# 6 Conservation Laws and Symmetries

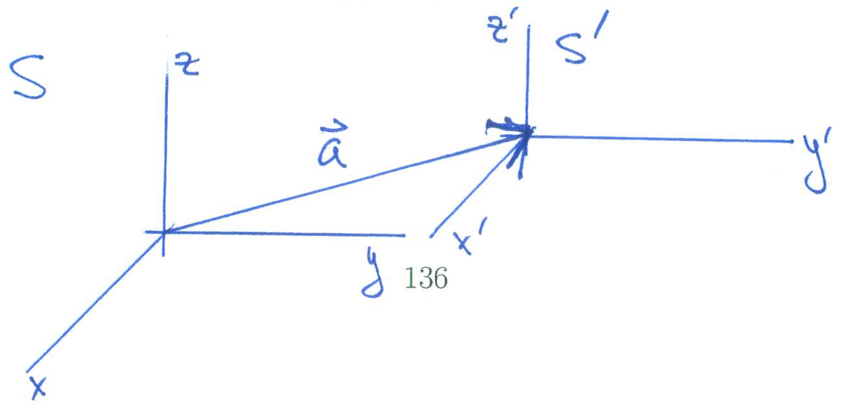
When considering Newton's Laws as the foundation of classical mechanics we found certain conservation laws when the forces on our system obeyed certain conditions. The Hamiltonian formulation of mechanics provides us with an even deeper insight into the nature of these conservation laws by relating them to symmetries of space-time that our system should have. Let's consider a system in which the fundamental forces are irrotational. Then we have that our dynamical principle is that of Hamilton's

$$\delta \int_{t_1}^{t_2} dt \; L(x_{\alpha\; i}, \dot{x}_{\alpha\; i}; t) = 0. \tag{6.1}$$

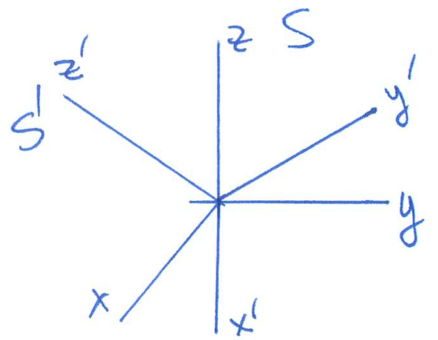
So $L$ determines the dynamical equations of the system. Hence we can relate our conservation equations to symmetries or invariances that $L$ must have; since now it determines the dynamics as $\vec{F}$ did for the Newtonian formulation. That is imagine we change the coordinates and time in our theory, we can then ask how that change might reflect itself in $L$. In particular, it is found experimentally over the centuries that our dynamical laws should be the same for whatever inertial frame we use.

If we are working in a closed system (a system which does not interact with anything outside the system) we expect that where the origin of our inertial frame is should not matter–that is the laws of physics (dynamics in this case) should be unchanged by a uniform translation of all the coordinates

$$x'_{\alpha\; i} = x_{\alpha\; i} + a_i, \tag{6.2}$$



where $\vec{a}$ is a constant translation vector.



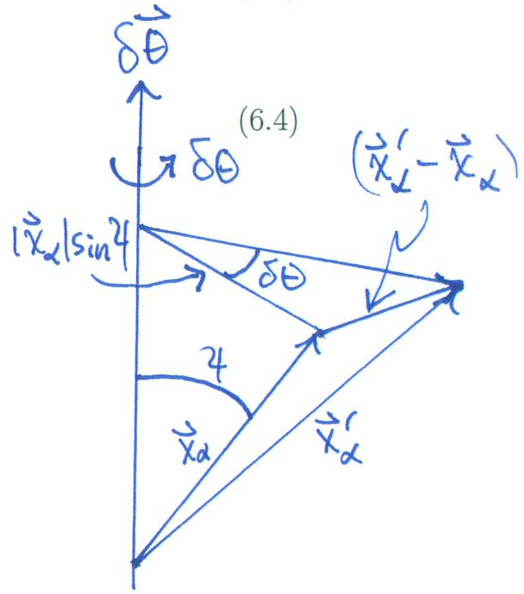

Nor should it matter if we change the orientation of our frame of reference, that is, apply a uniform rotation of all coordinates

