Physics 431: Review

All Classical Electromagnetic Phenomena are described by Maxwell's Equations:

1) $\vec{\nabla}\cdot\vec{D} = \rho$: Gauss's Law

2) $\vec{\nabla}\cdot\vec{B} = 0$: No Magnetic Monopoles

3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$: Faraday's Law

4) $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$: Ampere's Law

with $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ the charge and current densities. In matter in general

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \quad , \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \, ,$$

while in linear media $\vec{P} = \chi\vec{E} = \epsilon_0\chi_e\vec{E}$ and $\vec{M} = \chi_m\vec{H}$,

hence $\boxed{\vec{D} = \epsilon\vec{E} \quad , \quad \vec{B} = \mu\vec{H}}$. As well

for Ohmic media $\boxed{\vec{J} = \sigma\vec{E}}$.

Matter, i.e. the charges, react to E-M fields via the Lorentz Force Law

$\boxed{\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})}$, for point charge q.

Consequences of Maxwell's Equations:

I) Charge Conservation: Continuity Equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

II) Energy Conservation: Linear Media

A) $$\frac{d}{dt}\left[U + \sum_i T_i\right] = -\oint_S \vec{S}\cdot d\vec{a}$$

$U = \frac{1}{2}\int_V (\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\,d\tau$: Energy of E-M field in volume V

$\sum_i T_i$: total energy of the particles

($\frac{dT_i}{dt} = q_i \vec{E}(\vec{r}_i)\cdot\vec{v}_i$ and

$\frac{d}{dt}\sum_i T_i = \int_V \vec{J}\cdot\vec{E}\, dV$

= Rate of Joule heating)

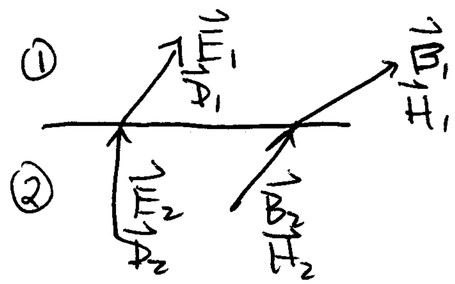
$\vec{S} = \vec{E}\times\vec{H}$ = Poynting Vector

= electromagnetic energy flow out (or into) of volume V bounded by S per unit area per unit time.

II B.) Locally: $u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$ = energy density

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

III) Boundary Conditions:



a) $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$

$$\boxed{B_{1n} = B_{2n}}$$

b) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow$

$$\boxed{E_{1t} = E_{2t}}$$

c) $\vec{\nabla}\cdot\vec{D} = \rho \Rightarrow$

$$\boxed{D_{1n} - D_{2n} = \sigma}$$

σ = surface charge density on boundary

d) $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \Rightarrow$

$$\hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{K}$$

$\vec{K}$ = surface current density in boundary

if $\vec{K} = 0 \Rightarrow$ $\boxed{H_{1t} = H_{2t}}$

IV) <u>Propagation of Electromagnetic Waves</u>:
Consider linear, charge-free, ohmic region, so

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \vec{J} = \sigma \vec{E}, \quad \rho = 0.$$

Maxwell's Eq's $\Rightarrow$ <u>Wave Equations</u>

$$\boxed{\left(\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} - \sigma\mu \frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = 0}$$

As well, solutions to this must also obey the Maxwell eq's.

A) Non-Conducting Media: $\boxed{\sigma = 0}$ $\Rightarrow$

$$\left(\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t)\end{pmatrix} = 0.$$

1) Monochromatic Plane Wave $\vec{E} = \vec{E}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$

$\vec{k} = k\hat{\epsilon}_3 \Rightarrow \boxed{k = \sqrt{\epsilon\mu}\,\omega = \frac{n\omega}{c}}, \quad \vec{k}\cdot\vec{E}_0 = 0$

$$\Rightarrow \vec{E}_0 = E_1\hat{\epsilon}_1 + E_2\hat{\epsilon}_2 = \left(|E_1|e^{i\varphi}\hat{\epsilon}_1 + |E_2|\hat{\epsilon}_2\right)e^{i\varphi_2}$$

Choose origin of time so that

$$\boxed{\vec{E}_0 = \left(|E_1|e^{i\varphi}\hat{\epsilon}_1 + |E_2|\hat{\epsilon}_2\right)}$$ general polarization

2) $\vec{B} = \vec{B}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$; $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$

$$\boxed{\vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0}$$

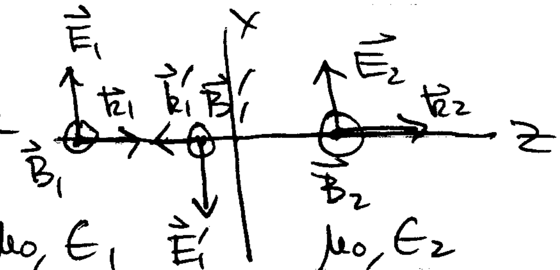
IV A 3) $u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$, $\vec{S} = \vec{E}\times\vec{H}$ for the <u>Real</u> physical E-M fields $\Rightarrow$

$$u = \epsilon(\mathrm{Re}\,\vec{E})\cdot(\mathrm{Re}\,\vec{E})$$

$$\vec{S} = \frac{c}{n} u\,\hat{\epsilon}_3 \qquad (n^2 = \mu\epsilon c^2)$$

$$\Rightarrow \boxed{\begin{aligned}\langle\vec{S}\rangle &= \frac{n}{\mu c}\frac{1}{2}\vec{E}^*\cdot\vec{E}\;\hat{\epsilon}_3 \\ &= \frac{1}{2}\frac{n}{\mu c}\left(|E_1|^2 + |E_2|^2\right)\hat{\epsilon}_3\end{aligned}}$$

4) Normal Incidence



$\vec{k}_1 = k_1\hat{z} = -\vec{k}_1'$

$\vec{k}_2 = k_2\hat{z}$

$k_1 = \frac{n_1\omega}{c}$, $k_2 = \frac{n_2\omega}{c}$

$$\vec{E}_1 = \hat{i}\,E_1\,e^{i(k_1z-\omega t)}$$
$$\vec{E}_1' = -\hat{i}\,E_1'\,e^{-i(k_1z+\omega t)}$$
$$\vec{E}_2 = \hat{i}\,E_2\,e^{i(k_2z-\omega t)}$$

B.C. : $E_t^{(1)} = E_t^{(2)}$

$B_t^{(1)} = B_t^{(2)}$

$$c\vec{B}_1 = n_1\hat{z}\times\vec{E}_1 = \hat{j}\,n_1E_1e^{i(k_1z-\omega t)}$$
$$c\vec{B}_1' = -n_1\hat{z}\times\vec{E}_1' = \hat{j}\,n_1E_1'e^{-i(k_1z+\omega t)}$$
$$c\vec{B}_2 = n_2\hat{z}\times\vec{E}_2 = \hat{j}\,n_2E_2e^{i(k_2z-\omega t)}$$

