Electrostatics Review

1) Coulomb's Law: Experimentally observed force between charges

$$\vec{F}_q = \frac{q}{4\pi\epsilon_0}\left[\int_V \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')\,dv' + \int_S \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')\,da'\right].$$

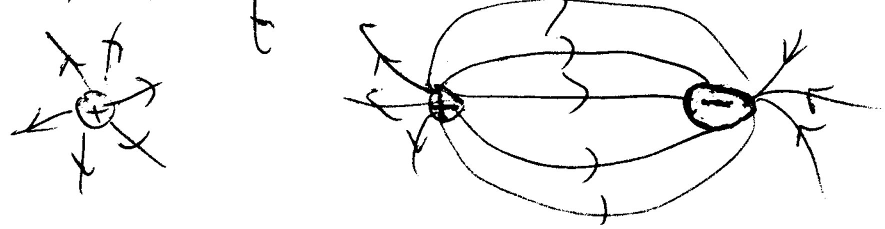
a) point charges q_i at $\vec{r}_i$

$$\rho(\vec{r}\,') = \sum_{i=1}^{N} q_i\,\delta(\vec{r}\,'-\vec{r}_i)$$

2) Electric Field: $\vec{E}(\vec{r})$ gives force/unit charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')\,dv' + \int_S \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')\,da'\right]$$

a) Lines of Force: end & start on charges



2.b) Since curl $\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)=0$

$\vec{E}$ is conservative $\Rightarrow$

1) $\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$

2) $\varphi(\vec{r}) - \varphi(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}} \vec{E}\cdot d\vec{l}$

independent of the path

φ = electrostatic potential.

$\vec{r}_1$ is chosen as a reference point so that $\varphi(\vec{r}_1)=0$.

$\Rightarrow$ Eq. for $\vec{E}(\vec{r})$ (with $\varphi(\infty)=0$)

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' + \int_S \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|}da'\right]$$

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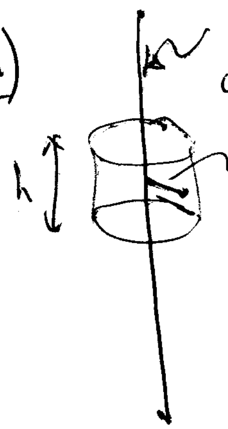
c) Gauss's Law

1) Integral form: $\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\underbrace{\int_V \rho\, dv}_{\text{total charge contained in } V}$ $+\int\sigma da$ if nec.

2) Differential form: $\vec{\nabla}\cdot\vec{E} = +\frac{1}{\epsilon_0}\rho$

2.c. ex. 1)



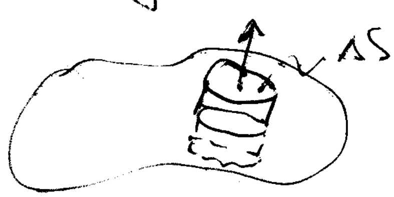
Gauss's law

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0} Q = \frac{h\lambda}{\epsilon_0}$$

$$= E(r)\, 2\pi r h$$

$$\Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}}$$

2) Charge Conductor with surface charge density σ
$\vec{E}$ just outside the surface



$$\vec{E} = -\vec{\nabla}\varphi \Rightarrow \vec{E} \sim \hat{n}$$

$$\oint_S \vec{E}\cdot d\vec{S} = E_n \Delta S = \frac{\Delta S\,\sigma}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

3) Point Charge

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho = \frac{q}{\epsilon_0}\delta(\vec{r})$$

$$\Rightarrow \nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$$

3) <u>Multipole Expansion for potential</u>

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dv'$$

expand denominator for $\frac{|\vec{r}\,'|}{|\vec{r}|} \ll 1$

$$\varphi(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\vec{r}\cdot\vec{p}}{4\pi\epsilon_0 r^3}$$