$$x'_{\alpha\ i} = x_{\alpha\ i} + \epsilon_{ijk}\delta\theta_j x_{\alpha\ k} \tag{6.3}$$

or in vector notation

$$\vec{x}'_\alpha = \vec{x}_\alpha + \delta\vec{\theta} \times \vec{x}_\alpha, \tag{6.4}$$

where $\delta\vec{\theta}$ is the constant infinitesimal rotation angle.




Further it should not matter when we start our clock in describing dynamics; that is the zero of time is arbitrary. So the dynamics should be unchanged by a translation in time

$$t' = t + \epsilon, \tag{6.5}$$

where $\epsilon$ is a constant shift in the zero of time. That is to say space is homogeneous so we can translate our origin of the frame of reference to any point in space and space is isotropic so we can rotate our frame of reference– directions do not matter and time is homogeneous so that the zero of time is arbitrary. We can make a Galilean transformation of our frame of reference and the laws of physics (dynamics) remain the same. Since $L$ determines the dynamics we would like to translate the above space-time symmetries to properties of the dynamics, that is $L$.

Now we would like to distinguish the different types of transformations between observers that can be made. In particular let's be more careful about our notation. The coordinates used by observer $S$ are just the time $t$ and the position $x_{\alpha\ i}(t)$ where we have made manifest the coordinates' dependence on the time. Likewise observer $S'$ uses time $t'$ and coordinates $x'_{\alpha\ i}(t')$ where the time $t'$ that $S'$ uses is made manifest. So the general transformations we want to consider are explicitly

$$\begin{aligned} t' &= t + \epsilon \\ \vec{x}_\alpha{}'(t') &= \vec{x}_\alpha(t) + \vec{a} + \delta\vec{\theta} \times \vec{x}_\alpha, \end{aligned} \tag{6.6}$$

where the infinitesimal transformation parameters are given by $\vec{a}$, for space translations, $\delta\vec{\theta}$, for space rotations, and $\epsilon$, for time translations. The transformation of the time is denoted by

$$\delta t \equiv t' - t = \epsilon. \tag{6.7}$$

On the other hand we have two ways to describe the variation of the coordinates. First is the *total variation* of $\vec{x}_\alpha$ defined as the difference between the coordinates $S'$ uses to describe the location of the particles in space and that of $S$, this is denoted by an upper case $\Delta$

$$\Delta\vec{x}_\alpha = \vec{x}'_\alpha(t') - \vec{x}_\alpha(t) = \vec{a} + \delta\vec{\theta} \times \vec{x}_\alpha. \tag{6.8}$$

So in the case of time translations, if a particle is located at $x = 1$ meter at time $t = 2$ seconds for observer $S$ then the particle is at the same location in space, $x' = 1$ meter, but this occurs at a different value of the argument that is at $t' = t + \epsilon = 2 + \epsilon$ seconds for observer $S'$. Thus $x'(t') = x(t)$, or the total variation of $x$ is zero in the case of translations of the zero of time, $\Delta \vec{x}_\alpha = \vec{x}'_\alpha(t') - \vec{x}_\alpha(t) = 0$, as indicated above.

Instead of the difference in coordinate functions evaluated at different numerical values of their arguments, we can consider the *intrinsic variation* of the coordinate functions themselves. This is just the difference in the two functions evaluated at the same numerical argument, and is denoted with a lower case $\delta$ (note: we have used $\delta$ and $d$ and now $\Delta$ in several different ways, which we are talking about should be made clear by the context)

$$\delta \vec{x}_\alpha \equiv \vec{x}'_\alpha(t) - \vec{x}_\alpha(t). \tag{6.9}$$

