Review

The Postulates of Quantum Mechanics are

1) The set of all possible states of a physical system is in 1-1 correspondence with the vector directions (rays) in a Hilbert space $H$. Since the state is described by the entire ray $|\psi\rangle$, we have that $|\psi\rangle$ and $|\psi\rangle$ with $\alpha \in \mathbb{C}$ describe the same state.

2) The physical observables of a system correspond 1-1 with the set of Hermitian operators on the state space $H$. That is, to each measurable quantity $Q$, there corresponds a Hermitian operator $A$ acting on $H$. Out of the set of all Hermitian operators, there is a subset which consists of mutually commuting operators and they are assumed to be complete (they form a CSCO).
3) Spectral Decomposition

a) The only possible result of the measurement of a physical observable \( A \) is one of the (real) eigenvalues of the corresponding Hermitian operator \( A \).

b) Let \( \phi_k \) be the simultaneous eigenstates of a CSCO so that

\[
A \phi_k = a_k \phi_k
\]

These states form an orthonormal basis for the state space \( \mathcal{H} \).

For a system in state \( |\psi\rangle \) (with \( \langle \psi | \psi \rangle = 1 \)) the probability of measuring the value \( a_k \) for the physical observable \( A \) is

\[
P_k = |\langle \phi_k | \psi \rangle|^2.
\]

Immediately following the measurement, the system is in the state \( |\phi_k\rangle \).
4) The time evolution of the physical states is given by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \]

where \( H = H(t) \) is the Hermitian Hamiltonian operator. \( H(t) \) is the total energy of the system. In general, the observables may also have explicit time dependence, \( A = A(t) \).

For an isolated system, the observables including the Hamiltonian are independent of time \( \frac{dA}{dt} = 0 \) and \( H \) is a constant operator.

**Comments:** 1) We extend the Hilbert space of states to include generalized basis with infinite (continuum) norms but whose scalar product with every basis of \( H \) is finite. So doing, we ensure bra-vector \( |\psi\rangle \) there corresponds a Ket-vector and vice-versa in the extended Hilbert space and its dual.
2) Then in postulate 3 we can have discrete as well as continuous (or even mixed) set of basis vectors. That is if the orthonormal basis is a discrete basis, then the simultaneous eigenvectors \( |\Phi_{ab\ldots}\rangle \), of the CSCO \( E_{\alpha\beta\ldots}\) obey

\[
|\Phi_{ab\ldots}\rangle = \delta_{\alpha\alpha} \delta_{\beta\beta} \ldots
\]

\[
1 = \sum_{a,b,\ldots} |\Phi_{ab\ldots}\rangle \langle \Phi_{ab\ldots}| = A |\Phi_{ab\ldots}\rangle \langle \Phi_{ab\ldots}| + B |\Phi_{ab\ldots}\rangle \langle \Phi_{ab\ldots}| + \ldots
\]

where

\[
A |\Phi_{ab\ldots}\rangle = a |\Phi_{ab\ldots}\rangle
\]

\[
B |\Phi_{ab\ldots}\rangle = b |\Phi_{ab\ldots}\rangle + \ldots
\]

with the eigenvalues \( E_{\alpha\beta\ldots}\) taking discrete values (in one correspondence with the integers).

An arbitrary state vector \( |\Psi\rangle \) has the expansion in terms of the \( |\Phi_{ab\ldots}\rangle \) basis

\[
|\Psi\rangle = \sum_{a,b,\ldots} c_{ab\ldots} |\Phi_{ab\ldots}\rangle
\]
with

\[ \Psi_{ab...} = \langle \phi_{ab...} | 12 \rangle. \]

This implies

\[ A_{12} = \sum_{a,b...} a \, \Psi_{ab...} \, \langle \phi_{ab...} | \rangle \]
\[ B_{12} = \sum_{a,b...} b \, \Psi_{ab...} \, \langle \phi_{ab...} | \rangle, \text{ etc.} \]

The probability of finding the values a,b,... when \( A, B \) are measured when the system is in state \( 12 \rangle \) is

\[ P_{ab...} = | \langle \phi_{ab...} | 12 \rangle |^2. \]

Note that

\[ \sum_{a,b...} P_{ab...} = \sum_{a,b...} | \langle \phi_{ab...} | 12 \rangle |^2 \]

\[ = \sum_{a,b...} \langle 12 | \phi_{ab...} \rangle \times \langle \phi_{ab...} | 12 \rangle \]

\[ = \langle 12 | 12 \rangle = 1 \] as required of a probability.