$$+ \frac{1}{2}\frac{x_i x_j}{4\pi\epsilon_0 r^5} Q_{ij} + \cdots$$

where

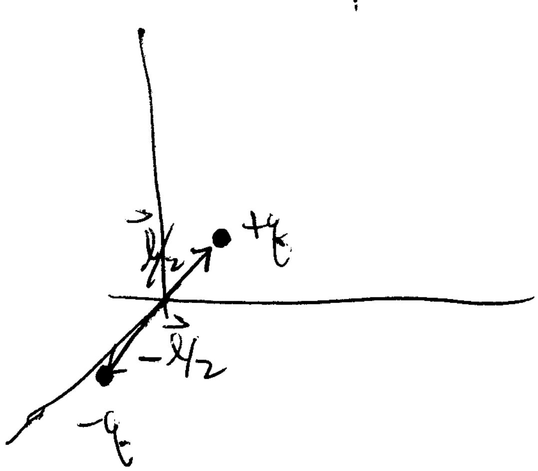
1) $Q = \int_V \rho(\vec{r}\,')\,dv$ monopole

2) Dipole $\vec{p} = \int_V \vec{r}\,'\rho(\vec{r}\,')\,dv'$

3) Quadrapole: $Q_{ij} = \int_V (3x_i' x_j' - \delta_{ij} r'^2)\rho(\vec{r}\,')\,dv$

ex.) <u>Electric</u> <u>Dipole</u>:

Suppose $\rho(\vec{r}\,') = q\left(\delta(\vec{r}\,' - \frac{\vec{l}}{2}) - \delta(\vec{r}\,' + \frac{\vec{l}}{2})\right)$



1) $Q=0$

2) $\vec{p} = \int_V \vec{r}'\rho(\vec{r}')dv' = q\left(\frac{\vec{l}}{2} - \left(-\frac{\vec{l}}{2}\right)\right)$
$= q\vec{l}$

3) $Q_{ij} \propto q\,O(l^2)$

...

Now: Let's take the limit $q \to \infty$ $|\vec{l}| \to 0$

Such that $q\vec{l} = \text{const.} \equiv \vec{p}$ (i.e. size → a pt.)

$Q_{ij} \to 0$ as all higher moments

So

$$\varphi(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} \quad \text{exactly}$$

This is the point dipole. $\vec{p}$ characterizes the point dipole completely, i.e. φ is determined once $\vec{p}$ is given!

Alternatively we could start with

q at $\vec{r}_0+\vec{\ell}$; and $-q$ at $\vec{r}_0$

Then

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}_0-\vec{\ell}|} - \frac{1}{|\vec{r}-\vec{r}_0|}\right]$$

and expand for $|\vec{r}-\vec{r}_0| \gg |\vec{\ell}|$

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\,\frac{(\vec{r}-\vec{r}_0)\cdot\vec{\ell}}{|\vec{r}-\vec{r}_0|^3} + \cdots$$

again in the point dipole limit $\vec{p} \equiv \lim\limits_{\substack{q\to\infty \\ |\vec{\ell}|\to 0 \\ q|\vec{\ell}|=|\vec{p}|}} q\vec{\ell}$

$$\boxed{\varphi(\vec{r}) = \frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}_0|^3}} \quad \text{exactly}$$

($\vec{r}_0 = \vec{0}$ the origin in multipole expansion).

4) Potential Energy of dipole in External $\vec{E}_{ext}$ field.

$$U = -\int_{\vec{r}_1}^{\vec{r}} \vec{F}\cdot d\vec{\ell} = q\left[\varphi_{ext}(\vec{r}) - \varphi_{ext}(\vec{r}_1)\right]$$

expand for dipole $\pm q$ at $(\vec{r}+\vec{\ell})$, $\vec{r}$

$$\boxed{U = -\vec{p}\cdot\vec{E}_{ext}(\vec{r})}$$

location of dipole

Solution to Electrostatic Problems
(especially when ρ, σ etc. is not given)

1) Poisson's & Laplace's Eq.

a) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$; $\vec{E} = -\vec{\nabla}\varphi$

$\Rightarrow \nabla^2\varphi = -\frac{1}{\epsilon_0}\rho$ Poisson's eq.

b.) $\nabla^2\varphi = 0$ Laplace's Eq.

c) Thm. 1. If φ_1 and φ_2 are solutions of Laplace's eq. then

$$\varphi = a\varphi_1 + b\varphi_2 \quad \text{with } a, b$$

arb. constants is also a solution.

d) Thm. 2.: Uniqueness Thm: Two solutions of Laplace's equation that satisfy the same boundary conditions differ at most by an additive constant.