Now for infinitesimal transformations we can relate the two types of variations as

$$\begin{aligned} \Delta \vec{x}_\alpha &= \vec{x}'_\alpha(t') - \vec{x}_\alpha(t) = \vec{x}'_\alpha(t') - \vec{x}_\alpha(t') + \vec{x}_\alpha(t') - \vec{x}_\alpha(t) \\ &= \delta \vec{x}_\alpha + \delta t \, \dot{\vec{x}}_\alpha(t). \end{aligned} \tag{6.10}$$

The second term above comes from Taylor expanding the coordinate for infinitesimal time shifts $\vec{x}_\alpha(t') = \vec{x}_\alpha(t + \delta t) = \vec{x}_\alpha(t) + \delta t \, \dot{\vec{x}}_\alpha(t)$. Putting this together with the total variation in equation (6.8) we have the intrinsic variation of the coordinate is

$$\delta \vec{x}_\alpha = \vec{a} + \delta \vec{\theta} \times \vec{x}_\alpha - \epsilon \dot{\vec{x}}_\alpha. \tag{6.11}$$

Note it is only when we change the zero of time that there is a difference between the total and intrinsic variations of the coordinates. For space translations and rotations the two types of variations are the same.

Now suppose we change from the unprimed to the primed coordinates and time as above. The Lagrangian depends on the coordinates, velocities and time which will change when evaluated in the primed frame of reference. So if observer $S$ uses the Lagrangian $L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t)$ then from observer $S'$'s frame of reference this function will become $L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t')$. The $S'$ observer uses the Lagrangian $L'$ obtained from $L$ by substituting the inverse transformation for the coordinates $x_{\alpha\ i}$ and velocities $\dot{x}_{\alpha\ i}$ in terms of the $x'_{\alpha\ i}$, $\dot{x}'_{\alpha\ i}$ and $t'$ on the right hand side of

$$L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t). \tag{6.12}$$

This is the definition of $L'$.

It should be noted that the definition of $S'$'s Lagrangian might require a slightly more general relation. As we have seen from Hamilton's principle it is the action that determines the dynamics, and so it is the action as written by the two observers that is the same, the transformations are just a change of variables. If the time varies in a more general manner than just shifting the zero of time, $t' = t + \delta t = t + \delta t(x, t)$, then the integral over time will transform with an additional factor so that the action for the two observers is related according to

$$\begin{aligned} \Gamma &= \int_{t'_1}^{t'_2} dt'\ L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') \\ &= \int_{t_1}^{t_2} dt\ L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t). \end{aligned} \tag{6.13}$$

Once again, this gives the definition of $S'$'s Lagrangian $L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = (dt/dt')L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t)$. In our case equation (6.12) defines $L'$ (we will only consider constant shifts in the time $\delta t = \epsilon$ and $\epsilon$ is a constant so that $d\delta t/dt = 0$, that is $dt'/dt = 1$).

So far we have given the definition of $L'$, equation (6.12). For there to be a symmetry of the system the laws of motion have to be of the same form for each observer in terms of their own coordinates and time, that is to be form invariant, then $L'$ must have the same functional form as $L$ so that each set of Euler-Lagrange equations have the same appearance in each observer's coordinates and time. That is if the transformation between the observers corresponds to a symmetry of the system so that the equations of motion are of the same form in either set of variables, we must have

$$L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') + \frac{dF}{dt'}, \tag{6.14}$$

where we have demanded equality up to a total time derivative of a function of the coordinates and time since such a term, $dF(x'_{\alpha\ i}(t'); t')/dt' = \sum_{\alpha\ i}(\partial F/\partial x'_{\alpha\ i}(t'))\dot{x}'_{\alpha\ i}(t') + \partial F/\partial t'$, does not contribute to the Euler-Lagange equations. Simply put $L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t)$, equation (6.14) then implies that $L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t)$. That is the Lagrangian is invariant when $S'$'s variables are substituted for those of $S$, $L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') = L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t)$. So equation (6.12) and equation (6.14) combine to define the Lagrangian $L'$ and to describe its invariance under symmetry transformations.

