7 Hamiltonian Dynamics

We have been treating L as a function of q, $\dot{q}$, and t where the variation in generalized velocities $\delta\dot{q}$ is not an independent function but just the time derivative of the variation of the generalized coordinate $\frac{d}{dt}\delta q$ and hence these are related in the same way that q and $\dot{q}$ are related. In this way we have obtained n second order differential equations for q^a, the Euler-Lagrange equations

$$\frac{\partial L(q,\dot{q};t)}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L(q,\dot{q};t)}{\partial \dot{q}^a}\right) = 0. \tag{7.1}$$

Any set of n second order differential equations can be converted into $2n$ first order differential equations by means of Legendre transforming the function L. That is we can treat q^a and $v^a = \dot{q}^a$ as independent variables, that is the variations δq^a and δv^a are independent. Rather than v^a it is customary to use the momentum p_a conjugate to q^a, that is $p_a \equiv \partial L/\partial \dot{q}^q$ as the other independent variable. Then we can eliminate $\dot{q}$ from our equations in favor of p by viewing this formula

$$p_a = \frac{\partial L}{\partial \dot{q}^a} \tag{7.2}$$

as implicitly yielding

$$\dot{q}^a = \dot{q}^a(q^b, p_b; t). \tag{7.3}$$

We can use our definition of the Hamiltonian to determine the new first order dynamical equations of motion

$$H(q^a, p_a; t) = \sum_{a=1}^{n} p_a \dot{q}^a(q^b, p_b; t) - L(q^a, \dot{q}^a(q^b, p_b; t); t), \tag{7.4}$$

where now $H = H(q^a, p_a; t)$ is viewed as a function of the independent variables $(q, p; t)$ and where $\dot{q}$ occurs we use $\dot{q}^a(q^b.p_b; t)$. This is called a Legendre transformation. It changes the variables from $(q, \dot{q}; t)$ to $(q, p; t)$.

To find the equations of motion consider the change in H caused by an increment of time dt

$$dH = \sum_{a=1}^{n} \left(\frac{\partial H}{\partial q^a} dq^a + \frac{\partial H}{\partial p_a} dp_a \right) + \frac{\partial H}{\partial t} dt, \tag{7.5}$$

where dq and dp are independent. On the other hand we can use the definition of $H = \sum_{a=1}^{n} p_a \dot{q}^a - L$ to calculate its change

$$dH = \sum_{a=1}^{n} \left(dp_a \dot{q}^a + p_a d\dot{q}^a - \frac{\partial L}{\partial q^a} dq^a - \frac{\partial L}{\partial \dot{q}^a} d\dot{q}^a \right) - \frac{\partial L}{\partial t} dt. \tag{7.6}$$

But we have that $p_a = \frac{\partial L}{\partial \dot{q}^a}$ and the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) = 0, \tag{7.7}$$

are just

$$\dot{p}_a = \frac{\partial L}{\partial q^a}. \tag{7.8}$$

So this dynamical input implies

$$\begin{aligned} dH &= \sum_{a=1}^{n} (dp_a \dot{q}^a + p_a d\dot{q}^a - \dot{p}_a dq^a - p_a d\dot{q}^a) - \frac{\partial L}{\partial t} dt \\ &= \sum_{a=1}^{n} (\dot{q}^a dp_a - \dot{p}_a dq^a) - \frac{\partial L}{\partial t} dt. \end{aligned} \tag{7.9}$$

Now this must be the same result as equation (7.5) for the change in H, dH, hence we find

$$\begin{aligned} \dot{q}^a &= \frac{\partial H}{\partial p_a} \\ -\dot{p}_a &= \frac{\partial H}{\partial q^a} \\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t}. \end{aligned} \tag{7.10}$$

Substituting this into equation (7.5) yields

$$dH = \sum_{a=1}^{n} \left(-\dot{p}_a dq^a + \dot{q}^a dp_a\right) + \frac{\partial H}{\partial t} dt, \tag{7.11}$$

which implies

$$\begin{aligned} \frac{dH}{dt} &= \sum_{a=1}^{n} \left(-\dot{p}_a \dot{q}^a + \dot{q}^a \dot{p}_a\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}. \end{aligned} \tag{7.12}$$

