Review of Magnetostatics

Stationary charges $\Rightarrow$ electrostatics
Steady currents $\Rightarrow$ magnetostatics

I) Electric Currents

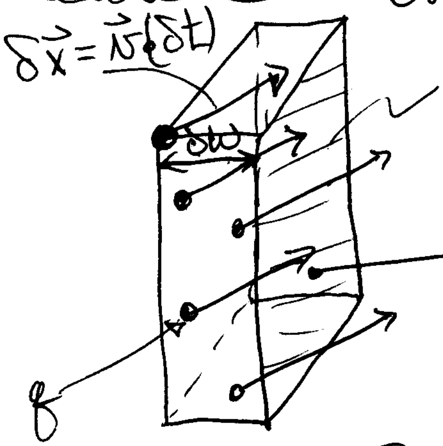
1) $I \equiv$ rate charge is transported thru a given surface

$$= \frac{dQ}{dt} = \frac{\text{coul}}{\text{sec}} = 1\text{ampere}$$

2) Current density (direction of positive charge flow)

Conducting medium $N = \frac{\#\text{ of charge carriers}}{\text{vol.}}$

each have charge q



$d\vec{S} = \hat{n}\, da$

$\vec{v}$ = drift velocity

Volume $= \delta w\, da = \hat{n} \cdot \vec{v}\, \delta t\, da$

δQ = charge pass thru da in δt

$$= (qN)(\text{volume}) = qN\, \vec{v} \cdot \hat{n}\, da\, \delta t$$

$$= qN\, \vec{v} \cdot d\vec{S}\, \delta t$$

Current flowing thru da $dI = \frac{\delta Q}{\delta t} = Nq\vec{v} \cdot d\vec{S}$

$$\equiv \vec{J} \cdot d\vec{S}$$

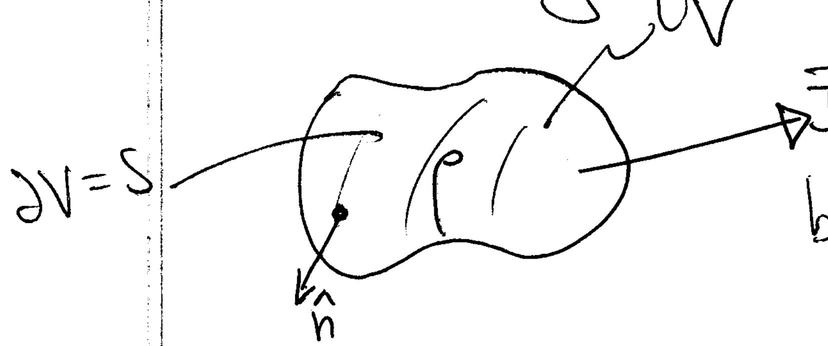
$\vec{J} = N q \vec{v} = \sum_i N_i q_i \vec{v}_i$ = current density
= Current/area
= flux rate of charge flow

$$\boxed{dI = \vec{J} \cdot d\vec{S}}$$

Current thru arb. surface

$$I = \int_S \vec{J} \cdot d\vec{S}$$

I.3.) Continuity equation



$I = +\oint_S \vec{J} \cdot \hat{n}\, da$

by Gauss's Thm.

$= +\int_V \vec{\nabla} \cdot \vec{J}\, dV$

$Q(t) = \int_V \rho(\vec{r}, t)\, dV$

$\vec{J}$ flows out, Q decreases $\xrightarrow{} \; = -\dfrac{dQ}{dt}$ in volume

$= -\int_V \dfrac{\partial \rho}{\partial t}\, dV$

V is arb.

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

II.4) In general $\vec{E}$ causes charges to drift $\Rightarrow \vec{J}$

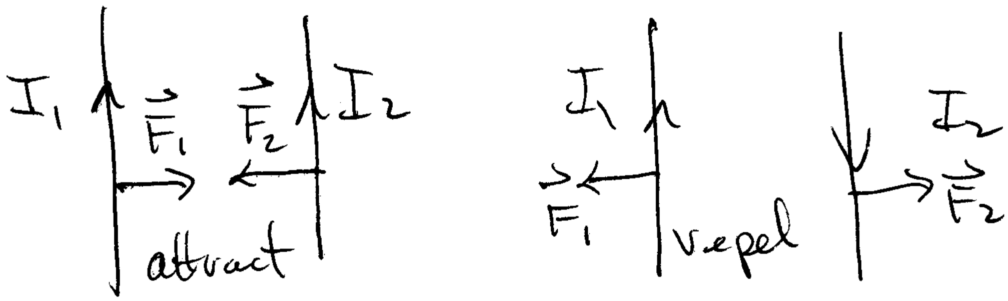
For Ohmic (linear) material

$$\vec{J} = \underbrace{g}_{\text{conductivity}} \vec{E}$$

$$\Rightarrow \quad \Delta\varphi = IR \, , \quad R = g\frac{A}{l} \quad \text{Ohm's law}$$

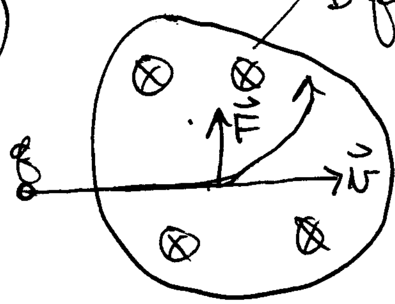
$$P = I^2 R = \frac{(\Delta\varphi)^2}{R} = I\Delta\varphi \, .$$

II) Lorentz Force Law



1) experiment $F_i \propto I_i \propto q v$ of charge in wire

2) $\vec{B}$ field into paper



q - moves $\perp$ to $\vec{v}$ & $\vec{B}$

$\Rightarrow \vec{F} \propto q\,\vec{v} \times \vec{B}$

Choose units of $\vec{B}$ so that

$\vec{F} = q\,\vec{v} \times \vec{B}$; add $\vec{E}$

Lorentz Force Law $\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$

units $B = \dfrac{\text{mass}}{\text{charge time}} = \text{Tesla}$

$(1\ \text{Tesla})(1\ \text{coul})\left(\dfrac{1\,\text{m}}{\text{sec}}\right) = 1\ \text{newton}$

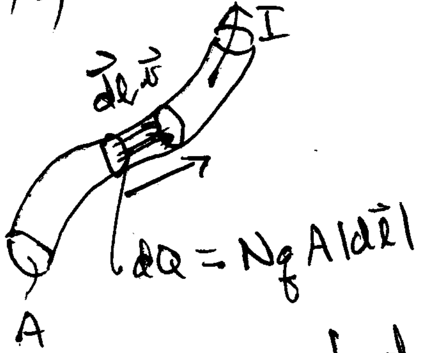
$10^{-4}\,T = 1\ \text{Gauss}$

Earth's B field $\sim .3 - .6\ G$

Electromagnets $\sim 2T \sim 100T$

nuclei $\sim 10^4 T$.