$\Rightarrow$

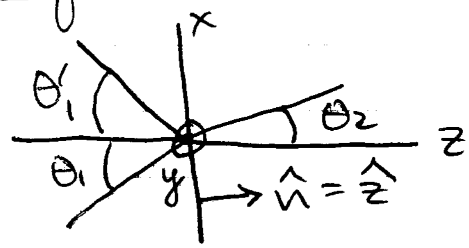
$$E_1' = \left(\frac{n_2-n_1}{n_1+n_2}\right)E_1$$

$$E_2 = \left(\frac{2n_1}{n_1+n_2}\right)E_1$$

$$R = \frac{|\langle\vec{S}_1'\rangle|}{|\langle\vec{S}_1\rangle|} = \frac{|E_1'|^2}{|E_1|^2} = \left[\frac{n_1-n_2}{n_1+n_2}\right]^2 ;\; R+T=1.$$

IV.A5) Quick Review of Oblique Incidence

Wave Equation:



$$\left(\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

Plane Wave: $\vec{E} = \vec{E}_0\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}$

$$\Rightarrow \quad \vec{k}\cdot\vec{k} - \epsilon\mu\,\omega^2 = 0 \quad , \quad \vec{k} = k\,\hat{\epsilon}_3$$

$$\Rightarrow \quad k = \sqrt{\epsilon\mu}\,\omega = \boxed{n\frac{\omega}{c} = k}$$

So Incident Wave $\quad \vec{E}_1 = \vec{E}_{10}\, e^{i(\vec{k}_1\cdot\vec{r} - \omega t)}$

Reflected Wave $\quad \vec{E}_1' = \vec{E}_{10}'\, e^{i(\vec{k}_1'\cdot\vec{r} - \omega t)}$

Transmitted Wave $\quad \vec{E}_2 = \vec{E}_{20}\, e^{i(\vec{k}_2\cdot\vec{r} - \omega t)}$

Maxwell's Eq. $\quad \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$

$$\Rightarrow \quad \vec{E}_{10} = E_{11}\,\hat{\epsilon}_{11}(\vec{k}_1) + E_{12}\,\hat{\epsilon}_{12}(\vec{k}_1)$$

$$\vec{E}_{10}' = E_{11}'\,\hat{\epsilon}_{11}'(\vec{k}_1') + E_{12}'\,\hat{\epsilon}_{12}'(\vec{k}_1')$$

$$\vec{E}_{20} = E_{21}\,\hat{\epsilon}_{21}(\vec{k}_2) + E_{22}\,\hat{\epsilon}_{22}(\vec{k}_2)$$

IV.A5.) Where $\{\hat{\epsilon}_1(\vec{k}), \hat{\epsilon}_2(\vec{k}), \hat{\epsilon}_3(\vec{k})\}$ form a right-handed orthonormal set of basis vectors.

Maxwell's Eq.: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0$

where

$$\vec{B} = \vec{B}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

and likewise $\vec{B}_0\cdot\vec{k} = 0 = \vec{B}_0\cdot\vec{E}_0$ so also $\{\vec{E}, \vec{B}, \vec{k}\}$ form a righthanded orthogonal set.

Boundary Conditions at $z=0$ i.e. on the x-y plane at all times

1) General: If E_t, B_t are to be continuous across the boundary (as well as B_n, E_n) then the phases of the plane waves on the boundary must be the same. $\Rightarrow$

a) ω the same

b) $\hat{n}, \vec{k}_1, \vec{k}_1', \vec{k}_2$ coplanar

so choose $\hat{\epsilon}_{12} = \hat{\epsilon}_{12}' = \hat{\epsilon}_{22} = \hat{j}$

c) $\theta_1 = \theta_1'$

d) $n_1 \sin\theta_1 = n_2 \sin\theta_2$

2) Relate $\hat{\epsilon}_{1,2,3}$ to $\hat{i}, \hat{j}, \hat{z}$:

$\hat{\epsilon}_{11} = \cos\theta_1\,\hat{i} - \sin\theta_1\,\hat{z}$	$\hat{\epsilon}_{11}' = -\cos\theta_1\,\hat{i} - \sin\theta_1\,\hat{z}$
$\hat{\epsilon}_{12} = \hat{j}$	$\hat{\epsilon}_{12}' = \hat{j}$
$\hat{\epsilon}_{13} = \sin\theta_1\,\hat{i} + \cos\theta_1\,\hat{z}$	$\hat{\epsilon}_{13}' = \sin\theta_1\,\hat{i} - \cos\theta_1\,\hat{z}$

IV.A.5) $\hat{\epsilon}_{21} = \cos\theta_2 \hat{i} - \sin\theta_2 \hat{z}$

$\hat{\epsilon}_{22} = \hat{j}$

$\hat{\epsilon}_{23} = \sin\theta_2 \hat{i} + \cos\theta_2 \hat{z}$

Solve BC for tangential components (normal compon. BC automatic)

$\Rightarrow$ i.e. $E_t^{①} = E_t^{②}$; $\frac{1}{\mu_1} B_t^{①} = \frac{1}{\mu_2} B_t^{②}$

$E'_{11} = r_{12}^{\parallel} E_{11}$ | $E_{21} = t_{12}^{\parallel} E_{11}$

$E'_{12} = r_{12}^{\perp} E_{12}$ | $E_{22} = t_{12}^{\perp} E_{12}$

Fresnel Coefficients

$$r_{12}^{\perp} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$r_{12}^{\parallel} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

$$t_{12}^{\perp} = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$t_{12}^{\parallel} = \frac{2 n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

Reflection & Transmission Coefficients:

$R^{\perp} = {r_{12}^{\perp}}^2$; $R^{\parallel} = {r_{12}^{\parallel}}^2$; $T^{\perp} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} {t_{12}^{\perp}}^2$

$T^{\parallel} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} {t_{12}^{\parallel}}^2$

IV A5) Properties:

1) $R^{\parallel}=0$ at $\theta_1=\theta_B$ Brewster's $\measuredangle$

$$\tan\theta_B=\frac{n_2}{n_1}$$

2) $R^{\parallel}=1=R^{\perp}$ at $\theta_1=\theta_c$ Critical $\measuredangle$

$$\sin\theta_c=\frac{n_2}{n_1}$$

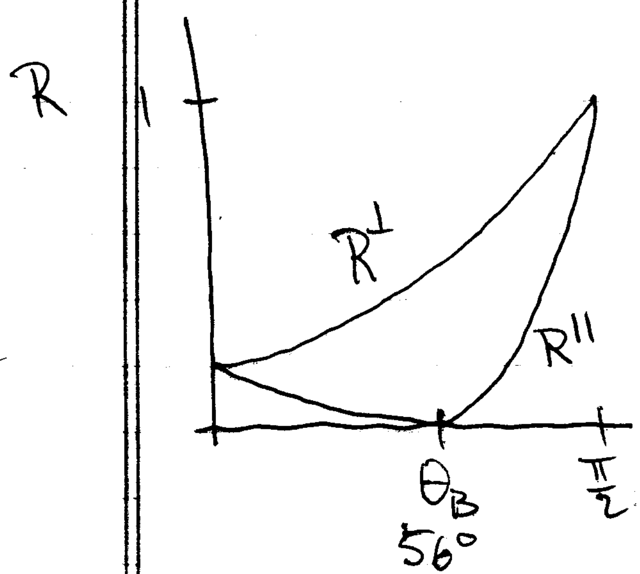
(since $\sin\theta_1\le 1$ critical $\measuredangle$ ocurrs for $n_1>n_2$)

For $\theta_1>\theta_c$ $\sin\theta_2=\frac{n_1}{n_2}\sin\theta_1>\frac{n_1}{n_2}\sin\theta_c$

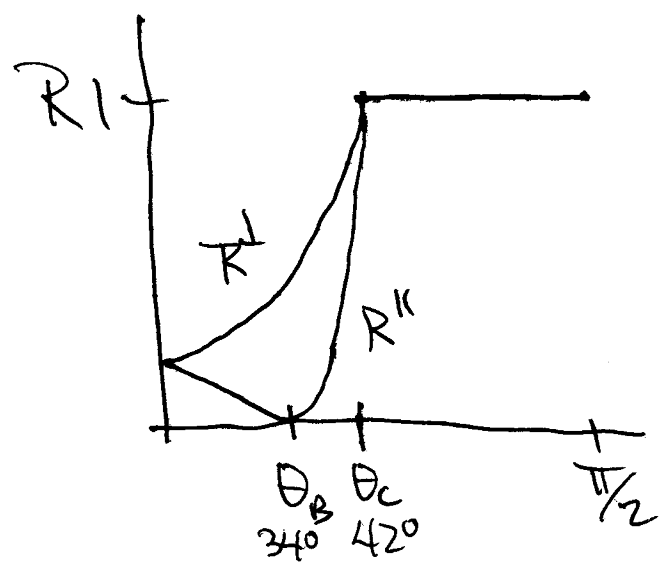
$$>\frac{n_1}{n_2}\frac{n_2}{n_1}>1 \quad !$$

θ_2 is complex ! $\Rightarrow$ reconsider the plane wave ansatz to include damping.

ex. $n_1=1$, $n_2=1.5$



ex. $n_1=1.5$, $n_2=1$



IV A. 5.) $\vec{E}_2 = \vec{E}_{20} e^{-k_2 Q z} e^{-i(\omega t - k_2 W x)}$ for $(\theta_1 \geq \theta_c)$

$\sin\theta_2 \equiv W \geq 1$; $\cos\theta_2 \equiv \sqrt{1-W^2} \equiv iQ$

$\Rightarrow \vec{E}_2 \sim e^{-k_2 Q z}$ damped in z-direction for $\theta_1 \geq \theta_c$

$$r_{12}^{\perp} = \frac{\cos\theta_1 - \frac{n_2}{n} iQ}{\cos\theta_1 + \frac{n}{n} iQ} \quad ; \quad r_{12}^{\parallel} = \frac{\cos\theta_1 - \frac{n_1}{n_2} iQ}{\cos\theta_1 + \frac{n_1}{n_2} iQ}$$

$$\Rightarrow \boxed{\begin{aligned} R^{\perp} &= |r_{12}^{\perp}|^2 = 1 \\ R^{\parallel} &= |r_{12}^{\parallel}|^2 = 1 \end{aligned}}$$

Total internal reflection (i.e. $T^{\perp} = 0 = T^{\parallel}$)

B) Conducting Media : $\boxed{\sigma_2 \neq 0}$

$\vec{E}_2(\vec{r},t) = e^{-i\omega t} \vec{E}_2(\vec{r}) \Rightarrow$

$[\nabla^2 + \omega^2 \epsilon\mu + i\omega\sigma\mu] \vec{E}_2(\vec{r}) = 0 \Rightarrow$

$\vec{E}_2(\vec{r}) = \vec{E}_{20} e^{i\vec{k}_2 \cdot \vec{r}}$ with $\vec{k}_2 = (k_{2r} + i k_{2i}) \hat{\epsilon}_{23}$
$= k_2 e^{i\phi_2} \hat{\epsilon}_{23}$

$\Rightarrow k_2 = \omega\sqrt{\epsilon_2 \mu_2} \left[1 + \left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2\right]^{1/4}$

$2\phi_2 = \tan^{-1}\left(\frac{\sigma_2}{\omega\epsilon_2}\right)$

IV B) or $$k_{2r} = \frac{\omega\sqrt{\epsilon_2\mu_2}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}+1\right]^{1/2}$$

$$k_{2i} = \frac{\omega\sqrt{\epsilon_2\mu_2}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}-1\right]^{1/2},$$

$$\vec{B}_2(\vec{r},t) = \vec{B}_{20}\, e^{-i(\omega t - \vec{k}_2\cdot\vec{r})} \; ; \; \vec{B}_{20} = \frac{\vec{k}_2}{\omega}\times\vec{E}_{20}$$

1) <u>Normal Incidence</u>: $\epsilon_1, \mu_1 = \mu_0$; $\epsilon_2, \mu_2 = \mu_0, \sigma_2$

$$\Rightarrow \quad E_1' = \left(\frac{k_2-k_1}{k_2+k_1}\right)E_1 \; ; \; E_2 = \left(\frac{2k_1}{k_2+k_1}\right)E_1$$