Finally, the expectation value of \( AB \)...

in state \( 12 \rangle \) is
\[ \langle \psi_1 | A | \psi_4 \rangle = \sum_{a,b,..}^i \alpha_{ab...} \langle \psi_1 | \phi_{ab...} \rangle \]
\[ = \sum_{a,b,..}^i \alpha_{ab...} | \langle \phi_{ab...} | \psi_4 \rangle |^2 \]
\[ = \sum_{a,b,..}^i \alpha_{ab...} | \langle \phi_{ab...} | \psi_4 \rangle |^2 \]

as expected for \( \langle \psi | \phi \rangle \) a probability.

On the other hand if the orthonormal basis is continuous, then
\[ \langle \phi_{\alpha \beta...} | \phi_{\alpha' \beta...} \rangle = \delta^{(\alpha,\alpha')} \delta^{(\beta,\beta')} \]
\[ I = \int d\alpha d\beta... \langle \phi_{\alpha \beta...} | \phi_{\alpha \beta...} \rangle \]

where
\[ A | \phi_{\alpha \beta...} \rangle = \alpha | \phi_{\alpha \beta...} \rangle \]
\[ B | \phi_{\alpha \beta...} \rangle = \beta | \phi_{\alpha \beta...} \rangle \]
\[ \text{with the eigenvalues } \Sigma \delta \beta... \text{ taking on a continuum of values.} \]
An arbitrary state vector \( |\Psi\rangle \) has the expansion in terms of the continuous basis \( \{ |\phi_{\alpha,\beta,\cdots}\rangle\} \) given by:

\[
|\Psi\rangle = \int d\alpha \, d\beta \cdots \, \zeta(\alpha, \beta, \cdots) \, |\phi_{\alpha,\beta,\cdots}\rangle
\]

with

\[
\zeta(\alpha, \beta, \cdots) = \langle \phi_{\alpha,\beta,\cdots} | \Psi \rangle.
\]

This implies

\[
A|\Psi\rangle = \int d\alpha \, d\beta \cdots \, \alpha \, \zeta(\alpha, \beta, \cdots) \, |\phi_{\alpha,\beta,\cdots}\rangle
\]

\[
B|\Psi\rangle = \int d\alpha \, d\beta \cdots \, \beta \, \zeta(\alpha, \beta, \cdots) \, |\phi_{\alpha,\beta,\cdots}\rangle \text{ etc.}
\]

For the system in state \( |\Psi\rangle \), the probability of measuring \( \alpha \) in the range \( \alpha \) to \( \alpha + d\alpha \), \( \beta \) in the range \( \beta \) to \( \beta + d\beta \), etc., is

\[
dP(\alpha, \beta, \cdots) = \langle \phi_{\alpha,\beta,\cdots} | |\Psi\rangle |^2 \, d\alpha \, d\beta \cdots.
\]

Note, as required of a probability density

\[
\int dP(\alpha, \beta, \cdots) = \int d\alpha \, d\beta \cdots \, \langle \phi_{\alpha,\beta,\cdots} | |\Psi\rangle |^2 = <\Psi | \Psi >= 1.
\]
\[ = \langle \psi | \phi \rangle = 1. \]

And finally, the expectation value of A, B, ... in state \( \langle \varphi \rangle \) is

\[
\langle \varphi | A | \psi \rangle = \int dx dp ... \times \left( \langle \varphi | \psi \rangle \right) \langle \varphi | \phi \rangle
\]

\[ = \langle \phi | \varphi \rangle \langle \varphi | \psi \rangle = \int \mathcal{D} \varphi (x, p, ...) \times \langle \varphi | \psi \rangle \]

\[ = \int \mathcal{D} \varphi (x, p, ...) \times \langle \varphi | \varphi \rangle = \int \mathcal{D} \varphi (x, p, ...) \times \left( \int \mathcal{D} \varphi (x, p, ...) \right) \]

