Besides inverting the space coordinates, we can imagine a transformation that reverses the direction of time. More accurately, we can imagine that the motion of a system in reversed momentum flowing in the opposite direction, directed angular momentum opposite its original direction, etc. This motion-reversal, as we shall see, is equivalent to letting $t 	o -t$ in our states.

A time-reversal transformation is defined by $\mathbf{r}' = \mathbf{r}$ but $t' = -t$. Then we define the operator that relates the states in these two frames as $\mathcal{T} \equiv T$

$|\psi\rangle = T |\psi\rangle$. The transformation is defined to leave the coordinates unchanged so we define $|\varphi\rangle = |\psi\rangle$

$T \mathcal{T}^{-1} = \mathcal{T}$. 
But \( t \mapsto -t \) thus velocities and hence momentum should be reversed (motion reversal), so we define

\[
T \equiv T' = -\hat{p}.
\]

Since \( E = \mathbb{R} \times \hat{p} \Rightarrow T \hat{\ell} T' = -\hat{\ell} \).

Hence if the commutation relations are to be the same in each frame we must have

\[
T [X^i, P^j] T^{-1} = T i \hbar \delta^i_j T^{-1}
\]

\[
= T X^i T^{-1} T P^j T^{-1}
- T P^j T^{-1} T X^i T^{-1}
\]

\[
= - X^i P^j + P^j X^i
\]

\[
= -[X^i, P^j]
\]

\[
= -i \delta^i_j \hbar = i \delta^i_j T i T^{-1}
\]

\[
\Rightarrow T i T^{-1} = -i \Rightarrow T\dot{c} = -i \dot{T}.
\]
It must be anti-linear, hence (by Wigner's Theorem) it is anti-unitary.

Since two time reversal operations result in the original system we have that:

\[ T^2 |14\rangle = e^{i\eta} |14\rangle; \quad \eta \in \mathbb{R}. \]

Using the associative property of operator multiplication we find:

\[ T^3 |12\rangle = T^2 (T |12\rangle) = T (T^2 |12\rangle) \]

\[ = T (e^{i\eta} |12\rangle) \]

But \( T \) is anti-linear so

\[ = e^{-i\eta} (T |12\rangle). \]

Now for the sum of 2 states \((12\rangle + T |14\rangle)\) we also have

\[ T^2 (12\rangle + T |14\rangle) = e^{i\eta} (12\rangle + T |14\rangle) \]

\[ = e^{i\eta} (12\rangle + e^{-i\eta} (T |12\rangle) \]

\[ T^2 |12\rangle + T^3 |14\rangle = e^{i\eta} |12\rangle + e^{-i\eta} T |12\rangle \]
\[ e^{i\varphi|\Psi\rangle} + e^{-i\varphi} = |
\\]
\[ = e^{i\varphi'} (|\alpha\rangle + T|\alpha\rangle) 
\\]
\[ = e^{i\varphi} e^{i\varphi'} e^{-i\varphi} \Rightarrow \varphi = 0 \text{ or } \pi \\
\text{for all phases} \]

Thus
\[ T^2|\Psi\rangle = \pm|\Psi\rangle \]

2 successive time reversal transformations need not be the identity.

Now consider the Schrödinger equation
\[ \frac{i}{\hbar} \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle \]

Operators with \( T \Rightarrow \)
\[ -i\hbar \frac{\partial}{\partial t} (T|\Psi(t)\rangle) = THT^{-1} (T|\Psi(t)\rangle) \]

\( T \) is anti-linear

Now let \( t \rightarrow -t \)
\[
+i\hbar \frac{d}{dt} \left( T \kett{\chi}(t) \right) = \left( HH^{-1} \right) \left( T \kett{\chi}(t) \right)
\]

Now if \( H' = H \) so that the Hamiltonian is time reversal invariant, then

\[
\Rightarrow \quad i\hbar \frac{d}{dt} \left( T \kett{\chi}(t) \right) = H \left( T \kett{\chi}(t) \right)
\]

\[
\Rightarrow \quad \text{if} \, \kett{\chi}(t) \text{ is a solution to Schrödinger's equation, so is} \, \left( T \kett{\chi}(t) \right) \text{ as well.}
\]

(Note: if the Hamiltonian is symmtry, the time reversal invariance of \( H' \) implies that the conserved quantity in the case of anti-unitary symmetry does not imply a conserved quantity, but instead, solutions of Schrödinger's equation come in pairs \( \kett{\chi}(t) \) and \( T \kett{\chi}(t) \) (i.e. for stationary states, some degeneracy).
Suppose $H = \frac{\mathbf{p}^2}{2m} + U(R); \ m = \text{real} > 0$. 

$H' = THT^{-1} = \frac{\mathbf{p}^2}{2m} + V^*(\vec{E})$

If $H' = H \Rightarrow V^*(\vec{E}) = U(\vec{E})$

The potential is real if $H$ is $T$-invariant.