III./3) Currents interact with known $\vec{B}$-field



$$d\vec{F} = dQ\,\vec{v}\times\vec{B}$$
$$= NA_q|\vec{d\ell}|\vec{v}\times\vec{B} \quad \text{but } \vec{d\ell} \parallel \vec{v}$$
$$= NA_q|\vec{v}|\,\vec{d\ell}\times\vec{B}$$

but $I = NA_q|\vec{v}|$

So
$$\boxed{d\vec{F} = I\,\vec{d\ell}\times\vec{B}}$$

$\vec{B}$ uniform: $\vec{F} = \oint_C d\vec{F} = 0$ for closed circuit

4) Torque $\quad d\vec{\tau} = \vec{r}\times d\vec{F}$
$$= I\,\vec{r}\times(d\vec{r}\times\vec{B})$$

$$\vec{\tau} = I\oint_C \vec{r}\times(d\vec{r}\times\vec{B})$$

If $\vec{B}$ is uniform: $\vec{\tau} = I\vec{A}\times\vec{B} = \vec{m}\times\vec{B}$

magnetic dipole moment

$$\vec{A} = \iint_S d\vec{S} = \frac{1}{2}\oint_C \vec{r}\times d\vec{r}.$$

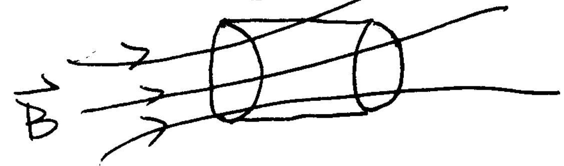
5) Current density $I = \vec{J}\cdot d\vec{S}$ So

$$I\,\vec{d\ell} = \vec{J}\cdot d\vec{S}\,\vec{d\ell} \quad \text{but } \vec{J}\parallel \vec{d\ell} \Rightarrow$$
$$= \vec{J}\,d\vec{S}\cdot\vec{d\ell} = \vec{J}\,dV$$

e.g. $d\vec{m} = \frac{1}{2}\vec{r}\times\vec{J}\,d\tau.$

III.) No Magnetic Charge:

Experimentally $\vec{B}$ lines are continuous never a source or sink $\Rightarrow$ flux of $\vec{B}$ thru any closed surface $= 0$



same # of lines enter as leave volume

$$\oint_S \vec{B}\cdot d\vec{S} = 0$$

$\parallel$ Gauss's thm.

$$\int_V \vec{\nabla}\cdot\vec{B}\,dV \text{ , V arbi} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Fundamental law of magnetism

no mag. monopoles.

IV.) Biot-Savart Law (1820) (Ampere experimental results lead to B-S Law)

B-S. found

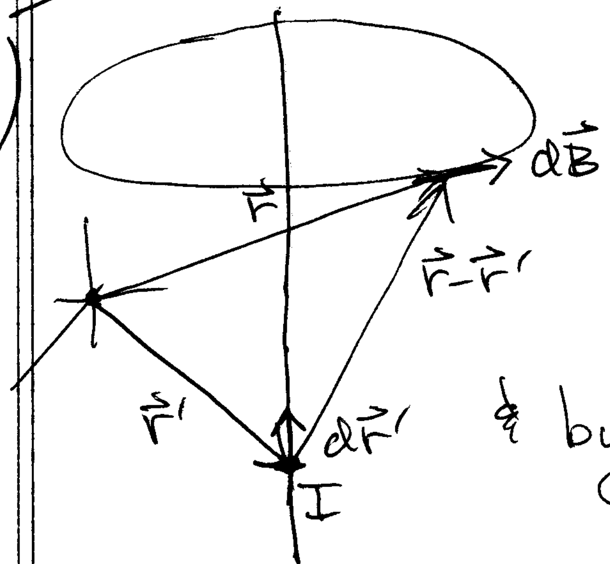
$$|d\vec{B}| \propto I\,|d\vec{r}\,'|$$

I current carrying line element $d\vec{r}\,'$

$$\propto \frac{1}{r^2}$$

$$\left.\begin{array}{l} d\vec{B} \perp \text{ to } d\vec{r}\,' \\ \perp \text{ to } (\vec{r}-\vec{r}\,') \end{array}\right\} \Rightarrow d\vec{B} \parallel d\vec{r}\,' \times (\vec{r}-\vec{r}\,')$$

IV.1) Law of B-S.



$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \frac{d\vec{r}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

& by superposition

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_C I \frac{d\vec{r}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} .$$

μ_0 = permeability of free space

$= 4\pi \times 10^{-7} \underbrace{\frac{\text{Kg-m}}{\text{coul}^2}}_{\frac{\text{nt}}{\text{amp}^2} = \frac{\text{Tesla m}}{\text{amp}}}$

$$\left(= \frac{\mu_0}{4\pi} \int_C I \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

If current flowing in matter

$I d\vec{r}' \longrightarrow \vec{J}(\vec{r}')dV'$

$I d\vec{r}' \to \vec{J}(\vec{r}')dV'$

Surface $I d\vec{r}' \longrightarrow \vec{K}(\vec{r}')dS'$

B-S law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dV'$$

IV. 2c) Consistency: $\vec{\nabla}\cdot\vec{B}=0$ check

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V dV' \quad \vec{\nabla}_{\vec{r}}\cdot\left[\frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]$$

Use $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = -\vec{A}\cdot(\vec{\nabla}\times\vec{B}) + \vec{B}\cdot(\vec{\nabla}\times\vec{A})$

but $\vec{\nabla}_{\vec{r}}\times\vec{J}(\vec{r}\,') = 0$ i.e. $\frac{\partial}{\partial x,y,z}$ not x',y',z'.

So $$\vec{\nabla}\cdot\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int_V dV'\,\vec{J}(\vec{r}\,')\cdot\vec{\nabla}_{\vec{r}}\times\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right)$$

Now $$\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

i.e. $$\frac{\partial}{\partial x}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= -\frac{1}{2}\frac{1}{|\vec{r}-\vec{r}\,'|^3}\,2(x-x')$$

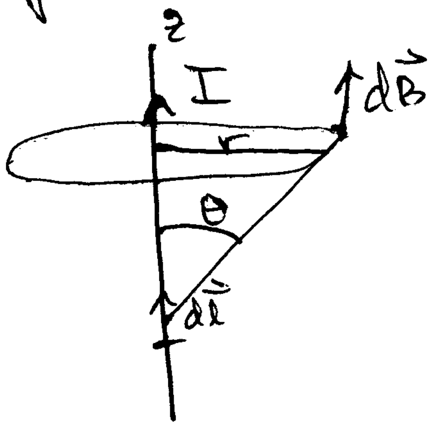
$$= -\frac{x-x'}{|\vec{r}-\vec{r}\,'|^3} \quad \text{etc.}$$

But $\vec{\nabla}\times\vec{\nabla}f = 0$!

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0} \checkmark$$

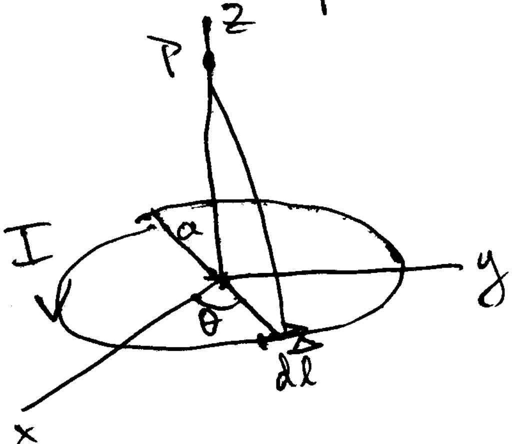
IV 3.) Examples:

1)



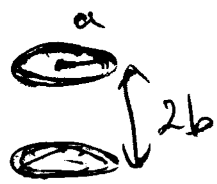
$$|\vec{B}| = \frac{\mu_0 I}{2\pi r}$$

2) Circular loop



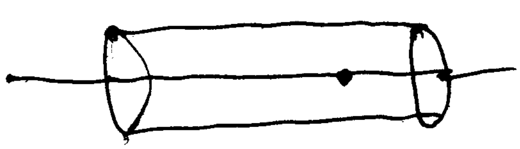
$$\vec{B}(z) = \frac{\mu_0 I}{2} \frac{a^2}{(z^2+a^2)^{3/2}} \hat{k}$$

3) Helmholtz Coils



use circular loop case

4) Solenoid: field on axis



messy: again use circular loop case.

V.1) Ampere's (Circuital) Law

B-S law was for <u>steady currents</u>:
current flows in wire steadily — no charge builds up just keeps flowing $\frac{\partial \rho}{\partial t} = 0$
$\Rightarrow$ $\vec{\nabla} \cdot \vec{J} = 0$ from continuity eq.