Good Conductor $\frac{\sigma_2}{\omega\epsilon_2} >> 1 \Rightarrow k_2 \simeq \frac{(1+i)}{\delta}$, $\phi_2 \simeq \frac{\pi}{4}$

$$\delta = \sqrt{\frac{2}{\mu_2\sigma_2\omega}} = \text{skin depth}$$

$\left(\langle\vec{S}\rangle \sim \vec{E}_0^*\cdot\vec{E}_0\right)$

$$\boxed{R = \frac{|E_1'|^2}{|E_1|^2} \simeq 1 - 2n_1\frac{\omega\delta}{c}}$$

2) <u>Oblique Incidence</u>: $\theta_2 = $ complex

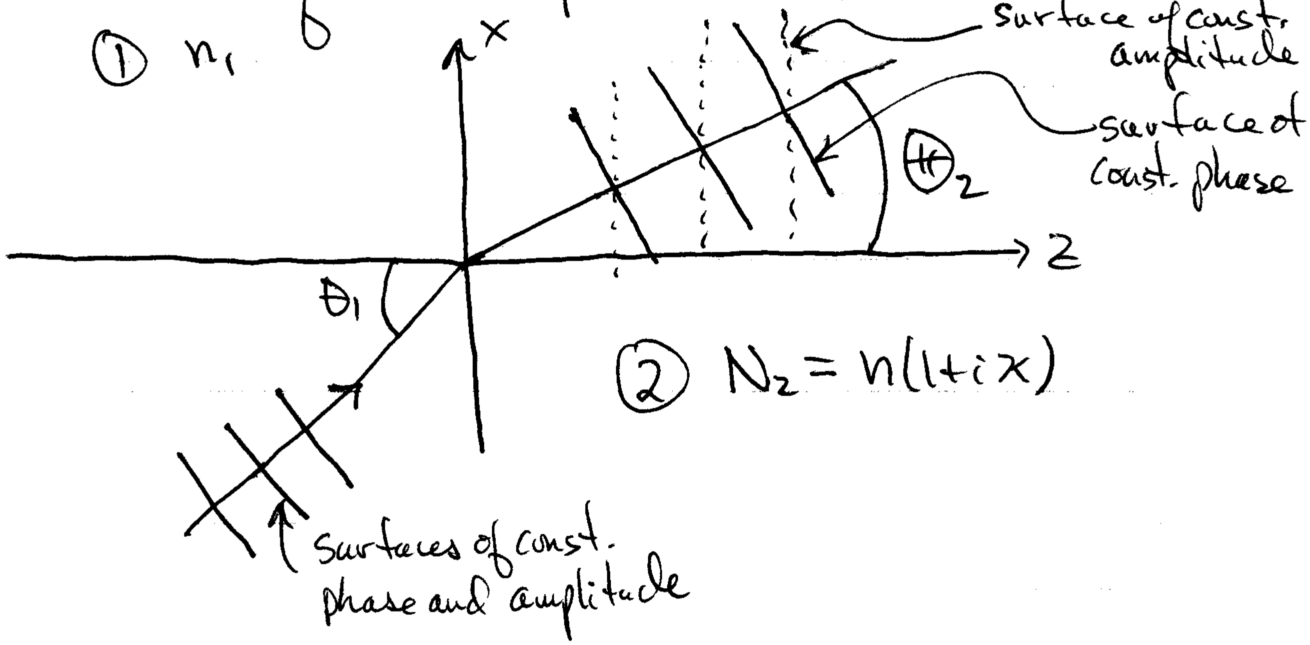
$$\vec{k}_2 = \frac{\omega n}{c}(1+i\kappa)\hat{\epsilon}_{23} \; ; \; k_{2r} = \frac{n\omega}{c} \; ; \; k_{2i} = \frac{n\omega}{c}\kappa$$

$$\boxed{n_1\sin\theta_1 = n(1+i\kappa)\sin\theta_2} \Rightarrow$$

$$\vec{E}_2 \sim e^{-z\left[\frac{\omega}{c}n\alpha(\kappa\cos\phi+\sin\phi)\right]}$$

(Recall class notes for details)

IV.B.2.) $\Rightarrow$ surface of constant amplitude : $z =$ constant.
while surface of constant phase make $\measuredangle\ \Theta_2$ with z-axis



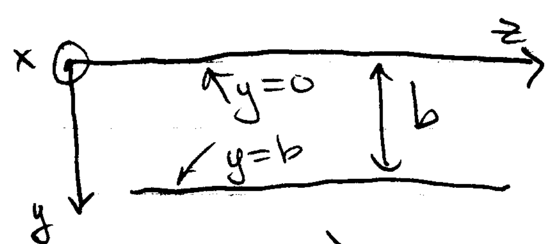
C) Propagation Between Perfect Conductors :

Consider plates $\parallel$ z-axis ; between plates vacuum : $\epsilon_0, \mu_0, \rho = \vec{J} = \sigma = 0$.

TE : $\vec{E}_1 = E\,\hat{i}\,e^{i(\vec{k}_0\cdot\vec{r}-\omega t)}$; $\vec{E}_1' = E'\,\hat{i}\,e^{i(\vec{k}_0'\cdot\vec{r}-\omega t)}$

$\vec{E}_2 = 0$: Perfect Conductors.

B.C. $E_t\Big|_{\text{surface}} = 0 = B_n\Big|_{\text{surface}}$



$y=0 \Rightarrow$ 1) $E' = -E$ since $\vec{E} = \vec{E}_1 + \vec{E}_1'$

$y=b$ 2) $kb\cos\theta = n\pi$, $n = 0, 1, 2, \ldots$

IV.C) cutoff wave length : $\lambda_c \equiv \frac{2b}{n}$

cutoff frequency : $\omega_n \equiv n\left(\frac{\pi c}{b\cos\theta}\right)$

guide wavelength : $\lambda_g = \frac{\lambda}{\sin\theta}$

guide propagation const. : $k_g = \frac{2\pi}{\lambda_g}$

$$\Rightarrow \quad k^2 = k_c^2 + k_g^2 \quad \text{or} \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2}.$$

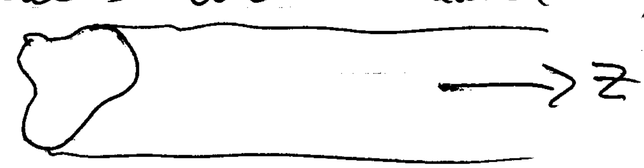
group vel. $v_{group} = c\sin\theta < c$

phase vel. $v_{phase} = \frac{\omega}{k_g} = \frac{c}{\sin\theta} > c$

$$= \frac{c\,\lambda_c}{\sqrt{\lambda_c^2 - \lambda^2}}$$

for $\lambda > \lambda_c$, v_{phase} is imaginary $\Rightarrow$ wave is attenuated in z direction.

D.) Wave Guides: Hollow conductors with perfect conducting walls and vacuum (air) inside.



