As a start lets ignore the extrinsic variables and just focus on the spin dynamics.

As an illustration of the use of the spin basis, consider the interaction of a spin $\frac{1}{2}$ particle with an external magnetic field. Classically, a spinning charge has a magnetic moment $\mu$ which interacts with the external magnetic field $B$. The classical Hamiltonian describing the interaction is simply $H = -\mu \cdot B$.

If the spinning charge has angular momentum $S$ due to its spin then the magnetic moment is proportional to it:

$\mu = \gamma S$

where $\gamma$ is the constant of proportionality $\gamma$, the gyro magnetic ratio, depends on the makeup of the spinning charge.

Quantum mechanically, a particle with spin $S$ has magnetic moment:

$\vec{\mu} = \gamma S = \gamma g \frac{e\hbar}{2m} \frac{1}{\hbar} S$

$\equiv g (\frac{\gamma}{\hbar}) \mu_B \frac{1}{\hbar} S$

where $\mu_B = \frac{e\hbar}{2m}$ is the Bohr magneton and $e$ is the magnitude of the electron charge. $g$ is the electron charge of the particle, $g = \gamma$ depends on the particle being considered. (For neutral particles like neutrin $\gamma = 0$ $g = \frac{e\hbar}{2m} = g \mu_B$.) $g$ is called the Landé $g$-factor.
The interaction of the spin $S$ with the external magnetic field $B$ is again described by the Hamiltonian

$$H = -\vec{\mu}_B \cdot \vec{B} = -g_\text{\mu_B} \vec{S} \cdot \vec{B}$$

$$= -g_\text{\mu_B} \frac{\mu_B}{\hbar} S_z \vec{B}.$$ 

For $\vec{B} = B \hat{z}$ this yields

$$H = -g_\text{\mu_B} \frac{\mu_B}{\hbar} S_z$$

$$\equiv \omega_B S_z$$

with the Bohr frequency $\omega_B = -g_\text{\mu_B} \frac{\mu_B}{\hbar}$. The spin basis vectors are the eigenstates of $H$:

$$H |S = \frac{1}{2}, m_S = \frac{1}{2}\rangle = \omega_B |S = \frac{1}{2}, m_S = \frac{1}{2}\rangle$$

with $m_S = \pm \frac{1}{2}$. Hence the energy eigenvalues are

$$E_{m_S} = \omega_B m_S = \omega_B \left(-g_\text{\mu_B} \frac{\mu_B}{\hbar}\right)$$

For example, for an electron $g = -1$ and $\hbar \omega_B = g_\text{\mu_B}$. Experimentally, we find

$$(g - 2) = 0.00119312$$

(i.e., $2 \approx 2.001159656$), which is known as the anomalous magnetic.
Hence

\[ E_\uparrow = \frac{g}{2} \mu_B B \approx \mu_B B \]
\[ E_\downarrow = -\frac{g}{2} \mu_B B \approx -\mu_B B \]
\[ m_s = -\frac{1}{2}, \frac{1}{2} \]

Suppose at time \( t = 0 \), the spin is in the state

\[ |2(0)\rangle = \cos \frac{\theta}{2} e^{-i\frac{\gamma}{2}t} \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sin \frac{\theta}{2} e^{i\frac{\gamma}{2}t} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \]

The state at time \( t \) is simply

\[ |2(t)\rangle = \cos \frac{\theta}{2} e^{-i\frac{\gamma}{2}t} e^{-i\frac{\hbar}{\mu_B} E_{\frac{1}{2}} t} \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sin \frac{\theta}{2} e^{i\frac{\gamma}{2}t} e^{i\frac{\hbar}{\mu_B} E_{-\frac{1}{2}} t} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \]

Since \( |\frac{1}{2}, m_s\rangle \) are energy eigenstates and

\[ i\hbar \frac{d}{dt} |2(t)\rangle = \hat{H} |2(t)\rangle. \]
So

\[ \left| 2(t) \right> = \cos \frac{\omega t}{2} \left| \frac{1}{2}, \frac{1}{2} \right> - i \frac{(\gamma + \omega B t)}{2} \left| \frac{1}{2}, \frac{1}{2} \right> + \sin \frac{\omega t}{2} \left| \frac{1}{2}, -\frac{1}{2} \right>. \]

The magnetic field \( B \) produces a phase shift between the coefficients of the spin eigenstates that depends on time.

Since \( [H, S_z] = 0 \), the observable \( S_z \) is a constant of the motion. The probability to measure spin \( \pm \frac{1}{2} \) at \( t \) is

\[
P_{\pm \frac{1}{2}} = \left| \left< \frac{1}{2}, \pm \frac{1}{2} \left| 2(t) \right> \right|^2 = \begin{cases} \cos^2 \frac{\omega t}{2} & \text{for } +\frac{1}{2} \\ \sin^2 \frac{\omega t}{2} & \text{for } -\frac{1}{2} \end{cases}
\]

independent of \( t \). However, \( S_x, y \) do not commute with \( H \); \( [H, S_x, y] \neq 0 \) and hence are not constants of the motion. Their expectation values are

\[
\left< S_x y \left| 2(t) \right> \right> = 
\]
\[
\begin{align*}
\frac{\hbar}{2} \sum_{j} & \frac{1}{2} \left( \cos \frac{\hbar}{2} e^{\frac{i}{2} (\varphi + \omega t)} + i \frac{\hbar}{2} e^{-\frac{i}{2} (\varphi + \omega t)} \right) \\
& \times \left( \cos \frac{\hbar}{2} e^{\frac{i}{2} (\varphi + \omega t)} + i \frac{\hbar}{2} e^{-\frac{i}{2} (\varphi + \omega t)} \right) \\
\Rightarrow \quad & \\
\langle \hat{S}_x | \hat{S}_x | \hat{S}_x \rangle & = \frac{\hbar}{2} \sin \Theta \cos (\varphi + \omega t) \\
\langle \hat{S}_y | \hat{S}_y | \hat{S}_y \rangle & = \frac{\hbar}{2} \sin \Theta \sin (\varphi + \omega t) \\
\langle \hat{S}_z | \hat{S}_z | \hat{S}_z \rangle & = \frac{\hbar}{2} \cos \Theta .
\end{align*}
\]

The expectation values of \( \hat{S} \) behave like classical angular momentum of magnitude \( \frac{\hbar}{2} \) undergoing Larmor precession with angular velocity \( \omega_B \).