Under these conditions we find simple expression for $\vec{\nabla} \times \vec{B}$:

$$\vec{\nabla}_{\vec{r}} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \, \vec{\nabla}_{\vec{r}} \times \left[\frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \right]$$

Use
$$\vec{\nabla} \times (\vec{F} \times \vec{G}) = (\vec{\nabla} \cdot \vec{G}) \vec{F} - (\vec{\nabla} \cdot \vec{F}) \vec{G} + (\vec{G} \cdot \vec{\nabla}) \vec{F} - (\vec{F} \cdot \vec{\nabla}) \vec{G}$$

with $\vec{F} = \vec{J}(\vec{r}\,') \Rightarrow \frac{\partial}{\partial x_i} \vec{J}(\vec{r}\,') = 0$

$\vec{G} = \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \quad \Rightarrow$

$$\vec{\nabla}_{\vec{r}} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \left\{ \vec{J}(\vec{r}\,') \left(\vec{\nabla}_{\vec{r}} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) - \vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}} \left(\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) \right\}$$

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V. 1) But 1) $\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

hence
$$\vec{\nabla}_{\vec{r}}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = -\nabla_{\vec{r}}^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$
$$= -\left(-4\pi\delta^3(\vec{r}-\vec{r}')\right)$$
$$= 4\pi\,\delta^3(\vec{r}-\vec{r}')$$

So the first term RHS $= \frac{\mu_0}{4\pi}\int dV'\,\vec{J}(\vec{r}')\cdot 4\pi\delta^3(\vec{r}-\vec{r}')$
$$= \mu_0\,\vec{J}(\vec{r}).$$

Second term:

2) $\vec{J}(\vec{r}')\cdot\vec{\nabla}_{\vec{r}}\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = -\vec{J}(\vec{r}')\cdot\vec{\nabla}_{\vec{r}'}\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$

$$= +\vec{J}(\vec{r}')\cdot\vec{\nabla}_{\vec{r}'}\,\frac{\vec{r}'-\vec{r}}{|\vec{r}'-\vec{r}|^3}$$

But $\vec{\nabla}\cdot(\varphi\vec{F}) = (\vec{\nabla}\varphi)\cdot\vec{F} + \varphi\vec{\nabla}\cdot\vec{F}$

$$\Rightarrow \vec{\nabla}_{\vec{r}'}\cdot\left(\vec{J}(\vec{r}')\,\frac{x'-x}{|\vec{r}'-\vec{r}|^3}\right) = \cancel{\frac{x'-x}{|\vec{r}'-\vec{r}|^3}\,\vec{\nabla}_{\vec{r}'}\cdot\vec{J}(\vec{r}')}^{\,0 \text{ Steady current}}$$
$$+\vec{J}(\vec{r}')\cdot\vec{\nabla}_{\vec{r}'}\left(\frac{x'-x}{|\vec{r}'-\vec{r}|^3}\right)$$

Similarly for y, z.

V 2) So

$$\int_V \vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} \left(\frac{x'-x}{|\vec{r}\,'-\vec{r}|^3} \right) dV'$$

$$= \int_V dV' \, \vec{\nabla}_{\vec{r}\,'} \cdot \left(\vec{J}(\vec{r}\,') \frac{x'-x}{|\vec{r}\,'-\vec{r}|^3} \right)$$

$$= \oint_S \frac{x'-x}{|\vec{r}\,'-\vec{r}|^3} \vec{J}(\vec{r}\,') \cdot d\vec{S}' \equiv 0 .$$

Let S lie outside volume So $\vec{J}=0$ on S.

Thus $\int_V \vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} \frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3} dV' = 0.$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})}$$ Ampere's Law (differential form)

3) Integrate over open surface & use Stokes Thm

$$\int_S \vec{\nabla} \times \vec{B} \cdot d\vec{S} = \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{J} \cdot d\vec{S}$$

open surface

closed curve = boundary of S

V. 3.) But $I = \int_S \vec{J} \cdot d\vec{S}$ = total current flowing thru S i.e thru C.

So $$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I}$$ integral form of Ampere's law.

VI. Magneto-statics: $\frac{\partial \rho}{\partial t} = 0 = \vec{\nabla} \cdot \vec{J}$

$$\vec{\nabla} \times \vec{B} = \mu_0 J$$ Ampere's law

& always $\vec{\nabla} \cdot \vec{B} = 0$ Fundamental law of magnetism

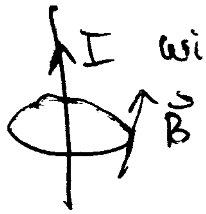
electrostatics $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ Gauss's law

$$\vec{\nabla} \times \vec{E} = 0$$

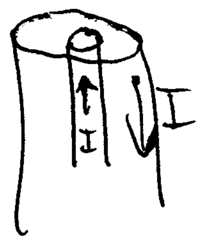
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

V. Ex. Ampere's law

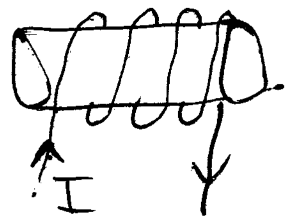
1) I wire



2) Coaxial cable



3) Solenoid



1) VII.) Magnetic Vector Potential

Thm. 1 p.57 $\vec{\nabla}\times\vec{E}=0 \Rightarrow \vec{E}=-\vec{\nabla}V$. scalar potential. V

Thm. 2 p.57 $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$ vector potential $\vec{A}$.

1) $\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\equiv 0$ ✓

2) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

but $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

Notice we can add $\vec{\nabla}\Lambda(\vec{r})$ to $\vec{A}$

$\vec{A}'\equiv\vec{A}+\vec{\nabla}\Lambda$ gauge transformation

$$\vec{B}'=\vec{\nabla}\times\vec{A}'=\underbrace{\vec{\nabla}\times\vec{A}}_{=\vec{B}}+\underbrace{\vec{\nabla}\times\vec{\nabla}\Lambda}_{=0}$$

$=\vec{B}$ ✓

We can choose Λ for our convenience — — We can use Λ to force $\vec{\nabla}\cdot\vec{A}'=0$ even if $\vec{\nabla}\cdot\vec{A}\neq 0$
$= f(\vec{r})$

So $\vec{\nabla}\cdot\vec{A}'=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{\nabla}\Lambda$

$0 \equiv f(\vec{r})+\nabla^2\Lambda$

VII 1) $\Rightarrow$ $\nabla^2 \Lambda(\vec{r}) = -f(\vec{r})$ Poisson's eq.

if $f \to 0$ as $|\vec{r}| \to \infty$ $\Rightarrow$ $\Lambda(\vec{r}) = \frac{1}{4\pi} \int dV' \frac{f(\vec{r}')}{|\vec{r}-\vec{r}'|}$

$$= \frac{1}{4\pi} \int dV' \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

(if $\vec{\nabla}\cdot\vec{A}$ does not drop off use other means to find $\Lambda(\vec{r})$ always possible).

Then we can always choose $\Lambda(\vec{r})$ so that $\boxed{\vec{\nabla}\cdot\vec{A} = 0 \quad \& \quad \vec{B} = \vec{\nabla}\times\vec{A}}$

This is called choosing a gauge; the gauge is called the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$.