Inside $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$

$$\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}\vec{E}_0(x,y)\\ \vec{B}_0(x,y)\end{pmatrix} e^{-i(\omega t - k_g z)}$$

IV b.) $\Rightarrow (\nabla_t^2 + k_c^2)\begin{pmatrix}\vec{E}_0\\ \vec{B}_0\end{pmatrix} = 0$; $\boxed{k_c^2 = k^2 - k_g^2}$, $\boxed{k = \frac{\omega}{c}}$

Let $\vec{E} = \vec{E}_z + \vec{E}_t$, $\vec{B} = \vec{B}_z + \vec{B}_t$ where

$$\begin{pmatrix}\vec{E}_z\\ \vec{B}_z\end{pmatrix} = \hat{z}\begin{pmatrix}E_z(x,y)\\ B_z(x,y)\end{pmatrix} e^{-i(\omega t - k_g z)}$$

$$\begin{pmatrix}\vec{E}_t\\ \vec{B}_t\end{pmatrix} = \begin{pmatrix}\overbrace{\hat{i}E_x(x,y) + \hat{j}E_y(x,y)}^{=\vec{E}_{t0}}\\ \underbrace{\hat{i}B_x(x,y) + \hat{j}B_y(x,y)}_{=\vec{B}_{t0}}\end{pmatrix} e^{-i(\omega t - k_g z)}$$

Maxwell's Eq.'s $\Rightarrow$ All transverse fields, E_x, E_y, B_x, B_y are given in terms of E_z, B_z, the field components $\parallel$ to the guide axis

1) TEM waves $\nexists$: $E_z = B_z = 0$ for waveguides with simply connected cross-sectional surfaces.

2) TE Waves $E_z = 0$, $B_z \neq 0$, with

$$\vec{B}_{t0} = \frac{ik_g}{k_c^2}\vec{\nabla} B_z$$

$$\vec{E}_{t0} = -\frac{\omega}{k_g}\hat{z}\times\vec{B}_{t0} = -i\frac{\omega}{k_c^2}\hat{z}\times\vec{\nabla} B_z$$

and B_z obeys the Helmholtz Eq. $\boxed{(\nabla_t^2 + k_c^2)B_z = 0}$

+ B.C. $\boxed{\left.\frac{\partial}{\partial n}B_z\right|_{\text{surface}} = 0 = \left.\hat{n}\cdot\vec{\nabla} B_z\right|_{\text{surface}}}$

III.D.3) TM Waves $B_z = 0$, $E_z \neq 0$, with

$$\vec{E}_{t_0} = \frac{i k_g}{k_c^2} \vec{\nabla} E_z$$

$$\vec{B}_{t_0} = \frac{\omega}{k_g} \hat{z} \times \vec{E}_{t_0} = i \frac{\omega}{k_c^2} \hat{z} \times \vec{\nabla} E_z$$

and E_z obeys the Helmholtz Eq.: $\boxed{(\nabla_t^2 + k_c^2) E_z = 0}$

+ B.C. $\boxed{\hat{n} \times \vec{E}\Big|_{\text{surface}} = 0}$

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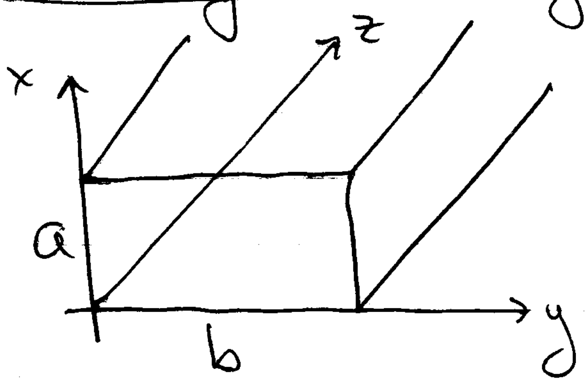
4) Rectangular Waveguides: TE Waves



Boundary Conditions:

$$\hat{n} \cdot \vec{\nabla} B_z \Big|_{\substack{\text{surface} \\ \text{of waveguide}}} = 0$$

$\Rightarrow$ 1) $\frac{\partial}{\partial x} B_z \Big|_{x=0} = 0$; 2) $\frac{\partial}{\partial x} B_z \Big|_{x=a} = 0$

3) $\frac{\partial}{\partial y} B_z \Big|_{y=0} = 0$; 4) $\frac{\partial}{\partial y} B_z \Big|_{y=b} = 0$.

IV D 4.) and $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)B_z=0$

$$\Rightarrow \boxed{B_z = B\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}},\ m,n=0,1,2,\ldots$$

with $\boxed{k_c^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}$

T_{mn} Mode Cutoff Frequency $\omega_{mn}\equiv ck_c \Rightarrow$

$$\boxed{\omega_{mn}=\pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\ ;\ m,n=0,1,2,\ldots$$

a) $m=n=0 \Rightarrow \vec{B}=0=\vec{E}$ so $TE_{00}\ \nexists$.

b) For $a<b$ TE_{01} is the Principal Mode (or Dominant Mode)

all waves with $\omega<\omega_{01}$ will be attenuated $\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}\sim e^{-|k_g|z}$ since

$$k_g=\frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2}\,.$$

TE_{01} Mode: $E_z=0$, $E_y=0$

$$E_x=-\frac{i\omega b}{\pi}B\sin\left(\frac{\pi y}{b}\right)e^{-i(\omega t-k_g z)}$$

$B_x=0$

$$B_z=B\cos\left(\frac{\pi y}{b}\right)e^{-i(\omega t-k_g z)}$$

$$B_y=\frac{-ik_g b}{\pi}B\sin\left(\frac{\pi y}{b}\right)\times e^{-i(\omega t-k_g z)}$$

IV D.4.b.) $\langle\vec{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\vec{E}^* \times \vec{B})$, so for TE_{01} $\Rightarrow$

$$\langle\vec{S}\rangle_{01} = \frac{1}{2\mu_0}\hat{z}\, E_x^* B_y$$
$$= \hat{z}\,\frac{\omega k_g}{2\mu_0}\,\frac{b^2}{\pi^2}B^2 \sin^2\left(\frac{\pi y}{b}\right)$$

Hence the Power transmitted in the z-direction

$$P_{01} = \hat{z}\cdot\int_0^a dx\int_0^b dy\,\langle\vec{S}\rangle_{01}$$

$$\boxed{P_{01} = \frac{ab^3}{4\pi^2\mu_0}\,\omega k_g B^2}$$

V) Emission Of Electromagnetic Radiation:

We are now concerned with the production of E-M radiation from the prescribed sources $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ in linear media.

Re-express Maxwell's Eq.'s in terms of the scalar, φ, and vector, $\vec{A}$, potentials.