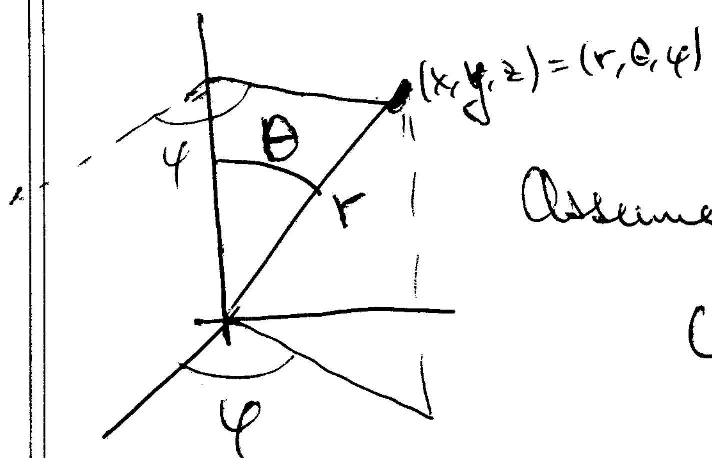
$$\nabla^2\varphi_1 = 0 = \nabla^2\varphi_2 \quad \text{in } V$$

& $\varphi_{\frac{1}{2}}$ or $\frac{\partial\varphi_{\frac{1}{2}}}{\partial n}$ on surfaces specified

Dirichlet   Neumann

1.d) Then $\varphi_1 = \varphi_2 + C \leftarrow$ constant.

2) Solutions of Laplace's Eq. in spherical coordinates: Zonal Harmonics:



Assume azimuthal symmetry
$$\varphi = \varphi(r,\theta) \text{ only}$$

$$\nabla^2 \varphi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right)$$

Solve by separation of variables
$$\varphi(r,\theta) = Z(r)\,P(\theta)$$

a) Legendre's Eq. for P
$$\frac{d}{d\theta}\left[\sin\theta \frac{dP}{d\theta}\right] + k \sin\theta\, P = 0$$

found $k = n(n+1)$ $n = 0, 1, 2, \ldots$

$$P_n(\cos\theta) = \frac{1}{2^n n!}\frac{d^n}{d(\cos\theta)^n}\left(\cos^2\theta - 1\right)^n$$

2 a) $P_0(\cos\theta) = 1$
$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$\vdots$

b.) Radial Eq. $Z_n = b_n r^n + C_n r^{-(n+1)}$

$\frac{d}{dr}\left(r^2 \frac{dZ_n}{dr}\right) = n(n+1) Z_n$

c.) Zonal Harmonics

for each n $\nabla^2 \varphi_n(r,\theta) = 0$

$$\varphi_n = Z_n(r) P_n(\theta)$$

$r^n P_n$ or $r^{-(n+1)} P_n$ are called Zonal Harmonics; they are complete so every function can be written in series expansion of φ_n:

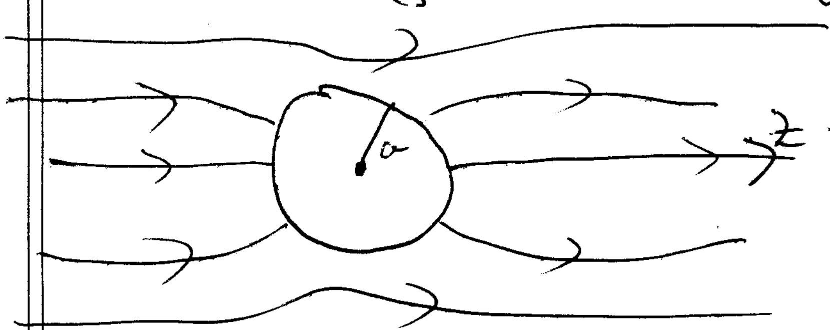
If $\nabla^2 \varphi = 0$ then

$$\varphi(r,\theta) = \sum_{n=0}^{\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}}\right] P_n(\cos\theta).$$

use B.C. to find A_n, B_n.