But $\vec{\nabla}\cdot\vec{A} = 0$ $\Rightarrow$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{0}) - \nabla^2\vec{A}$$

Thus $\boxed{\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}(\vec{r})}$

again 3 Poisson's equations "x, y, z components"

VII.1) Assuming $\vec{J} \to 0$ as $|\vec{r}| \to \infty$

$$\Rightarrow \quad \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

First

Alternate derivation: B-S-law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \, \frac{\vec{J}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

As usual $\dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\vec{\nabla}_{\vec{r}} \dfrac{1}{|\vec{r}-\vec{r}\,'|}$

& $\vec{\nabla} \times (\varphi \vec{F}) = \varphi \vec{\nabla} \times \vec{F} - \vec{F} \times \vec{\nabla}\varphi$

$$\Rightarrow \vec{\nabla}_{\vec{r}} \times \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \vec{J}(\vec{r}\,') \right] = -\vec{J}(\vec{r}\,') \times \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|}$$

So
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V dV' \, \vec{\nabla}_{\vec{r}} \times \left[\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right]$$

$$= \vec{\nabla}_{\vec{r}} \times \underbrace{\left[\frac{\mu_0}{4\pi} \int_V dV' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right]}_{\equiv \vec{A}(\vec{r})} \checkmark .$$

definition $\Rightarrow \vec{\nabla} \cdot \vec{A} = 0$ Coul. gauge.

VII. 2.) For line currents

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

Surface Currents

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}\,')\,dS'}{|\vec{r}-\vec{r}\,'|}$$

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3) Mathematical Aside: If $\vec{\nabla}\cdot\vec{B}=0$ then $\vec{B}=\vec{\nabla}\times\vec{A}$ for some $\vec{A}$.

Proof by Construction of $\vec{A}$

$$\vec{\nabla}\times\vec{A}=\vec{B} \quad \begin{cases} \partial_y A_z - \partial_z A_y = B_x \\ \partial_z A_x - \partial_x A_z = B_y \\ \partial_x A_y - \partial_y A_x = B_z \end{cases}$$

Given $\vec{B}$, $\vec{A}$ is not unique — gauge transformations

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \Rightarrow \vec{B}' = \vec{B}$$

So choose Λ so that $A_x = 0 \Rightarrow A_x' = \partial_x \Lambda$

$$\Rightarrow \quad \Lambda = \int_0^x dx'\, A_x'(x',y,z) + \lambda(y,z)$$

Then

$$\partial_x A_y = B_z \Rightarrow A_y = \int_0^x dx' B_z(x', y, z) + C_y(y,z)$$

$$-\partial_x A_z = B_y \Rightarrow -A_z = \int_0^x dx' B_y(x', y, z) - C_z(y,z)$$

Now $\partial_y A_z - \partial_z A_y = B_x$

$$= -\int_0^x dx' \partial_y B_y(x', y, z) + \partial_y C_z(y,z)$$

$$- \int_0^x dx' \partial_z B_z(x', y, z) - \partial_z C_y(y,z) = B_x$$

But $\vec{\nabla} \cdot \vec{B} = 0 = \partial_x B_x + \partial_y B_y + \partial_z B_z$

$\Rightarrow$

$$B_x = \int_0^x dx' \left(\partial_{x'} B_x(x', y, z) + \partial_y B_y(x', y, z) \right)$$

$$- \int_0^x dx' \partial_y B_y(x', y, z) + \partial_y C_z - \partial_z C_y$$

$$B_x(x,y,z) = B_x(x,y,z) - B_x(0,y,z) + \partial_y C_z - \partial_z C_y$$

$$\Rightarrow \boxed{\partial_y C_z(y,z) - \partial_z C_y(y,z) = +B_x(0,y,z)}$$

Again use λ-gauge invariance

$$\vec{A}'' = \vec{A} + \vec{\nabla}\lambda \begin{cases} A_x'' = A_x + \cancelto{0}{\partial_x \lambda} \\ A_y'' = A_y + \partial_y \lambda \\ A_z'' = A_z + \partial_z \lambda \end{cases} \qquad \lambda = \lambda(y,z)$$

to choose $C_y = 0$. That is the equation

$$\partial_y C_z - \partial_z C_y = B_x$$

remains unchanged if
$$C_z' = C_z + \partial_z \lambda$$
$$C_y' = C_y + \partial_y \lambda$$

$\lambda = \lambda(y,z)$; choose $\lambda \ni C_y' = 0$

i.e.

$$\partial_y \lambda = -C_y \Rightarrow \lambda(y,z) = -\int_0^y dy' \, C_y(y',z) + l(z)$$

So

$$\partial_y C_z(y,z) = B_x(0,y,z)$$

and

$$\boxed{C_z(y,z) = \int_0^y dy' \, B_x(0,y',z) + C(z)}$$

Hence

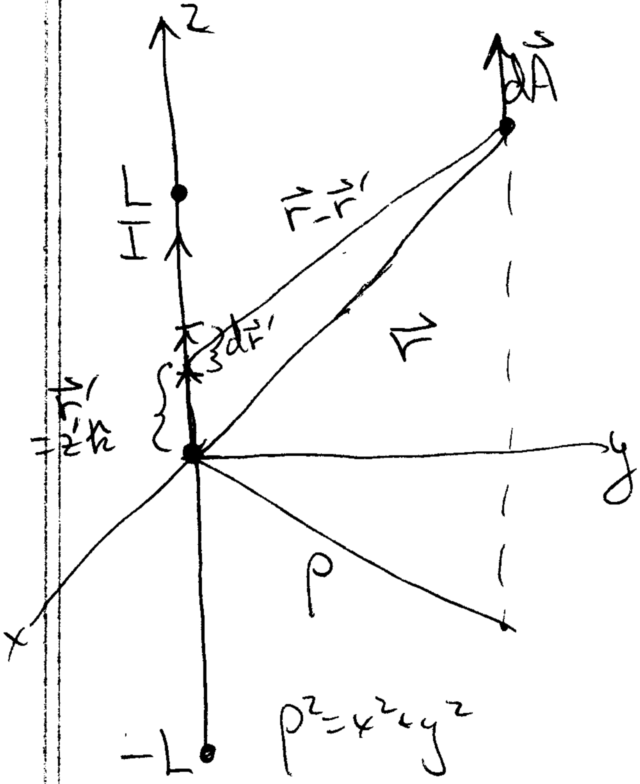
$$\boxed{\begin{aligned} A_x &= 0 \\ A_y &= \int_0^x dx' \, B_z(x',y,z) \\ A_z &= +\int_0^y dy' \, B_x(0,y',z) \\ &\quad - \int_0^x dx' \, B_y(x',y,z) + C(z) \end{aligned}}$$

Use final gauge invariance $l(z)$ to set $C = 0$.

So $\exists$ an $\vec{A}$ for every $\vec{B}$ with $\vec{\nabla}\cdot\vec{B}=0$
$\ni$ $\vec{B}=\vec{\nabla}\times\vec{A}$.

We finally can add to $\vec{A}$ any other $\vec{\nabla}\Lambda$ to recover general formula $\vec{B}=\vec{\nabla}\times\vec{A}$ for unconstrained $\vec{A}$ i.e. could choose Coulomb gauge for instance $\vec{\nabla}\cdot\vec{A}=0$.