The change in $L$ corresponding to a change in the time and space coordinates from one observer to another can be found by considering the total variation of the Lagrangian and expanding it to first order in the transformed terms

$$\begin{aligned}
\Delta L &= L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) \\
&= L(x_{\alpha\ i}(t) + \Delta x_{\alpha\ i}, \dot{x}_{\alpha\ i}(t) + \Delta \dot{x}_{\alpha\ i}; t + \delta t) - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) \\
&= \frac{\partial L}{\partial x_{\alpha\ i}} \Delta x_{\alpha\ i} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \Delta \dot{x}_{\alpha\ i} + \frac{\partial L}{\partial t} \delta t. \qquad (6.15)
\end{aligned}$$

Now the intrinsic variations of the velocities are just equal to the time derivative of the intrinsic variations of the coordinates since intrinsic variations are defined at the same time argument

$$\begin{aligned}
\delta \dot{x}_{\alpha\ i} &= \dot{x}'_{\alpha\ i}(t) - \dot{x}_{\alpha\ i}(t) \\
&= \frac{d}{dt}\left[x'_{\alpha\ i}(t) - x_{\alpha\ i}(t)\right] \\
&= \frac{d}{dt}\delta x_{\alpha\ i}. \qquad (6.16)
\end{aligned}$$

On the other hand the total variation of the velocities is in general not equal to the time derivative of the total variation of the coordinates since the time arguments are different in the definition of the total variation

$$\begin{aligned}
\Delta \dot{x}_{\alpha\ i} &= \frac{d}{dt'}x'_{\alpha\ i}(t') - \frac{d}{dt}x_{\alpha\ i}(t) \\
&= \frac{d}{dt'}x'_{\alpha\ i}(t') - \frac{dt'}{dt}\frac{d}{dt'}x_{\alpha\ i}(t) \\
&= \frac{d}{dt'}\left[x'_{\alpha\ i}(t') - x_{\alpha\ i}(t)\right] - \left[\frac{dt'}{dt} - 1\right]\frac{d}{dt'}x_{\alpha\ i}(t) \\
&= \frac{d}{dt}\Delta x_{\alpha\ i} - \left[\frac{dt'}{dt} - 1\right]\dot{x}_{\alpha\ i}, \qquad (6.17)
\end{aligned}$$

where in the last line we can set the primed time derivatives equal to the unprimed time derivatives since the $[dt'/dt - 1]$ factor as well as the $\Delta x_{\alpha\ i}$ factor are already first order in the variation. For our consideration $\delta t = \epsilon$ so that $dt'/dt = 1$ and $\Delta \dot{x}_{\alpha\ i} = d/dt \Delta x_{\alpha\ i}$.

Proceeding in the general case again, $\Delta L$ becomes

$$\begin{aligned}
\Delta L &= \frac{\partial L}{\partial x_{\alpha\ i}}\Delta x_{\alpha\ i} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{d}{dt}\Delta x_{\alpha\ i} + \frac{\partial L}{\partial t}\delta t \\
&= \left[\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right]\Delta x_{\alpha\ i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\Delta x_{\alpha\ i}\right] + \frac{\partial L}{\partial t}\delta t. \qquad (6.18)
\end{aligned}$$

The formula is just an algebraic identity at this point. It becomes a dynamical relation when we impose the Euler-Lagrange equations so that the first term on the right hand side vanishes. Thus we obtain the first form of Noether's Theorem

$$\Delta L = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \Delta x_{\alpha\ i}\right] + \frac{\partial L}{\partial t}\delta t. \tag{6.19}$$

If the transformation of the coordinates and time are a symmetry transformation of the system equation (6.14) is valid, then $\Delta L = L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) = L'(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) = 0$. We can stop here and utilize this general formula for our various transformations.