2d. ex.) Uncharged conducting sphere in initially uniform $\vec{E}$ field

φ for $r \geq a$.



$\vec{E} = \vec{E}_0$

B.C. 1) $\varphi(r,\theta)\Big|_{r\to\infty} = -E_0 r\cos\theta + \text{const.}$

$\Rightarrow A_2 = A_3 = \cdots = 0$ $\quad \boxed{A_1 = -E_0}$

2) $\varphi(a,\theta) = \varphi_0$ equipotential $\Rightarrow \boxed{A_0 = \varphi_0 \quad B_1 = E_0 a^3 \quad B_n = 0 \;\; n = 2,3,\ldots}$

3) uncharged so φ has no $\frac{1}{r}$ coulomb term.

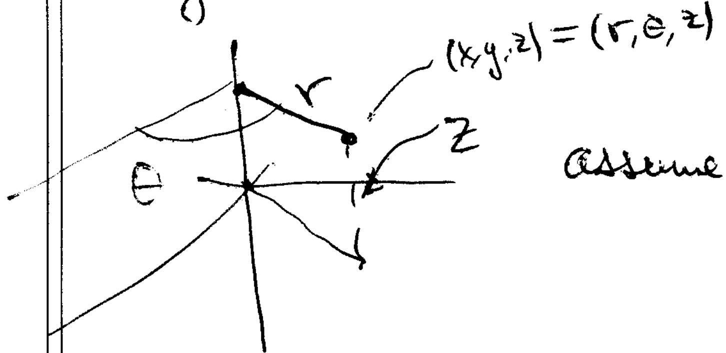
$\Rightarrow \boxed{B_0 = 0}$

$$\boxed{\varphi(r,\theta) = \varphi_0 - E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta}$$

$\vec{E} = -\vec{\nabla}\varphi$; $\quad E_r = \frac{\sigma}{\epsilon_0}$ conductor.

$\boxed{\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta}$; $Q_{net} = 0$.

3) Cylindrical Harmonics



assume $\varphi = \varphi(r,\theta)$ independent of z.

$$\nabla^2\varphi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

Solve by separation of variables

$$\varphi(r,\theta) = Y(r)\,S(\theta)$$

a) $S_n = a\cos n\theta + b\sin n\theta \qquad n = 0,1,2\ldots$

b) $Y_n = \begin{cases} a\ln r + b & n=0 \\ ar^n + br^{-n} & n=1,2,\ldots \end{cases}$

So c) $\nabla^2\varphi = 0 \Rightarrow$

$$\varphi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[A_n r^n\cos n\theta + B_n r^n \sin n\theta + \frac{C_n}{r^n}\cos n\theta + \frac{D_n}{r^n}\sin n\theta\right]$$

4.) Solution of Laplace's eq. By Electrostatic images.

$$\varphi(\vec{r}) = \varphi_1(\vec{r}) + \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r}')\,da'}{|\vec{r}-\vec{r}'|}$$