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3) Further, immediately following the measurement of A, the state of the system is reduced to that part of \( | \psi \rangle \) which is precisely that eigenvector of A. So if A is measured to yield \( a \), the system immediately afterwards is in state

\[ | \psi \rangle \rightarrow \sum_{b,c,...} a_{ab,c,...} | \phi \rangle_{b,c,...} \]

\[ \frac{1}{(\sum_{b,c,...} a_{ab,c,...} |^2)^{1/2}} \]

Normalized new state to 1.
In the case of continuous eigenvalues, the state of the system reduces to the state

\[
|\Phi + x\rangle \rightarrow \int dx \int d\beta \int d\gamma \cdots |\Phi(x, \beta, \gamma, \cdots)\rangle |\Phi(x, \beta, \gamma, \cdots)\rangle \frac{1}{\sqrt{\int dx \int d\beta \int d\gamma \cdots |\Phi(x, \beta, \gamma, \cdots)\rangle^2}}
\]

immediately after the measurement of \( A \) which yields \( x \) to within \( \Delta x \).

4) The Hilbert space is completely characterized by giving all algebraic relations (commutation relations) among the elements of any irreducible set of linear operators. An irreducible set of operators is one such that there is only one operator to commute with all the members of the set and the identity. For the case of a single particle with no other internal degrees of freedom (e.g., spin) we have that the position operator \( \hat{X} \) and its
Canonical conjugate momentum operator $\hat{P}$ formed an irreducible set. Their CCR are

$$[\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij}$$

$$[\hat{X}_i, \hat{X}_j] = 0 = [\hat{P}_i, \hat{P}_j].$$

So if an operator commutes with $\hat{X}_i$, it cannot be a function of $\hat{P}$, since they do not commute. If it also commutes with $\hat{P}$, likewise it cannot be a function of $\hat{X}_i$. Hence the operator is independent of $\hat{X}_i$ and $\hat{P}$ since there are no other degrees of freedom for the operator to act on. It must be a multiple of the identity. From the irreducible set we can extract a CSOC whose eigenvectors form a basis for the Hilbert space. Thus we are led to consider the coordinate basis and the momentum basis.
Coordinate Representation: Given a Cartesian coordinate system with the position denoted by \( \mathbf{x} \), we have
\[
X_i | \mathbf{x} \rangle = x_i | \mathbf{x} \rangle ; \quad (x_1, x_2, x_3) \in \mathbb{R}^3
\]
The states are continuum normalized
1) \( \langle \mathbf{x'} | \mathbf{1} \rangle = \delta^3 (\mathbf{x} - \mathbf{x'}) \)

and complete
2) \( \mathbf{1} = \int d^3x \ z^\dagger (\mathbf{x}) | \mathbf{x} \rangle < \mathbf{x} | \mathbf{1} \rangle \)

Hence any state \( | \psi \rangle \) has the expansion
\[
| \psi \rangle = \int d^3x \ z (\mathbf{x}) | \mathbf{x} \rangle
\]
with wavefunction \( z (\mathbf{x}) = \langle \mathbf{x} | \psi \rangle \).

Any function of \( X_i \) has expectation value
\[
\langle \mathbf{1} | F (\mathbf{x'}) | \psi \rangle = \int d^3x \ z^\dagger (\mathbf{x}) F (\mathbf{x}) z (\mathbf{x})
\]
Since \( F (\mathbf{x'}) | \mathbf{x} \rangle = F (\mathbf{x}) | \mathbf{x} \rangle \).
Consider the unitary operator (translational operator)

\[ U(\vec{a}) = e^{\frac{-i}{\hbar} \vec{P} \cdot \vec{a}} \]

U(\vec{a}) = U(-\vec{a}) = U^\dagger(\vec{a}) \text{ since } \vec{P} = -\vec{P}. \]

Since \([X_i, P_j] = i \hbar \delta_{ij}\) we have that

\[ [X_i, U(\vec{a})] = -\frac{i}{\hbar} (i\hbar a_i) U(\vec{a}) \]

\[ = a_i U(\vec{a}) \]

Then \(\hat{R} U(\vec{a}) = U(\vec{a})(\hat{R} + \hat{a})\).

