \bar{\xi}] = \int [dA^i_\mu][d\psi^\alpha][dc_i][d\bar{c}_i] \times$$

$$\times\, e^{+i\int d^4x \left[\mathcal{L}(\psi, A_\mu, c, \bar{c}) + J_\alpha \psi^\alpha + J^i_\mu A^{i\mu} + \bar{c}_i \xi^i + \bar{\xi}^i c_i\right]}$$

where

$$\mathcal{L} = \mathcal{L}_{inv}(\psi, A_\mu) + \mathcal{L}_f(A_\mu, \psi) + \mathcal{L}_{\phi\text{-}\pi}(A_\mu, \psi, c, \bar{c})$$

with

$$\mathcal{L}_{inv}(\psi, A_\mu) = \mathcal{L}_{matter} + \mathcal{L}_{Y\text{-}M} + \mathcal{L}_\theta$$

$$\mathcal{L}_{matter} = \mathcal{L}_{matter}(\psi, D_\mu\psi)$$

$$\mathcal{L}_{Y\text{-}M} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} = -\frac{1}{2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$$

$$\mathcal{L}_\theta = \frac{g^2}{32\pi^2} \theta F^i_{\mu\nu} \tilde{F}^{i\mu\nu} = \frac{g^2}{16\pi^2} \theta \mathrm{Tr}[F_{\mu\nu} \tilde{F}^{\mu\nu}]$$

$$\mathcal{L}_f = -\frac{\alpha}{2} f_i(A, \psi) f_i(A, \psi)$$

$$\mathcal{L}_{\phi\pi} = \int d^4y\, \bar{c}_i(x) M^{ij}_f(x,y) c_j(y)$$

with
$$M^{ij}_f(x,y) = \frac{\delta\, \delta_\Omega(\omega) f_i(A,\psi)(x)}{\delta \omega_j(y)}$$