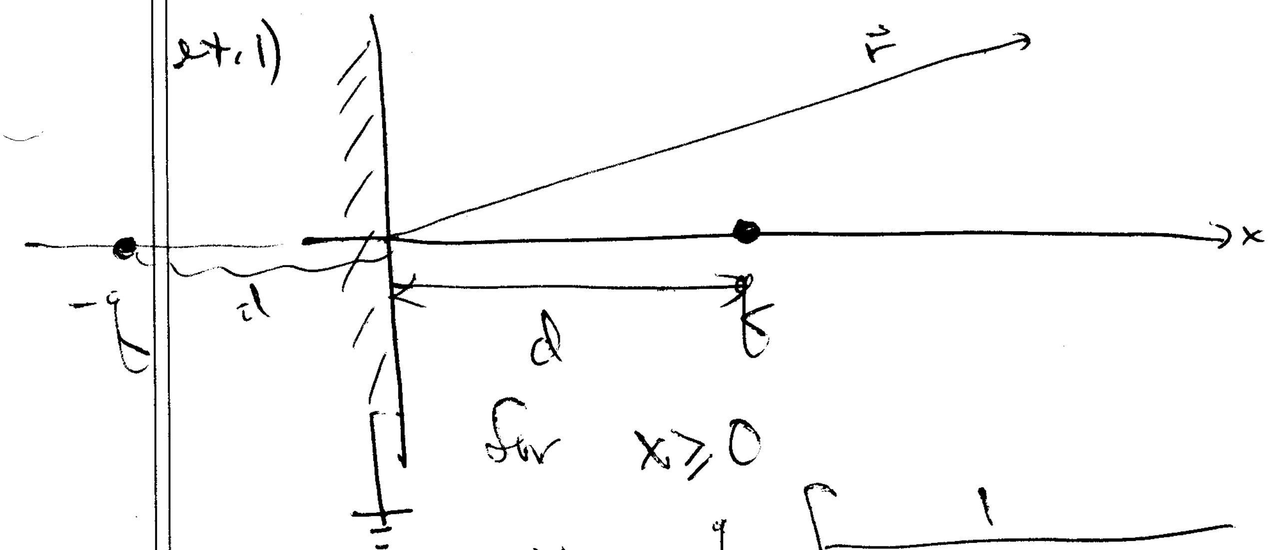
known

unknown

but can replace by $\varphi_2(\vec{r})$ from a specified charge distribution in region outside region of interest.

then $\varphi = \varphi_1 + \varphi_2$ is solution in region of interest.

ex. 1)



for $x \geq 0$

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right]$$

1) $\nabla^2\varphi = 0$ for $x \geq 0$

2) B.C. $\varphi(x=0) = 0$

$\Rightarrow$ φ is unique sol. to potential for $x \geq 0$.

See Class notes for more examples and HW.

5.) Coefficients of Potential: not always possible to determine potential in a conductor, but

$$\varphi_i = \sum_{j=1}^{N} p_{ij} Q_j$$

6.) Poisson's Eq $\nabla^2 \varphi = \rho$

$$\varphi = \varphi_L + \frac{1}{4\pi\epsilon_0}\int \frac{\rho \, dv'}{|\vec{r}-\vec{r}'|}$$

$\nabla^2 \varphi_L = 0$ chosen so φ obeys desired B.C.

ex. ||-plate diode.

Dielectric Media

1) Polarization of the media

$$\vec{P} = \frac{\Delta \vec{p}}{\Delta v} = \frac{1}{\Delta v} \sum_{\substack{\text{molecules} \\ \text{in } \Delta v}} \vec{p}_m$$

$\vec{P}$ = electric polarization.

The potential in the presence of a dielectric media with polarization $\vec{P}$ is

1)

a) $$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{V_0} \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dv'$$

b) Recall $$\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right)$$

$\Rightarrow$

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{S_0} \frac{\sigma_p(\vec{r}')\, da'}{|\vec{r} - \vec{r}'|}$$

$$+ \frac{1}{4\pi\epsilon_0} \int_{V_0} \frac{\rho_p(\vec{r}')\, dv'}{|\vec{r} - \vec{r}'|}$$

where

$$\left.\begin{aligned} \sigma_p(\vec{r}') &\equiv \vec{P}(\vec{r}') \cdot \hat{n}(\vec{r}') \\ \rho_p(\vec{r}') &\equiv -\vec{\nabla} \cdot \vec{P}(\vec{r}') \end{aligned}\right\} \text{Polarization charge densities.}$$

c) $$Q_{net} = \int_{V_0} \rho_p \, dv + \int_{S_0} \sigma_p \, da = 0.$$

d) $$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}).$$

2) Gauss's law in a dielectric: $\vec{P} = \vec{P}(\vec{E})$ so need more info, re-consider Gauss's law:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \left(\underset{\text{external}}{Q} + \underset{\text{polarization}}{Q_P} \right)$$

$$Q_p = -\oint_S \vec{P} \cdot d\vec{S}$$

$\Rightarrow$

a) $$\boxed{\oint_S \vec{D} \cdot d\vec{S} = Q}$$ Gauss's law in a dielectric (integral form)

$$\boxed{\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}}$$ the electric displacement