Applying this to the position eigenstate we find

\[ \hat{R}(U(\vec{a}) |F\rangle) = (\hat{R} + \hat{a})(U(\vec{a}) |F\rangle) \]

\[ \langle \vec{a} | \hat{a} \rangle |F\rangle = |F + \vec{a}\rangle. \]

Thus \( U(\vec{a}) |F\rangle = \langle F + \vec{a} | \vec{a} \rangle \).

Now

\[ \langle \vec{F} | U(\vec{a}) | F\rangle = \langle \vec{F} | U(-\vec{a}) | F\rangle \]

\[ = \langle \vec{F} + \vec{a} | F\rangle = \hat{a}(\vec{F} + \vec{a}) \]
but for infinitesimal \( \xi = \varepsilon \) we have

\[
U(\varepsilon) = 1 + \frac{i}{\hbar} \vec{P} \cdot \varepsilon + O(\varepsilon^2)
\]

So

\[
\langle \vec{r} | U(\varepsilon) | \vec{r}' \rangle = \langle \vec{r} | \vec{r}' \rangle + \frac{i}{\hbar} \langle \vec{r} | \vec{P} \cdot \varepsilon | \vec{r}' \rangle
\]

\[= \varepsilon \langle \vec{r} \rangle + \frac{i}{\hbar} \langle \vec{r} | \vec{P} \cdot \varepsilon | \vec{r}' \rangle
\]

but this

\[= 2 \varepsilon \langle \vec{r} + \vec{\varepsilon} \rangle = 2 \varepsilon \langle \vec{r} \rangle + \varepsilon \cdot \vec{\nabla} \varepsilon \langle \vec{r} \rangle
\]

\[
\Rightarrow \quad \langle \vec{r} | \vec{P} | \vec{r}' \rangle = -i \hbar \vec{\nabla} \varepsilon \langle \vec{r} | \vec{r}' \rangle
\]

\[
= -i \hbar \vec{\nabla} \langle \vec{r} | \vec{r}' \rangle
\]

Since \( |\vec{r}'\rangle \) was arbitrary \( \Rightarrow \)

\[
\langle \vec{r} | \vec{P} = -i \hbar \vec{\nabla} \langle \vec{r} | \vec{r}' \rangle , \quad \text{that is} \quad \vec{P} \text{ is represented by } \frac{i}{\hbar} \vec{\nabla} \text{ in the } \{|\vec{r}\rangle\}_B \text{ basis.}
\]
Suppose \( |\phi\rangle\) is a momentum eigenstate
\[
\hat{p} |\phi\rangle = \hat{p} |\phi\rangle ; \quad \hat{p} \in \mathbb{R}^3
\]
with continuum normalization

1) \[ \langle \hat{p}' | \hat{p} \rangle = (2\pi \hbar)^3 \delta^3(\hat{p}' - \hat{p}) \]

Then
\[
\langle \hat{p}' | \hat{p} | \phi \rangle = -i \hbar \nabla_{p'} \langle \hat{p}' | \phi \rangle
\]

\[
= \hat{p} \langle \hat{p}' | \phi \rangle
\]

\[
\Rightarrow \quad \langle \hat{p}' | \phi \rangle = N e^{i \cdot \hat{p} \cdot \hat{p}'}
\]

Since \( \langle \hat{p}' | \phi \rangle \) is complete we have
\[
\langle \hat{p} | \hat{p} \rangle = (2\pi \hbar)^3 \int \delta^3(\hat{p}' - \hat{p})
\]

\[
= \int d^3r \langle \hat{p} | \hat{F} | \hat{p} \rangle
\]

\[
= \int d^3r \left( \hbar \frac{\partial}{\partial r}(\hat{p} - \tilde{p}) \right) 101^2
\]

\[
= |10|^2 (2\pi \hbar)^3 \delta^3(\hat{p}' - \hat{p})
\]