However, another form of Noether's Theorem that proves very useful for deriving conservation theorems involves the intrinsic variation of the coordinates and Lagrangian rather than the total variation. Recalling equations (6.9), (6.10) and (6.11), the intrinsic variation of the Lagrangian is given by

$$\begin{aligned}
\delta L &= L(x'_{\alpha\ i}(t), \dot{x}'_{\alpha\ i}(t); t) - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) \\
&= L(x_{\alpha\ i}(t) + \delta x_{\alpha\ i}, \dot{x}_{\alpha\ i}(t) + \delta \dot{x}_{\alpha\ i}; t) - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) \\
&= \frac{\partial L}{\partial x_{\alpha\ i}}\delta x_{\alpha\ i} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\delta \dot{x}_{\alpha\ i}.
\end{aligned} \tag{6.20}$$

Using equation (6.16) this can be written as

$$\delta L = \left[\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right]\delta x_{\alpha\ i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \delta x_{\alpha\ i}\right]. \tag{6.21}$$

Now recall the total variation of $L$, equation (6.15),

$$\begin{aligned}
\Delta L &= L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') - L(x_{\alpha\ i}(t), \dot{x}_{\alpha\ i}(t); t) \\
&= [L(x'_{\alpha\ i}(t'), \dot{x}'_{\alpha\ i}(t'); t') - L(x_{\alpha\ i}(t'), \dot{x}_{\alpha\ i}(t'); t')]
\end{aligned}$$

$$+\left[L(x_{\alpha\ i}(t'),\dot{x}_{\alpha\ i}(t');t') - L(x_{\alpha\ i}(t),\dot{x}_{\alpha\ i}(t);t)\right]$$

$$= \delta L + \delta t \frac{dL}{dt}, \tag{6.22}$$

where we only keep first order in the variations. Hence we find the relation between the intrinsic and total variations of the Lagrangian

$$\delta L = \Delta L - \delta t \frac{dL}{dt}. \tag{6.23}$$

Substituting this above yields

$$\begin{aligned} \delta L &= \Delta L - \delta t \frac{dL}{dt} = \Delta L - \frac{d}{dt}\left[\delta t L\right] + \frac{d\delta t}{dt}\ L \\ &= \left[\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right]\delta x_{\alpha\ i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \delta x_{\alpha\ i}\right]. \end{aligned} \tag{6.24}$$

Hence we obtain the final form of the algebraic relation

$$\Delta L + \frac{d\delta t}{dt}\ L = \left[\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right]\delta x_{\alpha\ i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \delta x_{\alpha\ i} + \delta t\ L\right]. \tag{6.25}$$

The second form of Noether's Theorem is obtained by enforcing the Euler-Lagrange dynamical equations of motion $\left[\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right] = 0$ to give

$$\Delta L + \frac{d\delta t}{dt}\ L = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \delta x_{\alpha\ i} + \delta t\ L\right]. \tag{6.26}$$

We are now ready to apply Noether's Theorem (6.19) or (6.26) to the space and time transformations of equations (6.6), (6.7) and hence (6.8) and (6.11). First let's consider the invariance of the laws of physics under the change of the zero of time, that is the homogeneity of time: $\delta t = \epsilon$, where $\epsilon$ is an infinitesimal constant. We demand, as a result of centuries of experiments, that the dynamics should be independent of the origin of time, this means that $L$ should not depend on time explicitly and hence

$$L(x,\dot{x};t+\epsilon) = L(x,\dot{x};t), \tag{6.27}$$

that is

$$\frac{\partial L}{\partial t} = 0. \tag{6.28}$$

From (6.8) we see that $\Delta x_{\alpha\ i} = 0$ so that (6.19) implies that $\Delta L = 0$ for time translations. Since $\epsilon$ is a constant we have that $d\delta t/dt = 0$ and from (6.11) $\delta x_{\alpha\ i} = -\epsilon \dot{x}_{\alpha\ i}$, so (6.26) becomes

$$0 = -\epsilon \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \dot{x}_{\alpha\ i} - \ L\right]. \tag{6.29}$$

Thus we define the *Hamiltonian* $H$ by

$$H \equiv \left(\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\ \dot{x}_{\alpha\ i}\right) - \ L. \tag{6.30}$$