b) differential form of Gauss's law

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

3.) Linear, isotropic Dielectrics

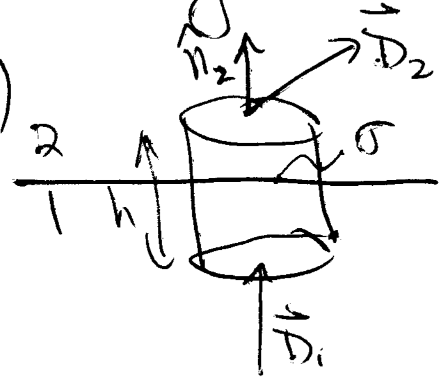
$$\vec{P} = \underset{\text{electric susceptibility}}{\chi} \vec{E}$$

$$\vec{D} = \epsilon \vec{E} \quad , \quad \epsilon = \epsilon_0 + \chi$$

$$K = \epsilon / \epsilon_0 = 1 + \chi / \epsilon_0$$ = dielectric constant.

4.) Boundary Conditions between dielectrics:

a)



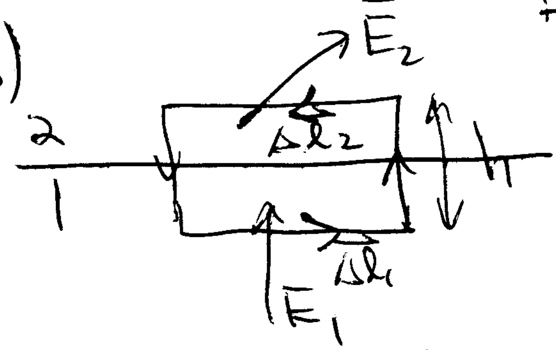
Gauss's law as $h \to 0$

$$\oint_S \vec{D} \cdot d\vec{S} = \sigma \Delta S$$

$$= (\vec{D}_2 - \vec{D}_1) \cdot \hat{n}\, \Delta S$$

$$\Rightarrow \boxed{D_{2n} - D_{1n} = \sigma}$$

b)



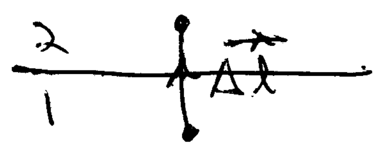
$\vec{E}$ is conservative

$$\oint_C \vec{E} \cdot d\vec{l} = 0$$

as $h \to 0$ $\quad (\vec{E}_1 - \vec{E}_2) \cdot \Delta l_1 = 0$

$$\Rightarrow \boxed{E_{1t} = E_{2t}}$$

c) Alternatively:



$\Delta\varphi = -\vec{E} \cdot \vec{\Delta l}$ as $\vec{\Delta l} \to 0$ $\quad \Delta\varphi \to 0$

So $\boxed{\varphi_2 = \varphi_1}$

5.) Laplace's Eq. in presence of dielectric media

a) Gauss's law $\vec{\nabla}\cdot\vec{D} = \rho$

linear, isotropic dielectric $\vec{D} = \epsilon\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon$$

$\vec{E}$ conservative $\vec{E} = -\vec{\nabla}\varphi \Rightarrow$

$$\boxed{\nabla^2\varphi = -\rho/\epsilon} \text{ Poisson's Eq.}$$

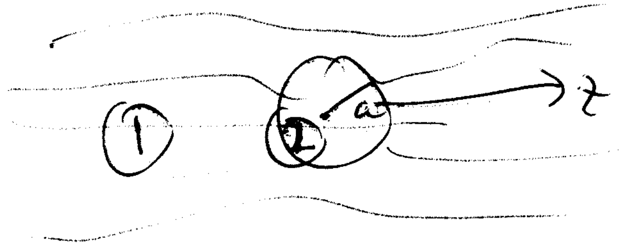
b.) Where $\rho = 0$ Laplace's eq. $\nabla^2\varphi = 0$ is the same — However we have the additional B.C. of

$$\boxed{D_{2n} - D_{1n} = \sigma \ \& \ \varphi_1 = \varphi_2}$$

c.) ex. Dielectric sphere in initially uniform $\vec{E}$ field.