VII.4) Example: Calculate $\vec{A}$ due to a finite length, steady current carrying wire



$$\vec{A}(\vec{r})=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz'\,\hat{k}}{|\vec{r}-\vec{r}'|}$$

$$=\frac{\mu_0 I\,\hat{k}}{4\pi}\int_{-L}^{+L}\frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$=\frac{\mu_0 I\,\hat{k}}{4\pi}\int_{u=-(L+z)}^{u=L-z}\frac{du}{\sqrt{x^2+y^2+u^2}}$$

$(u=z'-z\ ;\ du=dz')$

Now $\displaystyle\int \frac{du}{\sqrt{\rho^2+u^2}} = \ln\left[u+\sqrt{\rho^2+u^2}\right]$

check $\displaystyle\frac{d}{du}\left(\ln\left[u+\sqrt{\rho^2+u^2}\right]\right) = \frac{1}{u+\sqrt{\rho^2+u^2}}\left(1+\frac{1}{2}\frac{2u}{\sqrt{\rho^2+u^2}}\right)$

$$= \frac{1}{u+\sqrt{\rho^2+u^2}}\left(\frac{\sqrt{\rho^2+u^2}+u}{\sqrt{\rho^2+u^2}}\right)$$

$$= \frac{1}{\sqrt{\rho^2+u^2}} \quad \checkmark.$$

So

$$\vec{A}(\vec{r}) = \frac{\mu_0 I \hat{k}}{4\pi}\ln\left[u+\sqrt{\rho^2+u^2}\right]\Bigg|_{u=-(L+z)}^{u=+L-z}$$

$$= \frac{\mu_0 I \hat{k}}{4\pi}\ln\left[\frac{L-z+\sqrt{x^2+y^2+(L-z)^2}}{-L-z+\sqrt{x^2+y^2+(L+z)^2}}\right]$$

$$= \frac{\mu_0 I}{4\pi}\hat{k}\ln\left[\frac{(L-z)\left[1+\sqrt{1+\frac{x^2+y^2}{(L-z)^2}}\right]}{(L+z)\left[-1+\sqrt{1+\frac{x^2+y^2}{(L+z)^2}}\right]}\right]$$

Note for $L \gg x, y, z$

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0 I}{4\pi}\hat{k}\ln\left[\frac{L[1+1]}{L\left[-1+1+\frac{1}{2}\frac{x^2+y^2}{(L)^2}\right]}\right]$$

$$\simeq \frac{\mu_0 I}{4\pi}\hat{k}\ln\frac{4L^2}{(x^2+y^2)} \simeq \frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{\rho}\right)\hat{k}$$

So $\vec{A}(\vec{r}) \approx \frac{\mu_0 I \hat{k}}{2\pi} \ln\left(\frac{2L}{\rho}\right) \to \infty$ as $L \to \infty$.

But still $\vec{B}$ is finite as $L \to \infty$. We can find $\vec{B}$ for finite L, then take the limit

$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$; in cylindrical coordinates this is

$$\vec{B}(\vec{r}) = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\varphi}$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right]\hat{k}$$

But $A_\rho = A_\varphi = 0$ and

$$A_z = \frac{\mu_0 I}{4\pi} \ln\left[\frac{(L-z)\left[1+\sqrt{1+\frac{\rho^2}{(L-z)^2}}\right]}{(L+z)\left[-1+\sqrt{1+\frac{\rho^2}{(L+z)^2}}\right]}\right]$$

Since $\frac{\partial A_z}{\partial \varphi} = 0 \Rightarrow \boxed{\vec{B}(\vec{r}) = -\frac{\partial A_z}{\partial \rho}\hat{\varphi}}$

For $L \gg \rho, z$

$$A_z \approx \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{\rho}\right) \Rightarrow \vec{B}(\vec{r}) \approx -\frac{\mu_0 I \hat{\varphi}}{2\pi}\frac{\rho}{2L}\left(-\frac{2L}{\rho^2}\right)$$

$$\approx \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$$

So $\vec{B}(\vec{r}) \simeq \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$ and as $L\to\infty$ is finite and becomes $\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\hat{\varphi}$ as we obtained directly earlier.

VII.5) Finally, notice that $\vec{\nabla}\cdot\vec{B}=0$; $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{B} = \mu_0\vec{\nabla}\times\vec{J}$$

$$= \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B} = -\nabla^2\vec{B} = \mu_0\vec{\nabla}\times\vec{J}$$

So
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int dV' \frac{\vec{\nabla}_{\vec{r}'}\times\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Again $\vec{\nabla}\times(\varphi\vec{F}) = \varphi\vec{\nabla}\times\vec{F} - \vec{F}\times\vec{\nabla}\varphi$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int dV' \cancelto{0}{\vec{\nabla}_{\vec{r}'}\times\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)} \text{ surface term}$$

$$+ \frac{\mu_0}{4\pi}\int dV' \vec{J}(\vec{r}')\times\vec{\nabla}_{\vec{r}'}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= +\frac{\mu_0}{4\pi}\int dV' \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ The Biot-Savart law

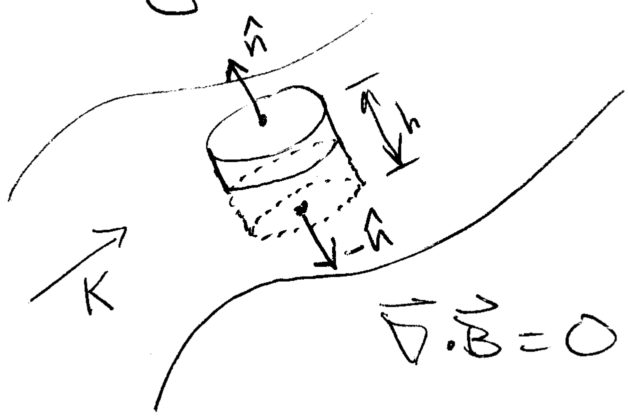
Likewise $\vec{\nabla}\cdot\vec{A}=0$ Coulomb gauge and

$$\vec{\nabla}\times\vec{A}=\vec{B}$$

$$\Rightarrow \quad \vec{A}(\vec{r})=\frac{1}{4\pi}\int dV'\,\frac{\vec{B}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

VIII.) Boundary Conditions;

1)



$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \oint_S \vec{B}\cdot d\vec{S}=0$$

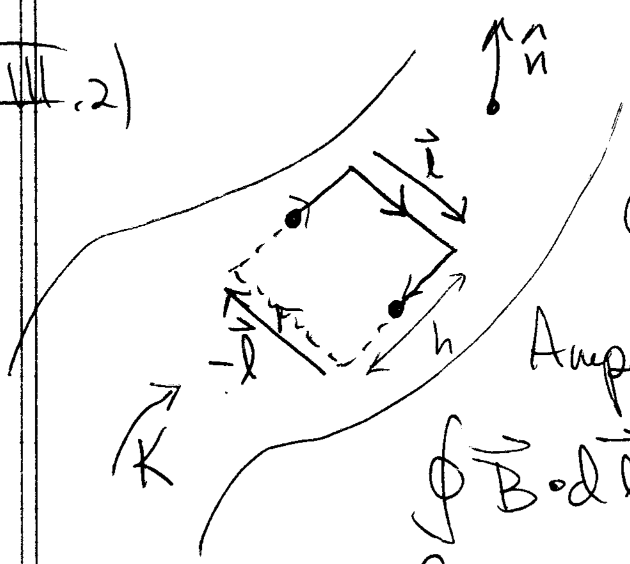
$$=\Big(\vec{B}\cdot\hat{n}\Big|_{\text{above}}-\vec{B}\cdot\hat{n}\Big|_{\text{below}}\Big)\Delta a$$

$$+\underbrace{\int_{\substack{\text{cylinder}\\ \text{height } h}}\vec{B}\cdot d\vec{S}}\sim O(h)\to 0$$

$$\Rightarrow \boxed{B_{\perp\,\text{above}}=B_{\perp\,\text{Below}}}$$

The normal component of $\vec{B}$ across any surface is continuous.

VIII.2)



Consider amperian loop

Ampere's law

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}.$$

$$I_{enc} = \vec{K}\cdot(\hat{n}\times\vec{l})$$

$$\oint_C \vec{B}\cdot d\vec{l} = \left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\vec{l} + \underbrace{\int_{sides} \vec{B}\cdot d\vec{l}}_{\sim O(h) \to 0.}$$

Thus $$\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\vec{l} = \mu_0 \vec{K}\cdot(\hat{n}\times\vec{l})$$

using $\vec{K}\cdot(\hat{n}\times\vec{l}) = \vec{l}\cdot(\vec{K}\times\hat{n})$

$$\Rightarrow \boxed{\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\vec{l} = \vec{l}\cdot\left(\mu_0 \vec{K}\times\hat{n}\right)}$$

The tangential component of $\vec{B}$ is discontinuous across a surface with current density.