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad (\Rightarrow \vec{\nabla}\cdot\vec{B}=0)$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} \quad \left(\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right)$$

Given $(\varphi,\vec{A})$, $\vec{E}$ & $\vec{B}$ are given above; But given $\vec{E},\vec{B}$, the potentials $\varphi,\vec{A}$ are only defined up to a gauge transformation

Let $\Lambda = \Lambda(\vec{r},t)$ be an arbitrary scalar function then let

$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\Lambda \\ \varphi' &= \varphi - \frac{\partial\Lambda}{\partial t} \end{aligned}\right\} \text{Gauge Transformation of potentials by } \Lambda$$

Then $\vec{E}$ and $\vec{B}$ are unchanged

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}$$

$$\vec{E}' = -\vec{\nabla}\varphi' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} = \vec{E}.$$

V) Choose Λ for convenience: Radiation Problems choose the <u>Lorentz</u> <u>Gauge</u>

$$\boxed{\vec{\nabla}\cdot\vec{A} + \epsilon\mu\frac{\partial\varphi}{\partial t} = 0.}$$

The remaining Maxwell Eq.'s, $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ and $\vec{\nabla}\cdot\vec{D} = \rho$ become the <u>Inhomogeneous</u> <u>Wave</u> <u>Equations</u> for the potentials

$$\boxed{\begin{aligned}\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\varphi &= -\frac{1}{\epsilon}\rho \\ \left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\vec{A} &= -\mu\vec{J}\end{aligned}}$$

(Note: If $\vec{\nabla}\cdot\vec{A} + \epsilon\mu\frac{\partial\varphi}{\partial t} = a(\vec{r},t) \neq 0$, then define $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, $\varphi' = \varphi - \frac{\partial\Lambda}{\partial t}$ and choose Λ so that $\vec{\nabla}\cdot\vec{A}' + \epsilon\mu\frac{\partial\varphi'}{\partial t} = 0$. That is

$$0 = \vec{\nabla}\cdot\vec{A}' + \epsilon\mu\frac{\partial\varphi'}{\partial t} = \underbrace{\vec{\nabla}\cdot\vec{A} + \epsilon\mu\frac{\partial\varphi}{\partial t}}_{a(\vec{r},t)} + \left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\Lambda$$

$\Rightarrow$

$$\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\Lambda(\vec{r},t) = -a(\vec{r},t).$$

Thus, for Λ a solution to this inhomogeneous wave equation, then φ' and $\vec{A}'$ are in the Lorentz Gauge.)

VI) The solution to the inhomogeneous wave equation can be found by Green function techniques

$$\left(\nabla_{\vec{r}}^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right) G(\vec{r},t;\vec{r}\,',t') = -\delta^3(\vec{r}-\vec{r}\,')\delta(t-t')$$

G = Green function. + B.C. outgoing waves

$$\Rightarrow \quad G(\vec{r},t;\vec{r}\,',t') = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\delta\left(t-t'-\sqrt{\epsilon\mu}\,|\vec{r}-\vec{r}\,'|\right)$$

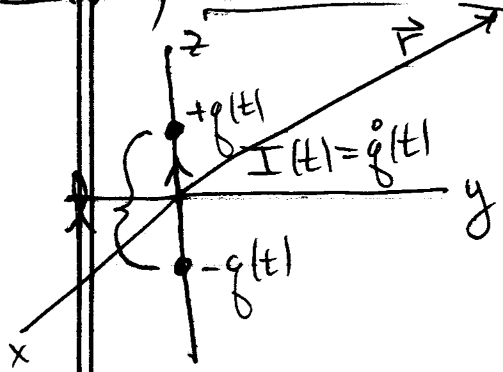
where $t_R \equiv t - \sqrt{\epsilon\mu}\,|\vec{r}-\vec{r}\,'|$ is called the retarded time

$$\Rightarrow \quad \boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{1}{4\pi\epsilon}\int_V \frac{\rho(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,d\tau' \\ \vec{A}(\vec{r},t) &= \frac{\mu}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,d\tau' \end{aligned}}$$

$$\left(\Lambda(\vec{r},t) = \frac{1}{4\pi}\int_V \frac{\left[\vec{\nabla}\cdot\vec{A} + \epsilon\mu\frac{\partial\varphi}{\partial t}\right](\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,d\tau' \right)$$

$$\boxed{\text{where } t' = t_R = t - \sqrt{\epsilon\mu}\,|\vec{r}-\vec{r}\,'| \text{ the retarded time}}$$

V. A.) Radiation From An Oscillating Dipole:



$$\vec{J}\,d\tau' \rightarrow \hat{z}\, I\, dz' \; ; \; I = \dot{q}$$

$$A_x = 0 = A_y$$

$$A_z = \frac{\mu_0}{4\pi}\int_{-l/2}^{+l/2} \frac{I\left(t - \frac{|\vec{r} - z'\hat{z}|}{c}\right)}{|\vec{r} - z'\hat{z}|}\, dz'$$

1) $l \ll r$; 2) $\frac{l}{2} \ll cT$ $\Rightarrow$

$$\boxed{A_z(\vec{r},t) \approx \frac{\mu_0}{4\pi}\frac{l}{r} I\left(t - \frac{r}{c}\right)}$$

use Lorentz gauge condition $\frac{\partial \varphi}{\partial t} = -c^2 \vec{\nabla}\cdot\vec{A}$

$\Rightarrow$

$$\boxed{\varphi(\vec{r},t) \approx \frac{l}{4\pi\epsilon_0}\frac{\cos\theta}{r}\left[\frac{q\left(t-\frac{r}{c}\right)}{r} + \frac{I\left(t-\frac{r}{c}\right)}{c}\right]}$$

Let $q(t) = q_0 \cos\omega t \Rightarrow I(t) = -q_0\omega \sin\omega t$

$$\equiv I_0 \sin\omega t$$

V.A.) $\Rightarrow$ $$A_z = \frac{\mu_0}{4\pi}\frac{l}{r} I_0 \sin\omega(t-\tfrac{r}{c})$$

$$\varphi = \frac{l}{4\pi\epsilon_0}\frac{\cos\theta}{r}\left[\frac{q_0\cos\omega(t-\frac{r}{c})}{r} + \frac{I_0}{c}\sin\omega(t-\tfrac{r}{c})\right]$$

In spherical polar coordinates: $A_r = A_z\cos\theta$
$A_\theta = -A_z\sin\theta$
$A_\varphi = 0$.

In the <u>radiation zone</u> $(r \gg \lambda = \frac{c}{\omega})$

$$B_r = B_\theta = 0 \qquad E_r = E_\varphi = 0$$

$$B_\varphi = \frac{\mu_0}{4\pi}\frac{l I_0 \sin\theta}{r}\frac{\omega}{c}\cos\omega(t-\tfrac{r}{c})$$

$$E_\theta = \frac{l I_0 \sin\theta}{4\pi\epsilon_0 r}\frac{\omega}{c^2}\cos\omega(t-\tfrac{r}{c})$$

Hence the Power radiated through a sphere of radius R in the radiation zone is

$$P = \oint_{\substack{\text{sphere of}\\ \text{radius } R}} \vec{S}\cdot\hat{r}\,da \qquad (da = R^2 d\Omega)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}E_\theta B_\varphi\,\hat{r}$$

