Scattering Cross-Sections: Review

In the previous chapter we concerned ourselves with determining the discrete energy eigenvalues of a time-independent Hamiltonian. These bound state energies were taken conventionally to be negative. In general the Hamiltonian also has a positive energy or continuous part to its spectrum. These values do not correspond to bound states but are referred to as scattering states. Since any value of E is in the spectrum it is not a question of determining it, but rather of choosing the energy E in such a situation it usually corresponds to the initial energy of some incoming projectile particle. This particle will when it approaches closely enough interact with a target particle (or alternatively stated, the particles will collide) and get off at some direction. If then is our aim to predict the probability that the particle will after interacting be located at some point in space?
This probability of course is related to the scattered state energy eigenfunction. Hence scattering theory will tend to itself with predictions for the scattering probability through the determination of the eigenfunction for a given (continued) energy eigenvalue \( E > 0 \).

This is just what the Dirac equation simply did for us.

In general there are 4 types of collision processes:

1) Elastic scattering: Particles \( A \) and \( B \) collide and keep their same energies relative to the center of mass (and do not alter their internal state). This is represented by \( A + B \rightarrow A + B \).

2) Inelastic scattering: Particles \( A \) and \( B \) change their internal states. That is, kinetic energy of \( A \) and \( B \) can be absorbed by either \( A \) or \( B \) during their collision to alter their internal state. This is represented by \( A + B \rightarrow A^* + B \).
For example, an electron can scatter off a hydrogen atom and leave the hydrogen atom in the same state it was initially, say the ground state. This is elastic scattering. Or it could leave the hydrogen atom in some excited state while changing its energy. This is inelastic scattering.

3) Rearrangement collisions: Particles A and B are exchanged for other particles in an interaction. The outgoing particles may be of a different type than the incoming particles. This often occurs in chemical and nuclear reactions. It is represented by

\[ A + B \rightarrow C + D. \]

4) Particle Production: The collision of 2 particles produces 3 or more outgoing particles. This is common in elementary particle physics. It is represented by

\[ A + B \rightarrow C + D + E + \ldots \]

In these collisions, each final set of particles possible is called a different channel. Scattering theory will also predict the
probabilities for each type of channel.

In our case we will deal only with elastic and inelastic scattering. Indeed in a scattering process the angular distribution of the scattered particles is described in terms of a differential cross section \( \sigma(\theta, \phi) \). Initially we have a flux \( J_{\text{in}} \) of incoming particles per unit area per unit time incident on the target (or for collision). The number of particles per unit time scattered into a solid angle \( d\sigma \) centered about spherical polar angles \( \theta, \phi \) is proportional to the incident flux \( J_{\text{in}} \) and the angular opening \( d\Omega \).

\[
dn(\theta, \phi) = \sigma(\theta, \phi) J_{\text{in}} d\Omega.
\]

The constant of proportionality \( \sigma(\theta, \phi) \) is the differential cross section.
Suppose the particle detector is located in the direction \((\theta, \phi)\) at a distance \(r\) from the target and well outside the incident beam of particles. If the detector subtends a solid angle \(d\Omega\), it will receive \(dn = J_{\text{in}} \frac{d\Omega}{r^2 d\Omega}\) scattered particles per unit time. The area of the detector is \((r^2 d\Omega)\). Then we have a flux of outgoing scattered particles at the detector of

\[
J_{\text{out}} = \frac{dn}{r^2 d\Omega} = J_{\text{in}} \sigma(\theta, \phi) \frac{d\Omega}{r^2 d\Omega},
\]

Hence the differential cross section is also given by

\[
\sigma(\theta, \phi) = \frac{r^2 J_{\text{out}}}{J_{\text{in}}},
\]

\(\sigma(\theta, \phi)\) has units of area. We also have that \(J_{\text{out}}\) will decrease like \(1/r^2\) so that \(\sigma(\theta, \phi)\) is actually independent of \(r\). By summing over all directions, we obtain the total cross section.
\[ \sigma = \int d\Omega \sigma(\theta, \phi) \]
\[ = \int_{0}^{\pi} \int_{0}^{2\pi} d\theta \sin \theta \sigma(\theta, \phi). \]

Note that the cross section \( \sigma(\theta, \phi) \) is given in terms of a ratio of fluxes, quantities that are easily measured as well as calculated. Of course, this could be classical or quantum mechanics, we have not specified any dynamics as yet.

Finally, we have been treating the target particle as if it were stationary, but in fact particles often collide with each other with equal momentum. The two most common frames in which to work then give the "laboratory" and "CM" frames.

[Diagram of particle trajectories in laboratory and CM frames]
In the lab frame the incoming particle has mass \( m_1 \), velocity \( \vec{V} \), while the target particle of mass \( m_2 \) is at rest. The velocity of the center of mass is then
\[
\vec{V} = \left( \frac{m_1}{m_1 + m_2} \right) \vec{V}_1.
\]

In the CM frame, the center of mass is at rest. Here we subtract \( \vec{V} \) from all velocities. So the incoming particle 1 now has velocity
\[
\vec{V}_{1\text{cm}} = \vec{V}_1 - \vec{V} = \left( \frac{m_2}{m_1 + m_2} \right) \vec{V}_1,
\]
while the target particle 2 now has velocity
\[
\vec{V}_{2\text{cm}} = -\left( \frac{m_1}{m_1 + m_2} \right) \vec{V}_1 \quad (= -\vec{V}).
\]