Since $\vec{B}\cdot\hat{n}\big|_{above} = \vec{B}\cdot\hat{n}\big|_{below}$ we can summarize all BC conditions as $\vec{B}_{above} - \vec{B}_{below} = \mu_0(\vec{K}\times\hat{n})$

That is since $\vec{l}$ is any vector in boundary surface

$$\left(\vec{B}_{above} - \vec{B}_{below}\right)_{\substack{\text{tangential}\\ \text{component}\\ \text{vector}}} = \mu_0 \vec{K}\times\hat{n}$$

*

or taking the cross-product with $\hat{n}$

$$\boxed{\hat{n}\times(\vec{B}_{above} - \vec{B}_{below}) = \mu_0\vec{K}}$$

VIII.3.) Vector potential $\vec{A}$ is continuous across the boundary

1) $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow (\vec{A}_{above} - \vec{A}_{below})\cdot\hat{n} = 0$

2) $\vec{\nabla}\times\vec{A} = \vec{B}$

$$\Rightarrow \oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot d\vec{S} = \Phi = \text{magnetic flux}$$

but for an amperian loop of vanishing thickness the flux is zero through it since area $\rightarrow 0$

$$\Rightarrow \left(\vec{A}_{above} - \vec{A}_{below}\right)\Big|_{\text{tangential}} = 0$$

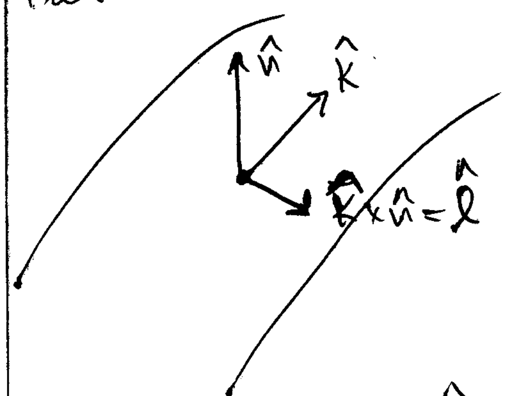
$$\Rightarrow \boxed{\vec{A}_{above} = \vec{A}_{below}}$$

(like V in electrostatics)

VIII. 3) Now since $\vec{B} = \vec{\nabla}\times\vec{A}$ we have

$$\vec{B}_{above} - \vec{B}_{below} = \vec{\nabla}\times\vec{A}_{above} - \vec{\nabla}\times\vec{A}_{below}$$
$$= \mu_0(\vec{K}\times\hat{n})$$

But



$\vec{A}$ is continuous so moving along surface

$$\vec{A}_{above}(\vec{r}+\vec{\Delta r}) = \vec{A}_{above}(\vec{r}) + \vec{\Delta r}\cdot\vec{\nabla}\vec{A}_{above}\Big|_{surface}$$
$$= \vec{A}_{below}(\vec{r}+\vec{\Delta r})$$
$$= \vec{A}_{below}(\vec{r}) + \vec{\Delta r}\cdot\vec{\nabla}\vec{A}_{below}\Big|_{surface}$$

but $\vec{\Delta r} = \Delta_k r\,\hat{k} + \Delta_{kn} r\,\hat{k}\times\hat{n}$

$$\Rightarrow \quad \frac{\partial\vec{A}}{\partial k},\ \frac{\partial\vec{A}}{\partial(k\times n)} = \frac{\partial\vec{A}}{\partial l} \text{ is continuous}$$

So only $\dfrac{\partial\vec{A}}{\partial n}$ can be discontinuous. But

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{l} & \hat{k} & \hat{n} \\ \frac{\partial}{\partial l} & \frac{\partial}{\partial k} & \frac{\partial}{\partial n} \\ A_l & A_k & A_n \end{vmatrix} = \hat{l}\left(\frac{\partial A_n}{\partial k} - \frac{\partial A_k}{\partial n}\right) - \hat{k}\left(\frac{\partial A_n}{\partial l} - \frac{\partial A_l}{\partial n}\right) + \hat{n}\left(\frac{\partial A_k}{\partial l} - \frac{\partial A_l}{\partial k}\right)$$

So

$$\vec{\nabla}\times\vec{A}_{above} - \vec{\nabla}\times\vec{A}_{below}$$
$$= \hat{l}\left(-\frac{\partial A_k}{\partial n}\right)\Bigg|_{\substack{above\\-below}} + \hat{k}\frac{\partial A_l}{\partial n}\Bigg|_{\substack{above\\-below}} = \mu_0 K\hat{l} \Rightarrow \quad \frac{\partial A_l}{\partial n}\Bigg|_{\substack{above\\-below}} = 0 \qquad \frac{\partial A_k}{\partial n}\Bigg|_{above-below} = -\mu_0 K$$

VIII.3.) So

$$\left.\frac{\partial \vec{A}}{\partial n}\right|_{\text{above}} = -\mu_0 \vec{K} + \left.\frac{\partial \vec{A}}{\partial n}\right|_{\text{below}}$$

So

$$\boxed{\left.\frac{\partial \vec{A}}{\partial n}\right|_{\text{above}} - \left.\frac{\partial \vec{A}}{\partial n}\right|_{\text{below}} = -\mu_0 \vec{K}}$$

Note: we used Coulomb gauge

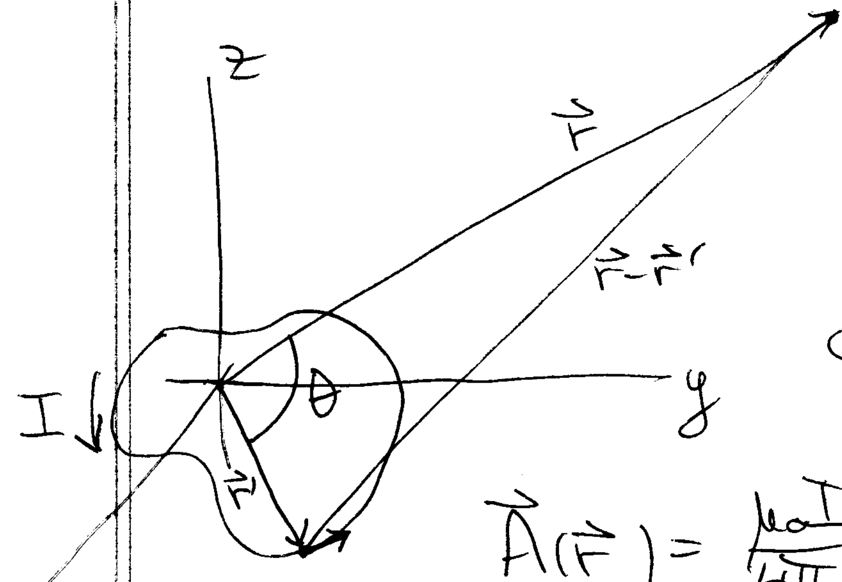
$$0 = \vec{\nabla}\cdot\vec{A} = \left.\left(\frac{\partial A_\ell}{\partial \ell} + \frac{\partial A_K}{\partial K} + \frac{\partial A_n}{\partial n}\right)\right|_{\text{above}}$$

$$= \left.\left(\frac{\partial A_\ell}{\partial \ell} + \frac{\partial A_K}{\partial K} + \frac{\partial A_n}{\partial n}\right)\right|_{\text{below}}$$

but $\frac{\partial \vec{A}}{\partial \ell}, \frac{\partial \vec{A}}{\partial K}$ are continuous $\Rightarrow$

$$\left.\frac{\partial A_n}{\partial n}\right|_{\text{above}} = \left.\frac{\partial A_n}{\partial n}\right|_{\text{below}}.$$

IX.) Multipole Expansion For the Vector Potential



Let us consider $|\vec{r}| >> |\vec{r}'|$

Since

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

we can consider expanding $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\frac{|\vec{r}'|}{|\vec{r}|}$

Now

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{\vec{r}\cdot\vec{r} + \vec{r}'\cdot\vec{r}' - 2\vec{r}\cdot\vec{r}'}}$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= \frac{1}{r}\frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta}}$$

But the square root we are familiar with ~~from~~ the multipole expansion in electrostatics

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^{k} P_k(\cos\theta)$$

$P_n(\cos\theta)$ = Legendre Polynomials.