$$\varphi = \begin{cases} \varphi_1 & r \geq a \\ \varphi_2 & r \leq a \end{cases} \quad \nabla^2\varphi_i = 0$$

B.C. 1) $\varphi_1\Big|_{r\to\infty} = -E_0 r\cos\theta$



2) no charge so coulomb term absent in φ_2 ($\varphi_2(0)$ finite)

5c) 3) $\varphi_1(a,\theta) = \varphi_2(a,\theta)$

4) $D_{1r} = D_{2r} \Rightarrow +\epsilon_0 \left.\frac{\partial \varphi_1}{\partial r}\right|_{r=a} = \epsilon \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=a}$.

Expand $\varphi_{1,2}$ in Zonal Harmonics

$$\varphi_1 = A_1 r\cos\theta + \frac{C_1}{r^2}\cos\theta$$

etc.

$$\varphi_2 = A_2 r\cos\theta + \frac{C_2}{r^2}\cos\theta$$

$(1,2,3,4) \Rightarrow$

$$\boxed{\begin{aligned} \varphi_1 &= -E_0 r\cos\theta + \frac{(K-1)}{(K+2)} E_0 \frac{a^3}{r^2}\cos\theta \\ \varphi_2 &= -\frac{3}{(K+2)} E_0 r\cos\theta \end{aligned}}$$

$\vec{E} = -\vec{\nabla}\varphi$, $\vec{D} = \epsilon\vec{E}$, $\vec{P} = \chi\vec{E}$

$\rho_p = -\vec{\nabla}\cdot\vec{P}$; $\sigma_p = \vec{P}\cdot\hat{n}\Big|_{r=a}$; $Q_p = 0$.

Electrostatic Energy

1) $U = -$Work done by force

$$= -\int_A^B \vec{F}\cdot d\vec{l} = -q\int_A^B \vec{E}\cdot d\vec{l} = q(\varphi_B - \varphi_A)$$

a) for an assembly of charges

$$U = \frac{1}{2}\int_V \rho(\vec{r})\varphi(\vec{r})\,dv + \frac{1}{2}\int_S \sigma(\vec{r})\varphi(\vec{r})\,da.$$

b) Applied to point charges we must exclude ∞ self-energy $\rho(\vec{r}_j)\varphi(r_j) = \infty$.

$$\rho(\vec{r}) = \sum_{j=1}^{m} q_j\,\delta(\vec{r}-\vec{r}_j)$$

$$U = \frac{1}{2}\sum_{j=1}^{m} q_j\,\varphi(\vec{r}_j)$$

$$\varphi(\vec{r}_j) = \sum_{\substack{k=1 \\ k\neq j}}^{m} \frac{q_k}{4\pi\epsilon_0|\vec{r}_j-\vec{r}_k|}$$

exclude ∞ self-energy.

$$\Rightarrow\quad U = \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1,\,k\neq j}^{m} \frac{q_j q_k}{4\pi\epsilon_0|\vec{r}_j-\vec{r}_k|}$$

1) b) We can derive this directly by bringing each point charge in from ∞; we will see that self-energy is really excluded.

bring q_1 from ∞ $\qquad U_1 = 0$

" q_2 " ∞ $\qquad U_2 = \dfrac{q_2 q_1}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|}$

⋮

$$U = \sum_{j=1}^{m} U_j = \sum_{j=1}^{m} \sum_{k=1}^{j-1} \frac{q_j q_k}{4\pi\epsilon_0 |\vec{r}_j - \vec{r}_k|}$$

$$= \frac{1}{2} \sum_{j=1}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \frac{q_j q_k}{4\pi\epsilon_0 |\vec{r}_j - \vec{r}_k|} .$$