The time derivatives of q^a and p_a are the $2n$ first order differential equations of motion for the (q^a, p_a) system and are known as *Hamilton's equations*

$$\begin{aligned} \dot{q}^a &= \frac{\partial H}{\partial p_a} \\ -\dot{p}_a &= \frac{\partial H}{\partial q^a}. \end{aligned} \tag{7.13}$$

q^a and p_a are called *canonically conjugate variables.* Recall the Euler-Lagrange equations of motion are n second order differential equations for q^a. Note if $\partial H/\partial t = 0$ then $H =$ constant. Further, if U is velocity independent and $x = x(q)$ with not explicit t dependence then

$$H = E. \tag{7.14}$$

Consider the example of a single particle in a conservative force field with potential energy $U(x, y, z)$

$$L = T - U = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z). \tag{7.15}$$

The Euler-Lagrange equations of motion are

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \tag{7.16}$$

which yields Newton's 2^{nd} law

$$m\ddot{x}_i = -\partial_i U. \tag{7.17}$$

Now Legendre transform to the Hamiltonian

$$H(\vec{x}, \vec{p}; t) = \vec{p} \cdot \dot{\vec{x}} - L(\vec{x}, \dot{\vec{x}}(\vec{x}, \vec{p}; t); t), \tag{7.18}$$

where we eliminate $\dot{\vec{x}}$ in terms of $\vec{x}$, $\vec{p}$ and t, $\dot{\vec{x}} = \dot{\vec{x}}(\vec{x}, \vec{p}; t)$ by the definition of $\vec{p}$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, \tag{7.19}$$

so that

$$\dot{\vec{x}} = \frac{1}{m}\vec{p}. \tag{7.20}$$

Hence

$$\begin{aligned} H(\vec{x}, \vec{p}; t) &= \frac{1}{m}\vec{p} \cdot \vec{p} - T + U \\ &= \frac{\vec{p}^2}{m} - \frac{1}{2}m\frac{1}{m^2}\vec{p} \cdot \vec{p} + U \\ &= \frac{\vec{p}^2}{2m} + U(\vec{x}). \end{aligned} \tag{7.21}$$

The dynamical equations of motion are Hamilton's equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i} = \frac{1}{m}p_i \\ -\dot{p}_i &= \frac{\partial H}{\partial x_i} = \partial_i U. \end{aligned} \tag{7.22}$$

Thus we have two first order differential equations. Note we obtain the usual second order differential equation Newton's law (Euler-Lagrange equation of motion) by substituting the $p_i = m\dot{x}_i$ Hamilton equation into the $\dot{p}_i = -\partial_i U$

Hamilton equation, implying $m\ddot{x}_i = -\partial_i U$. We can then solve this for x_i and $p_i = m\dot{x}_i$.

In general it is easier to obtain the equations of motion by Lagrangian methods. However since q and p are independent this sometimes leads to simplification of the analysis of the problem. In particular if q^a does not appear in H then

$$-\dot{p}_a = \frac{\partial H}{\partial q^a} = 0, \tag{7.23}$$

and $p_a = \text{constant} \equiv \pi_a$. q^a is then called *cyclic*. So

$$H = H(q^1, \ldots, \not{q}^a, \ldots, q^n, p_1, \ldots, \pi_a, \ldots, p_n; t) \tag{7.24}$$

depends on $(2n-2)$ variables now; we have reduced the number of degrees of freedom. q^a is said to be *ignorable*. Indeed $\dot{q}^a = \partial H/\partial \pi_a \equiv \omega^a$ and hence $q^a(t) = \int^t \omega^a dt$. This could lead to a practical simplification of the problem. If q^a is cyclic in H it is also cyclic in L, that is $\partial L/\partial q^a = 0$ ($H = p\dot{q} - L$ so that $\partial H/\partial q = p\partial\dot{q}/\partial q - \partial L/\partial q - \partial L/\partial\dot{q}\partial\dot{q}/\partial q = -\partial L/\partial q = 0$.) So we see that $p_a = \partial L/\partial\dot{q}^a = \text{constant}$ for cyclic variables.