Proof: let $\xi = \cos\theta$ and $t \equiv \frac{r'}{r} < 1$

So

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\frac{r'}{r}\cos\theta+\left(\frac{r'}{r}\right)^2}}$$

$$= \frac{1}{r}\frac{1}{\sqrt{1-2t\xi+t^2}} = \frac{1}{r}\frac{1}{\sqrt{1-u}}$$

with $u \equiv (2t\xi - t^2)$ here

Now

$$\frac{1}{\sqrt{1-2t\xi+t^2}} = Z(t) = \sum_{l=0}^{\infty} t^l P_l(\xi)$$

is the generating function for Legendre polynomials

$$P_l(\xi) = \left.\frac{d^l Z(t)}{dt^l}\right|_{t=0}$$

Check explicitly by Taylor expanding about $t=0$

$$(1-2t\xi+t^2)^{-1/2} = 1 - \tfrac{1}{2}(1-2t\xi+t^2)^{-3/2}(-2\xi+2t)\Big|_{t=0}(t)$$

$$+\Big[-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)(1-2t\xi+t^2)^{-5/2}(-2\xi+2t)^2\Big|_{t=0} - \tfrac{1}{2}(1-2t\xi+t^2)^{-3/2}\,2\Big|_{t=0}\Big]\frac{t^2}{2}$$

$$+\Big[-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)(1-2t\xi+t^2)^{-7/2}(-2\xi+2t)^3$$
$$-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)(1-2t\xi+t^2)^{-5/2}\,2(-2\xi+2t)(2)$$
$$-\tfrac{1}{2}(1-2t\xi+t^2)^{-5/2}\left(-\tfrac{3}{2}\right)2(-2\xi+2t)\Big]\Big|_{t=0}\frac{t^3}{6}$$

$$+\dots$$

$$= 1-\tfrac{1}{2}(-2\xi)\,t + \left[-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)(-2\xi)^2 - 1\right]\frac{t^2}{2}$$

$$+\Big[-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)(-2\xi)^3 - \tfrac{1}{2}\left(-\tfrac{3}{2}\right)2^2(-2\xi)$$
$$-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)2(-2\xi)\Big]\,t^3/6 \quad + \dots$$

$$Z(t) = 1 + \xi t + [3\xi^2 - 1]\frac{t^2}{2!} + [15\xi^3 - 9\xi]\frac{t^3}{3!} + \cdots$$

So we see that

$$P_0 = 1 \qquad P_2 = 3\xi^2 - 1$$

$$P_1 = \xi \qquad P_3 = 15\xi^3 - 9\xi \qquad \cdots$$

So

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \quad Z\left(\frac{r'}{r}\right)$$

$$= \frac{1}{r}\left\{1 + \xi\frac{r'}{r} + [3\xi^2 - 1]\frac{\left(\frac{r'}{r}\right)^2}{2!} + \cdots\right\}$$

So

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi r}\left[\sum_{l=0}^{\infty} \oint_C d\vec{r}'\left(\frac{r'}{r}\right)^l P_l(\cos\theta)\right]$$

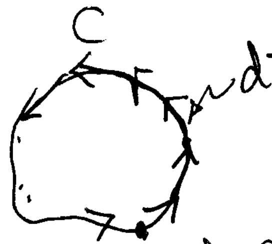
monopole, dipole

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint_C d\vec{r}' + \frac{1}{r^2}\oint_C d\vec{r}'\, r'\cos\theta \right.$$

quadrupole ...

$$\left. + \frac{1}{r^3}\oint_C d\vec{r}'\, r'^2\left[\frac{3}{2}\cos^2\theta - \frac{1}{2}\right] + \cdots\right]$$

Now



So $\oint_C d\vec r' = 0$

The monopole term is zero.
This is a reflection of no magnetic charges,

$\vec A_{dipole}(\vec r)$

Now $$\frac{1}{|\vec r-\vec r'|} = \frac{1}{r}\left[\left(1+\frac{r'^2}{r^2}-\frac{2}{r^2}\vec r\cdot\vec r'\right)\right]^{-1/2}$$

$$= \frac{1}{r}\left[1-\frac{1}{2}\frac{1}{r^2}\left(r'^2-2\vec r\cdot\vec r'\right)+\cdots\right]$$

$$= \left[\frac{1}{r}+\frac{\vec r\cdot\vec r'}{r^3}+O\left(\frac{r'^2}{r^3}\right)\right]$$

$$\vec A_{dipole}(\vec r) = \frac{\mu_0 I}{4\pi}\oint_C d\vec r'\,\frac{1}{r^2}\,r'\cos\theta$$

$$= \frac{\mu_0 I}{4\pi}\oint_C d\vec r'\,\frac{\vec r'\cdot\vec r}{r^3}$$

So recall

$$(\vec{r}\,'\times d\vec{r}\,')\times\vec{r} = -\vec{r}\,'(\vec{r}\cdot d\vec{r}\,') + d\vec{r}\,'(\vec{r}\cdot\vec{r}\,')$$

Also

$$d\left[\vec{r}\,'(\vec{r}\cdot\vec{r}\,')\right] = \vec{r}\,'(\vec{r}\cdot d\vec{r}\,') + d\vec{r}\,'(\vec{r}\cdot\vec{r}\,')$$

So add and $\div 2$

$$\boxed{d\vec{r}\,'(\vec{r}\cdot\vec{r}\,') = \frac{1}{2}(\vec{r}\,'\times d\vec{r}\,')\times\vec{r} + \frac{1}{2}d\left[\vec{r}\,'(\vec{r}\cdot\vec{r}\,')\right]}$$

Now

$$\oint_C d\left[\vec{r}\,'(\vec{r}\cdot\vec{r}\,')\right] = 0$$

since it is the total change of a vector around a closed loop is zero — it comes back to itself.

So

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi}\,\frac{1}{2r^3}\oint_C (\vec{r}\,'\times d\vec{r}\,')\times\vec{r}$$

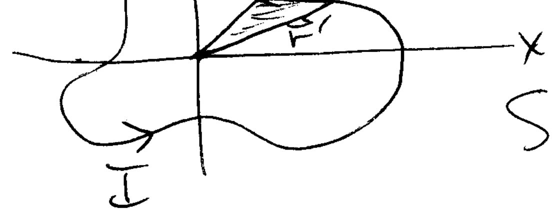
$$= \frac{\mu_0}{4\pi}\underbrace{\left[\frac{I}{2}\oint_C \vec{r}\,'\times d\vec{r}\,'\right]}_{\equiv\ \vec{m}\ \text{the magnetic dipole moment}}\times\frac{\vec{r}}{|\vec{r}|^3}$$

So

$$\boxed{\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}}$$

Valid far from coil (or let $C \to$ point, $I \to \infty$ so that $\vec{m}$ = const.
$\vec{m}$ = const.
and this is exact for $\vec{A}$)

(For special case of a plane loop



$$\frac{1}{2}\vec{r}\,'\times d\vec{r}\,' = \overset{\text{area}}{\Delta a}\,\hat{k}$$