Thus we take \( N = 1 \).
Otherwise the completeness relation for the momentum basis $\tilde{\psi}(p)$ is
\[ 1 = \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{\psi}(p) \tilde{\psi}^*(p). \]

Any state $|\Psi\rangle$ has the expansion in terms of momentum eigenstates
\[ |\Psi\rangle = \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{\Psi}(p) \tilde{\psi}(p) |p\rangle \]
with
\[ \tilde{\Psi}(p) = \langle \Psi | p \rangle. \]

Then
\[ \tilde{\Psi}(\vec{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{\Psi}(p) \frac{-\hbar^2}{2m} \tilde{\psi}(p) \]
and
\[ \tilde{\psi}(p) = \int d^3r e^{-i \frac{\vec{p} \cdot \vec{r}}{\hbar}} \tilde{\Psi}(\vec{r}). \]

Finally, for $H = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})$, the Schrödinger equation in the $\tilde{\psi}(p)$ basis takes the form of the wave equation Schrödinger
\[ \frac{i}{\hbar} \frac{\partial}{\partial t} \tilde{\Psi}(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right] \tilde{\Psi}(\vec{r}, t) \]
with
\[ \tilde{\Psi}(\vec{r}, t) = \langle \vec{r} | \Psi(t) \rangle. \]
In addition to single particles moving in a potential, we also consider many particle systems with inter-particle as well as external potentials. In such cases, the wave equation had the form

\[ \frac{\hbar}{2i} \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; t) = \hat{H} \Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; t) \]

\[ = \sum_{i=1}^{N} \left( \frac{-\hbar^2}{2m_i} \nabla_i^2 \Psi(\vec{r}_i; t) \right) + V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; t) \Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; t) . \]

In the case of the two-body central potential \( V(\vec{r}_1, \vec{r}_2; t) = V(1r_1 - r_2) \) this reduces to

\[ \frac{\hbar}{2i} \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2; t) = \frac{-\hbar^2}{2m_1} \nabla_1^2 \Psi(\vec{r}_1, \vec{r}_2; t) - \frac{\hbar^2}{2m_2} \nabla_2^2 \Psi(\vec{r}_1, \vec{r}_2; t) + V(1r_1 - r_2) \Psi(\vec{r}_1, \vec{r}_2; t) \]

and we were able to separate the variables to CM and relative coordinates in order to have 2-1"single" particle problems.

**Ex.** Hydrogen atom \( V = -\frac{e^2}{4\pi \varepsilon_0} \frac{1}{r_{12}} \)
Important to realize central potential problems is the translational and rotational symmetries of the Hamiltonian. In particular the rotational invariance of the central potential implied that the orbital angular momentum commuted with $\mathcal{H}$. Since

$$[L_i, L_j] = i \hbar \epsilon_{ijk} L_k$$

we found that $\mathcal{H}, L^2, L_z, L_z^3$ were a set of CSOC and we used the eigenfunctions of $L^2$ and $L_z$ to expand our stationary states of the Hydrogen atom,

$$\Psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}^{\ell m}(\theta, \phi)$$

with

$$\mathcal{H} \Psi_{n\ell m} = E_n \Psi_{n\ell m},$$

$$L_z \Psi_{n\ell m} = m \hbar \Psi_{n\ell m},$$

$$L^2 \Psi_{n\ell m} = (\ell + 1) \hbar^2 \Psi_{n\ell m}$$

and

$$E_n = -\left[ \frac{\epsilon^2}{4\hbar^2} \right] \frac{m^2}{2\hbar^2} \int \frac{1}{\mathcal{E}^4}$$

$$n=1,2,3,\ldots, \ell=0,1,\ldots,n-1, m=-\ell,\ldots,\ell.$$
As we see, symmetries of the Hamiltonian are important since the symmetry implies an operator related to it will commute with the H, and hence will be part of a CSCO along with H. Hence, we will like to undertake a more systematic discussion of group transformations, symmetries, and conservation laws in QM.