Consider the two body central force problem with Lagrangian

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|). \tag{7.25}$$

Using new center of mass coordinates

$$\begin{aligned} \vec{R} &= \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2 \\ \vec{r} &= \vec{r}_1 - \vec{r}_2, \end{aligned} \tag{7.26}$$

and the inverse relations

$$\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M}\vec{r} \end{aligned} \tag{7.27}$$

where the total mass of the two bodies is $M = m_1 + m_2$ and the reduced mass is defined as

$$\mu \equiv \frac{m_1\, m_2}{m_1 + m_2}. \tag{7.28}$$

Hence the Lagrangian becomes

$$L = \frac{1}{2} M \dot{\vec{R}}^{\,2} + \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - U(|\vec{r}|). \tag{7.29}$$

If we choose our inertial frame to be at the center of mass, then the location of the center of mass is $\vec{R} = 0$ in that frame. Or more generally $\vec{R}$ is cyclic. So $\vec{P} = \partial L / \partial \dot{\vec{R}} = M\dot{\vec{R}} =$ constant. Thus we find

$$\vec{R} = \frac{\vec{P}}{M} t + \vec{R}_0. \tag{7.30}$$

So without loss of generality we can ignore $\vec{R}$. Hence the Lagrangian for the "relative particle" is

$$L = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - U(|\vec{r}|). \tag{7.31}$$

Next we can work in cylindrical coordinates (r, θ, z) in which L becomes

$$L = \frac{1}{2}\mu \left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U(\sqrt{r^2 + z^2}). \tag{7.32}$$

We see that θ is cyclic so that

$$\frac{\partial L}{\partial \theta} = 0, \tag{7.33}$$

which implies that

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant} \equiv l. \tag{7.34}$$

The Euler-Lagrange equations become

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) \;=\; 0 = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r} - \mu \ddot{r}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 = -\frac{\partial U}{\partial z} - \mu \ddot{z}$$
$$\text{and} \qquad l = \mu r^2 \dot{\theta}. \tag{7.35}$$

Since the two particles move in the same plane we can orient the coordinate axes so that there is no force in the z-direction and hence $z(t)$ remains at its initial value taken to be zero (along with its z component of velocity) so $z(t) = 0$. This implies that the potential energy is a function of r only $U = U(r)$. The radial Euler-Lagrange equation becomes

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{dU}{dr}, \tag{7.36}$$

where the angular momentum equation for $\dot{\theta} = l/\mu r^2$ has been used. For the $1/r$ gravitational potential energy, the orbits will be conic sections as we have seen and will see again.

Finally we can state our variational dynamical principle in terms of q and p as independent variations

$$\delta \int_{t_1}^{t_2} dt \left(\sum_{a=1}^{n} p_a \dot{q}^a - H(q, p; t) \right) = 0, \tag{7.37}$$

where δp_a and δq^a are independent but $\delta \dot{q}^a = \frac{d}{dt}\delta q^a$ is not independent. Hence

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} dt \sum_{a=1}^{n} \left(\delta p_a \dot{q}^a + \frac{d}{dt}(p_a \delta q^a) - \dot{p}_a \delta q^a - \frac{\partial H}{\partial q^a}\delta q^a - \frac{\partial H}{\partial p_a}\delta p_a \right) \\ &= \int_{t_1}^{t_2} dt \sum_{a=1}^{n} \left[\left(\dot{q}^a - \frac{\partial H}{\partial p_a} \right) \delta p_a - \left(\dot{p}_a + \frac{\partial H}{\partial q^a} \right) \delta q^a \right]. \end{aligned} \tag{7.38}$$

Since δq^a and δp_a are independent we obtain Hamilton's equations

$$\begin{aligned} \dot{q}^a &= \frac{\partial H}{\partial p_a} \\ -\dot{p}_a &= \frac{\partial H}{\partial q^a}. \end{aligned} \tag{7.39}$$