So $\oint_C \frac{1}{2}\vec{r}\,'\times d\vec{r}\,' = a\hat{k} = \vec{a}$

area of plane loop

and $\vec{m} = I\vec{a}$ independent of origin)

The magnetic field for a dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{A}(\vec{r})$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \times \left[\frac{\vec{m} \times \vec{r}}{r^3} \right]$$

Using $\vec{\nabla} \times (\vec{F} \times \vec{G}) = \cancelto{0}{(\vec{G} \cdot \vec{\nabla})\vec{F}} - (\vec{F} \cdot \vec{\nabla})\vec{G}$

$$+ \vec{F}(\vec{\nabla} \cdot \vec{G}) - \vec{G}\cancelto{0}{(\vec{\nabla} \cdot \vec{F})}$$

for $\vec{F} = \vec{m}$

$$\vec{B} = \frac{\mu_0}{4\pi} \left[-(\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}}{r^3} + \vec{m}\, \vec{\nabla} \cdot \frac{\vec{r}}{r^3} \right]$$

Now $\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \frac{3}{r^3} - \vec{r} \cdot \frac{3\vec{r}}{r^5} = \frac{3}{r^3} - \frac{3r^2}{r^5} = 0.$

So

$$(m \cdot \vec{\nabla}) \frac{\vec{r}}{r^3} = \frac{\vec{m}}{r^3} - \frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5}$$

$$\Rightarrow \boxed{\vec{B}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} \right]}$$

(Recall electric dipole

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right])$$

So notice that in dipole case

$$\vec{E} = -\vec{\nabla} V \quad ; V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

So we can write

$$\vec{B}_{dipole} = -\mu_0 \vec{\nabla} V^*$$

$$V^* = \frac{\vec{m}\cdot\vec{r}}{4\pi r^3}$$ magnetic scalar potential.

XI) Magnetic Material & Magnetic Intensity

a) $\vec{M} = N \langle \vec{m} \rangle = \dfrac{\text{average dipole moment}}{\text{volume}}$

magnetization $= \underbrace{\dfrac{1}{\Delta v}}_{\text{macro. pt.}} \sum_{\substack{\text{atoms}\\ i}} \vec{m}_i$ all macroscopic magnetic properties can be described by $\vec{M}$

b) Magnetization Currents

$\vec{J}_b = \vec{\nabla} \times \vec{M}$ volume current density

$\vec{K}_b = \vec{M} \times \hat{n}$ surface current density

c) Magnetic potential & field produced by magnetized material

$\Delta \vec{m} = \vec{M} \Delta v$

X) c) and
$$d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\Delta\vec{m}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$
$$= \frac{\mu_0}{4\pi}\frac{\vec{M}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv'$$

So
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \vec{M}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv'$$
$$= \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{j}_M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,da'$$

as you expect from B-S law.

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d) Now $\vec{B} = \vec{\nabla}\times\vec{A}$ so
$$\vec{B}(\vec{r}) = \mu_0\vec{M}(\vec{r}) - \mu_0\vec{\nabla}\varphi^*(\vec{r})$$
$$\varphi^*(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\vec{M}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv'$$

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$\vec{\nabla}\cdot\vec{B}=0$

$\vec{\nabla}\times\vec{B}=\mu_0(\vec{J}+\vec{J}_M)$

e) In general when we have ext. currents also
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv' + \mu_0\vec{M}(\vec{r}) - \mu_0\vec{\nabla}\varphi^*(\vec{r})$$

But we need $\vec{M}=\vec{M}(\vec{B})$ so introduce

X.) e.) The magnetic intensity $\vec{H}$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}.$$ but still φ^* depends on $\vec{M}$

Let's try differential approach to relate $\vec{H}$ directly to $\vec{J}$!

1) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \vec{J}_M)$ $\quad \vec{J}_M = \vec{\nabla}\times\vec{M}$

So $\Rightarrow$ $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}}$ Ampere's law in presence of mag. material.

So $\boxed{\oint_C \vec{H}\cdot d\vec{l} = I}$ ← free or ext. current only.

f.) Linear, isotropic Mag. material:

$$\vec{M} = \chi_m \vec{H}$$

χ_m mag. susceptibility.

$$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H} \equiv \mu\vec{H}$$

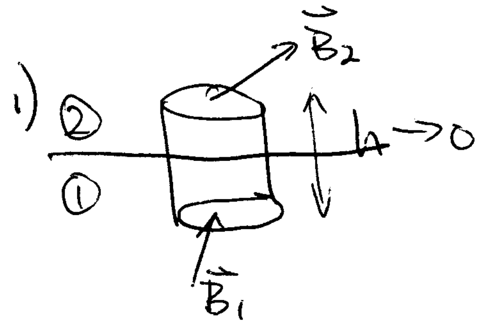
μ permeability.

$$K_m = \frac{\mu}{\mu_0} = 1 + \chi_m$$

K_m relative permeability.

X) g.) B.C.

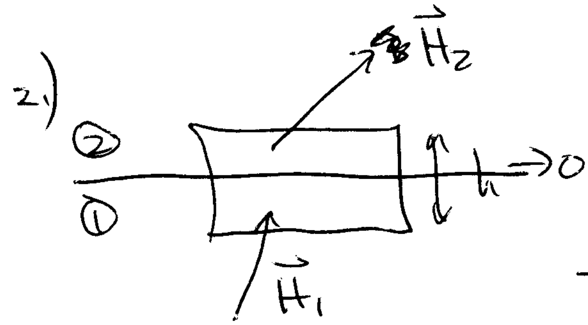
1)



$\vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow \boxed{B_{2n} = B_{1n}}$

2)



$\oint_C \vec{H}\cdot d\vec{l} = I$

$\Rightarrow \boxed{H_{2t} = H_{1t}}$ (no surface currents)

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h.). BVP : 1) $\vec{\nabla}\cdot\vec{B} = 0$
2) $\vec{\nabla}\times\vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}\varphi^*$

1) $\vec{B} = \mu\vec{H}$ linear $\Rightarrow \vec{\nabla}\cdot\vec{H} = 0$
2) $\vec{\nabla}\cdot\vec{M} = 0$ uniform $\vec{M}$. $\Rightarrow \vec{\nabla}\cdot\vec{H} = 0$

$\Rightarrow$ $\boxed{\nabla^2\varphi^* = 0 \text{ Laplace eq.} + \text{B.C.} = \text{BVP}}$

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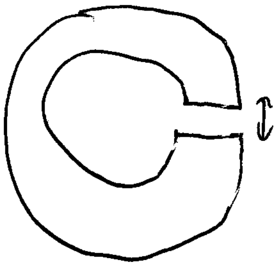
i.) Problems with known current distributions:

1.) Torus with N turns & I & linear material

$\oint_C \vec{H}\cdot d\vec{l} = I \Rightarrow H = \frac{NI}{2\pi r} \Rightarrow B = \mu\frac{NI}{2\pi r}$

X.) (i) 2.) Torus with gap:



linear material $\vec{B} = \mu \vec{H}$

$\oint_C \vec{H} \cdot d\vec{l} = NI$ + B.C.

$\Rightarrow$ B = same everywhere

$$B = \frac{\mu \mu_0 N I}{\mu_0 2\pi r + d(\mu - \mu_0)}$$

End of Statics

XI.) Electromagnetic Induction: $\frac{\partial \vec{B}}{\partial t} \neq 0.$

a) $\text{Emf} \equiv \mathcal{E} = \oint_C \vec{E} \cdot d\vec{l} \neq 0$ when $\frac{\partial \vec{B}}{\partial t} \neq 0.$

b.) Faraday's law new exp. law: 1831

$\mathcal{E} = -\frac{d\phi}{dt}$ $\quad \phi = \int_S \vec{B} \cdot d\vec{S}$ integral form.

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ diff. form.

(− sign = Lenz's law)