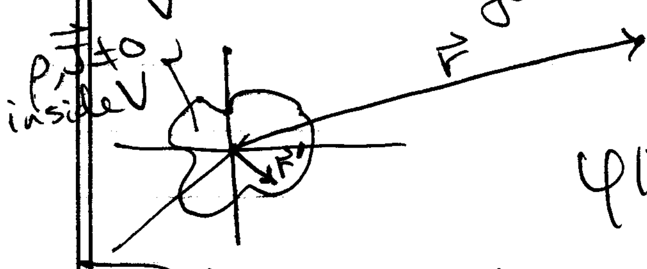
V.A) $\Rightarrow$

$$P = \frac{l^2 I_0^2}{6\pi\epsilon_0}\frac{\omega^2}{c^3}\cos^2\omega\left(t-\frac{R}{c}\right)$$

Time average Power radiated away from the dipole is ~~is~~

$$\boxed{\langle P\rangle = \frac{l^2\omega^2}{6\pi\epsilon_0 c^3}\frac{I_0^2}{2}}$$

B) Radiation from a group of moving charges:
(i.e. charge-current distribution)



$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\,d\tau'$$

Taylor expand $\Rightarrow$

$$\boxed{\varphi(\vec{r},t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\vec{r}\cdot\vec{p}(t-\frac{r}{c})}{r^3} + \frac{\vec{r}\cdot\dot{\vec{p}}(t-\frac{r}{c})}{cr^2} + \dots\right]}$$

where $Q = \int_V \rho(\vec{r}', t-\frac{r}{c})\,d\tau'$ = constant charge enclosed in V.

$$\vec{p}(t-\tfrac{r}{c}) = \int_V \vec{r}'\,\rho(\vec{r}', t-\tfrac{r}{c})\,d\tau',$$

The dipole moment of the charge distribution.

VB.) Likewise $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} d\tau'$

Taylor Expand $\Rightarrow$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r} \underbrace{\int_V \vec{J}(\vec{r}', t - \tfrac{r}{c}) d\tau'}_{= \dot{\vec{p}}(t - \frac{r}{c})} + \cdots$$

by continuity eq. $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0.$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r}\dot{\vec{p}}(t - \tfrac{r}{c}) + \cdots}$$

(Agrees with Lorentz Condition $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$)

In <u>Radiation Zone</u> $r \gg \lambda$ only $\frac{1}{r}$ parts of $\vec{E}, \vec{B}$ fields retained

$\Rightarrow$ $\vec{B} = \vec{\nabla}\times\vec{A}$

$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$ $\Rightarrow$

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= -\frac{c}{r}\,\vec{r}\times\vec{B}(\vec{r},t) \\ \vec{B}(\vec{r},t) &= -\frac{\mu_0}{4\pi c r^2}\,\vec{r}\times\ddot{\vec{p}}(t - \tfrac{r}{c}) \end{aligned}}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{c}{\mu_0 r}\,\vec{r}\,(\vec{B}^2)$$

V.B) Hence

$$\boxed{\vec{S} = \frac{1}{16\pi^2 \epsilon_0 c^3 r^5}\, \vec{r}\,(\vec{r}\times\ddot{\vec{p}})^2}$$

Let θ be the $\measuredangle$ between $\vec{r}$ & $\ddot{\vec{p}}$ (ex. $\ddot{\vec{p}}$ is along z-axis)

$$\boxed{\vec{S} = \frac{(\ddot{\vec{p}}^{\,2})\sin^2\theta}{16\pi^2 \epsilon_0 c^3 r^2}\,\frac{\vec{r}}{r}}$$

Hence, the power radiated through a large sphere of radius R is

$$\boxed{P_R = \oint_{\substack{\text{sphere of}\\ \text{radius } R}} \vec{S}\cdot\hat{r}\,da = \frac{1}{4\pi\epsilon_0}\,\frac{2}{3}\,\frac{\ddot{\vec{p}}^{\,2}(t-\frac{R}{c})}{c^3}}$$

In the case of the oscillating dipole

$$\vec{p} = \hat{z}\,l\,q(t) = -\hat{z}\,\frac{I_0 l}{\omega}\cos\omega t$$

$\Rightarrow$

$$P_R = \frac{1}{6\pi\epsilon_0}\,\frac{(I_0 l\omega)^2}{c^3}\cos^2\omega(t-\tfrac{R}{c})$$

as on page -23-

V. C.) Radiation From A Point Charge: Liénard-Wiechert Potentials

On page -20- the potentials have integrands evaluated at the retarded time $t' = t_R = t - \frac{|\vec{r}-\vec{r}'|}{c}$. Thus we can use the properties of the Dirac δ-function to write them as

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_V d\tau' \int_{-\infty}^{+\infty} dt'\, \delta\left(t' - \left[t - \frac{|\vec{r}-\vec{r}'|}{c}\right]\right)\frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V d\tau' \int_{-\infty}^{+\infty} dt'\, \delta\left(t' - \left[t - \frac{|\vec{r}-\vec{r}'|}{c}\right]\right)\frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

For a point charge q, $\rho(\vec{r},t) = q\,\delta^3(\vec{r}-\vec{r}_q(t))$ and $\vec{J}(\vec{r},t) = \vec{v}_q(t)\rho(\vec{r},t) = q\vec{v}_q(t)\,\delta^3(\vec{r}-\vec{r}_q(t))$

where $\vec{r}_q(t)$, $\vec{v}_q(t) = \dot{\vec{r}}_q(t)$ are the point charge's position and velocity at time t.

Hence $$\varphi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} dt'\, \frac{\delta\left(t' - \left[t - \frac{|\vec{r}-\vec{r}_q(t')|}{c}\right]\right)}{|\vec{r}-\vec{r}_q(t')|}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}q\int_{-\infty}^{+\infty} dt'\, \vec{v}_q(t')\,\frac{\delta\left(t' - \left[t - \frac{|\vec{r}-\vec{r}_q(t')|}{c}\right]\right)}{|\vec{r}-\vec{r}_q(t')|}$$

V. c.) Using the properties of the Dirac δ-function

i.e. $$\delta(f(t')) = \frac{\delta(t'-t_0)}{f'(t_0)} \quad \text{where } f(t_0)=0,$$

we find the Liénard-Wiechert Potentials

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}_q(t_R)| - \frac{\vec{v}_q(t_R)}{c}\cdot(\vec{r}-\vec{r}_q(t_R))}\right] \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\, q\left[\frac{\vec{v}_q(t_R)}{|\vec{r}-\vec{r}_q(t_R)| - \frac{\vec{v}_q(t_R)}{c}\cdot(\vec{r}-\vec{r}_q(t_R))}\right] \\ &\text{where } t_R \equiv t - \frac{|\vec{r}-\vec{r}_q(t_R)|}{c} \text{ is the retarded time} \end{aligned}}$$

i) Uniform Motion: $\vec{v}_q$ = constant:

Let $\vec{v}_q = v\,\hat{i}$ and $\vec{r}_q(t) = v\,t\,\hat{i}$

Then $t_R \equiv t - \frac{|\vec{r} - v t_R \hat{i}|}{c}$ yields a quadratic equation for t_R

$$(c^2-v^2)t_R^2 - 2(tc^2 - xv)t_R + (t^2c^2 - r^2) = 0$$

V.C (1.) $\Rightarrow$
$$t_R(c^2-v^2) = (tc^2 - xv) - \sqrt{r^2(c^2-v^2)+x^2v^2-2xvtc^2+t^2c^2v^2}$$

(minus sign is chosen since $t_R\big|_{v=0} = t - \frac{r}{c}$)

This yields

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x-vt)^2+\left(1-\frac{v^2}{c^2}\right)(y^2+z^2)}} \\ \vec{A}(\vec{r},t) &= \frac{\vec{v}}{c^2}\varphi(\vec{r},t) \\ &= \frac{\mu_0}{4\pi}\frac{q\vec{v}}{\sqrt{(x-vt)^2+\left(1-\frac{v^2}{c^2}\right)(y^2+z^2)}} \end{aligned}}$$

Note: $v \to 0$ $\quad \varphi(\vec{r},t) \to \frac{q}{4\pi\epsilon_0 r}$

$\vec{A}(\vec{r},t) \to 0$

as it should.

V C.1.) The $\vec{E}$ & $\vec{B}$ fields are determined, as usual, from

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \frac{q}{4\pi\epsilon_0} \frac{\left(1-\frac{v^2}{c^2}\right)\left[(x-vt)\hat{i} + y\hat{j} + z\hat{z}\right]}{\left[(x-vt)^2 + \left(1-\frac{v^2}{c^2}\right)(y^2+z^2)\right]^{3/2}} \\ \vec{B}(\vec{r},t) &= \frac{1}{c^2}\, \vec{v} \times \vec{E}(\vec{r},t) \end{aligned}}$$

Remark: In the radiation zone $\vec{E}, \vec{B} \sim \frac{1}{r^2}$, hence No radiation is emitted from a uniformly moving charge.

2.) General Motion: $\vec{r}_q(t)$, $\vec{v}_q(t) = \dot{\vec{r}}_q(t)$

$$\vec{a}_q(t) = \ddot{\vec{r}}_q(t)$$

$t_R = t - \frac{|\vec{r} - \vec{r}_q(t_R)|}{c}$ implies that

$$\frac{\partial t_R}{\partial t} = \left[1 - \frac{\vec{v}_q(t_R)}{c} \cdot \frac{(\vec{r} - \vec{r}_q(t_R))}{|\vec{r} - \vec{r}_q(t_R)|}\right]^{-1}$$

V.C.2.) and

$$\vec{\nabla}_{\vec{r}}\, t_R = -\frac{1}{c}\frac{(\vec{r}-\vec{r}_q(t_R))}{|\vec{r}-\vec{r}_q(t_R)|}\left[1-\frac{\vec{v}_q(t_R)}{c}\cdot\frac{(\vec{r}-\vec{r}_q(t_R))}{|\vec{r}-\vec{r}_q(t_R)|}\right]^{-1}$$

Using the Liénard-Wiechert potentials on page -27- and the t_R derivatives we find, after a lengthy calculation, the $\vec{E}$ and $\vec{B}$ fields using

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t),$$

$$\boxed{\begin{aligned}\vec{E}(\vec{r},t) = {} & \frac{q}{4\pi\epsilon_0}\frac{1}{\left[|\vec{r}-\vec{r}_q(t_R)| - \frac{\vec{v}_q(t_R)}{c}\cdot(\vec{r}-\vec{r}_q(t_R))\right]^3}\times \\ & \times\Bigg(\left(1-\frac{\vec{v}_q^{\,2}(t_R)}{c^2}\right)\left((\vec{r}-\vec{r}_q(t_R)) - \frac{|\vec{r}-\vec{r}_q(t_R)|\,\vec{v}_q(t_R)}{c}\right) \\ & + \frac{1}{c^2}(\vec{r}-\vec{r}_q(t_R))\times\left[\left((\vec{r}-\vec{r}_q(t_R)) - \frac{|\vec{r}-\vec{r}_q(t_R)|\,\vec{v}_q(t_R)}{c}\right)\times\vec{a}_q(t_R)\right]\Bigg) \\ \vec{B}(\vec{r},t) = {} & \frac{1}{c}\frac{(\vec{r}-\vec{r}_q(t_R))}{|\vec{r}-\vec{r}_q(t_R)|}\times\vec{E}(\vec{r},t)\end{aligned}}$$

V.C.2.) Note: $\vec{E} \perp \vec{B}$ and $\vec{B} \perp (\vec{r} - \vec{r}_q(t_R))$.

In the radiation zone only the second set of terms in $\vec{E}$ contribute since they go as $\frac{1}{|\vec{r} - \vec{r}_q(t_R)|}$, they go like the $\vec{a}_q(t_R)$. Hence, $\vec{E} \perp (\vec{r} - \vec{r}_q(t_R))$ in the radiation zone also and $\vec{E}, \vec{B}, (\vec{r} - \vec{r}_q(t_R))$ form an orthogonal system. Thus

$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ becomes, in the radiation zone

$$\vec{S} = \frac{1}{\mu_0 c} (\vec{E}^2) \frac{(\vec{r} - \vec{r}_q(t_R))}{|\vec{r} - \vec{r}_q(t_R)|}.$$

Further for a slowly moving, but accelerating, charge, i.e. $\frac{|\vec{v}_q|}{c} \ll 1$, we find

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &\approx \frac{q}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}_q(t_R)) \times [(\vec{r} - \vec{r}_q(t_R)) \times \vec{a}_q(t_R)]}{|\vec{r} - \vec{r}_q(t_R)|^3 c^2} \\ \vec{B}(\vec{r},t) &\approx \frac{q}{4\pi\epsilon_0 c} \frac{\vec{a}_q(t_R) \times (\vec{r} - \vec{r}_q(t_R))}{|\vec{r} - \vec{r}_q(t_R)|^2 c^2} \end{aligned}}$$

VI.C.2.) Hence, the Poynting Vector becomes

$$\boxed{\vec{S} = \frac{q^2}{16\pi^2 \epsilon_0 c^3} \frac{(\vec{r}-\vec{r}_q(t_R))\left[(\vec{r}-\vec{r}_q(t_R))\times\vec{a}_q(t_R)\right]^2}{|\vec{r}-\vec{r}_q(t_R)|^5}}$$

Integrating over a large sphere of radius R centered on the retarded position of the charge, we obtain the <u>Larmor</u> (1897) <u>Formula</u> for power radiated by a non-relativistic, accelerating charged particle

$$P_R = \int_R \vec{S}\cdot d\vec{a}$$

$$\boxed{P_R = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3} \frac{\vec{a}_q(t_R)^2}{c^3}}$$