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c) Conductors $1, \ldots, m$ at potential $\varphi_1 \cdots \varphi_m$

$$U = \frac{1}{2}\int_S \sigma \varphi \, da = \frac{1}{2}\sum_{j=1}^{m} \varphi_j \int_{S_j} \sigma_j \, da$$

$$\boxed{U = \frac{1}{2}\sum_{j=1}^{m} Q_j \varphi_j} .$$

2) Gauss's law applied to U:

$$\vec{\nabla}\cdot\vec{D}=\rho \quad \& \text{ B.C. } \sigma=\vec{D}\cdot\hat{n} \quad \text{for charged conductors.}$$

$$U=\frac{1}{2}\int_V \rho\varphi\, dv+\frac{1}{2}\int_S \sigma\varphi\, da$$

$$=\frac{1}{2}\int \varphi\,\vec{\nabla}\cdot\vec{D}\, dv+\frac{1}{2}\int_S \varphi\,\vec{D}\cdot d\vec{S}$$

$\Rightarrow$

$$\boxed{U=\frac{1}{2}\int_V \vec{E}\cdot\vec{D}\, dv}$$

Use identities, & $D\sim\frac{1}{r^2}$, $\varphi\sim\frac{1}{r}$, $\int_S D\varphi\, dS \xrightarrow[r\to\infty]{} 0$.

energy is stored in Electrostatic fields!

$$u=\frac{1}{2}\vec{E}\cdot\vec{D} \quad \text{energy density.}$$

3) Capacitors

$$\varphi_i=\sum_{j=1}^{N} p_{ij}Q_j \quad \text{on conductors}$$

So $$U=\frac{1}{2}\sum_{i=1}^{N} Q_i\varphi_i=\frac{1}{2}\sum_{i,j=1}^{N} p_{ij}Q_iQ_j$$

for Equal & oppositely charged conductors

$$\Delta\varphi=\varphi_1-\varphi_2=(p_{11}+p_{22}-2p_{12})Q$$

$$Q=C\Delta\varphi$$

$$C=\frac{1}{p_{11}+p_{22}-2p_{12}}=\text{Capacitance.}$$

3) and $U = \frac{1}{2} Q \Delta\varphi = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C (\Delta\varphi)^2$

4.) Examples. ‖-plate capacitor

$$\text{Gauss's law} \Rightarrow D = \frac{Q}{A}$$

$$D = \epsilon E \Rightarrow E = \frac{Q}{\epsilon A}$$

$$\Delta\varphi = Ed \Rightarrow C = \frac{Q}{\Delta\varphi} = \frac{\epsilon A}{d},$$

5.) Forces & Torques on charges

a.) Isolated system $Q = \text{fixed}$.

$$\underbrace{dW}_{\text{work performed by } \vec{E} \text{ field}} = \vec{F} \cdot d\vec{r} = \underbrace{-dU}_{\text{loss in potential energy}} = -\vec{\nabla} U \cdot d\vec{r}$$

$$\vec{F} = -\left(\vec{\nabla} U\right)_Q .$$

Similarly $dW = \vec{\tau} \cdot d\vec{\theta} = -dU$

$$\Rightarrow \tau_i = -\left(\frac{\partial U}{\partial \theta_i}\right)_Q ,$$

5.b) Suppose charges on conductors at <u>fixed potential</u> (external source battery does work)

$$dW = dW_b - dU$$

work performed by battery & $\vec{E}$ field

for conductors $dU = \frac{1}{2}\sum_j \varphi_j dQ_j$

& $dW_b = \sum_j \varphi_j dQ_j$

$\Rightarrow dW_b = 2\,dU$

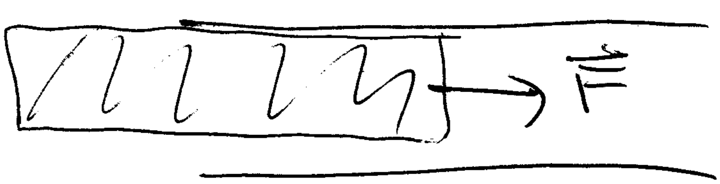
$\Rightarrow dW = \vec{F}\cdot d\vec{r} = +dU = \vec{\nabla}U\cdot d\vec{r}$

$$\boxed{\vec{F} = +\left(\vec{\nabla}U\right)_\varphi}$$

Similarly

$$\tau_i = +\left(\frac{\partial U}{\partial \theta_i}\right)_\varphi .$$

c) <u>ex.</u> || plate capacitors with dielectric slab



1) Q fixed

2) φ fixed