Now the final velocity of particle 1 in the 2 frames is related by
\[
N_1' \cos \Theta_{1\text{lab}} = N_{1\text{cm}} \cos \Theta_{1\text{cm}} + V
\]
\[
N_1' \sin \Theta_{1\text{lab}} = N_{1\text{cm}} \sin \Theta_{1\text{cm}}
\]
Taking the ratio, we have
\[ \tan \theta_{\text{lab}} = \frac{\sin \theta_{\text{cm}}}{\cos \theta_{\text{cm}} + \frac{V}{\sqrt{S_{\text{cm}}}}} \quad \text{(general)} \]

For **elastic scattering**, the energies of the particles and therefore their speeds relative to the CM, are unchanged. Thus

\[ q_{\text{1cm}} \equiv q_{\text{1cm}}' = \frac{M_1}{M_1 + m_2} \eta \quad \text{elastic} \]

hence

\[ \frac{V}{q_{\text{1cm}}} = \frac{M_1}{m_2} \quad \text{and} \]

\[ \left[ \tan \theta_{\text{lab}} = \frac{\sin \theta_{\text{cm}}}{\cos \theta_{\text{cm}} + \frac{M_1}{m_2}} \quad \text{elastic} \right] \]

So, as usual, there is no need to calculate the **differential cross section** in both frames, we usually calculate it in the CM frame and then relate it to the lab frame cross section by a transformation. The relation is determined from the fact that the number of
Particles scattered into a particular angular cone is the same in either frame. Also the incident flux (# of particles per unit area per unit time) is the same in both frames, thus

\[ \sigma_{\text{lab}}(\Theta_{\text{lab}}, \Phi_{\text{lab}}) \, d\Omega_{\text{lab}} = \sigma_{\text{cm}}(\Theta_{\text{cm}}, \Phi_{\text{cm}}) \, d\Omega_{\text{cm}} \]

with \( \Phi_{\text{lab}} = \Phi_{\text{cm}} \) and \( \tan \Theta_{\text{lab}} = \frac{\sin \Theta_{\text{cm}}}{\cos \Theta_{\text{cm}} + \beta} \)

(\( \beta = \frac{V}{c} \)). Thus

\[ \sigma_{\text{lab}}(\Theta_{\text{lab}}, \Phi_{\text{lab}}) = \sigma_{\text{cm}}(\Theta_{\text{cm}}, \Phi_{\text{cm}}) \, \frac{\sin \Theta_{\text{cm}} \, d\Theta_{\text{cm}}}{\sin \Theta_{\text{lab}} \, d\Theta_{\text{lab}}} \]

Now from \( \tan \Theta_{\text{lab}} = \frac{\sin \Theta_{\text{cm}}}{\cos \Theta_{\text{cm}} + \beta} \), we have taking the differentiated

\[ (1 + \tan^2 \Theta_{\text{lab}}) \, d\Theta_{\text{lab}} = \left[ \frac{\cos \Theta_{\text{cm}}}{\cos \Theta_{\text{cm}} + \beta} + \frac{\sin^2 \Theta_{\text{cm}}}{(\cos \Theta_{\text{cm}} + \beta)^2} \right] \times d\Theta_{\text{cm}} \]

So
\[
\frac{d\theta_{cm}}{d\theta_{lab}} = \frac{(\cos \theta_{cm} + \beta)^2 \left[ 1 + \tan^2 \theta_{lab} \right]}{(\cos \theta_{cm} (\cos \theta_{cm} + \beta) + \sin^2 \theta_{cm})} \\
= \frac{(\cos \theta_{cm} + \beta)^2 \left[ (\cos \theta_{cm} + \beta)^2 + \sin^2 \theta_{cm} \right]}{(\cos \theta_{cm} + \beta)^2} \\
\left( 1 + \beta \cos \theta_{cm} \right)
\]

\[
\frac{d\theta_{cm}}{d\theta_{lab}} = \frac{(1 + 2 \beta \cos \theta_{cm} + \beta^2)}{(1 + \beta \cos \theta_{cm})}
\]

Further

\[
\sin \theta_{lab} = \sqrt{\frac{\sin^2 \theta_{cm} + (\cos \theta_{cm} + \beta)^2}{\sin^2 \theta_{cm} + (\cos \theta_{cm} + \beta)^2}} \\
= \frac{\sin \theta_{cm}}{\sqrt{\sin^2 \theta_{cm} + (\cos \theta_{cm} + \beta)^2}}
\]

\[
= \frac{\sin \theta_{cm}}{\sin \theta_{lab}} = \sqrt{1 + 2 \beta \cos \theta_{cm} + \beta^2}
\]
Thus we find
\[
\frac{\sin \Theta_{\text{cm}}}{\sin \Theta_{\text{lab}}} \frac{d\Theta_{\text{cm}}}{d\Theta_{\text{lab}}} = \frac{(1+2\beta \cos \Theta_{\text{cm}} + \beta^2)^{3/2}}{1 + \beta \cos \Theta_{\text{cm}}}.
\]

Therefore the cross sections in the lab and C-M frames are related by
\[
\sigma_{\text{lab}}(\Theta_{\text{lab}}, \phi_{\text{lab}}) = \sigma_{\text{cm}}(\Theta_{\text{cm}}, \phi_{\text{cm}}) \frac{(1+2\beta \cos \Theta_{\text{cm}} + \beta^2)^{3/2}}{1 + \beta \cos \Theta_{\text{cm}}}.
\]

Theoretically it is usually simpler to work in the C-M frame.