Noether's Theorem then tells us that due to the observed homogeneity of time, $\frac{\partial L}{\partial t} = 0$, the Hamiltonian is a constant

$$H = \text{constant}. \tag{6.31}$$

Note that if $\frac{\partial L}{\partial t} \neq 0$, then Noether's Theorem implies that $\Delta L = \epsilon \partial L/\partial t$ and so

$$\frac{d}{dt}H = -\frac{\partial L}{\partial t}. \tag{6.32}$$

Further if $U$ is independent of velocity, $\partial U/\partial \dot{x}_{\alpha\ i} = 0$, and time (conservative forces) then

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} &= \frac{\partial (T-U)}{\partial \dot{x}_{\alpha\ i}} = \frac{\partial T}{\partial \dot{x}_{\alpha\ i}} = m_\alpha \dot{x}_{\alpha\ i} \\ &= p_{\alpha\ i}. \end{aligned} \tag{6.33}$$

Since $T = 1/2 \sum_{\alpha\ i} m_\alpha (\dot{x}_{\alpha\ i})^2$, we have that

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \dot{x}_{\alpha\ i} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha (\dot{x}_{\alpha\ i})^2 = 2T. \tag{6.34}$$

Hence

$$H = 2T - L = 2T - (T - U) = T + U = E \qquad \text{the total energy.} \tag{6.35}$$

So we find that as long as $\partial U/\partial \dot{q}^A = 0$ and $x_{\alpha\ i} = x_{\alpha\ i}(q^A)$, we have that

$$\begin{aligned}
\sum_{A=1}^{3N} \frac{\partial L}{\partial \dot{q}^A}\dot{q}^A &= \sum_{A=1}^{3N}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \frac{\partial \dot{x}_{\alpha\ i}}{\partial \dot{q}^A}\dot{q}^A \\
&= \sum_{A=1}^{3N}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \frac{\partial x_{\alpha\ i}}{\partial q^A}\frac{dq^A}{dt} \\
&= \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\dot{x}_{\alpha\ i} \\
&= 2T,
\end{aligned} \tag{6.36}$$

having used $\dot{x} = (\partial x/\partial q)\dot{q}$. So homogeneity of time, $\partial L/\partial t = 0$, once again implies that

$$H \equiv \sum_{A=1}^{3N} \frac{\partial L}{\partial \dot{q}^A}\dot{q}^A - L = E = \text{constant}, \tag{6.37}$$

the total energy is conserved.

Next consider the homogeneity of space. That is the dynamics should remain unchanged under a translation of the origin of the frame of reference by a constant vector $\vec{a}$. From equation (6.7) we have that $\delta t = 0$ and from equations (6.11) and (6.10) we have that $\Delta x_{\alpha\ i} = \delta x_{\alpha\ i} = a_i$ and so $\Delta \dot{x}_{\alpha\ i} = \delta \dot{x}_{\alpha\ i} = 0$. The origin of space should not matter; so we require that

$$L(x_{\alpha\ i} + \Delta x_{\alpha\ i}, \dot{x}_{\alpha\ i}; t) = L(x_{\alpha\ i}, \dot{x}_{\alpha\ i}; t), \tag{6.38}$$

that is $L$ is left invariant under a translation of the origin of the frame of reference. Hence, applying this to the first line of equation (6.18), we have

$$\Delta L = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial x_{\alpha\ i}} a_i = 0. \tag{6.39}$$

The first form and second form of Noether's Theorem reduces to the same equation when $\delta t = 0$, so equation (6.19) becomes

$$\Delta L = \frac{d}{dt}\left(\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}a_i\right). \tag{6.40}$$

Since $\Delta L = 0$ and $a_i$ is an arbitrary constant, we find

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right) = 0 \tag{6.41}$$

for each $i = 1, 2, 3$. This implies that

$$\sum_{\alpha=1}^{N}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} = \text{constant}. \tag{6.42}$$

Now once again

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} &= \frac{\partial (T-U)}{\partial \dot{x}_{\alpha\ i}} = \frac{\partial T}{\partial \dot{x}_{\alpha\ i}} = m_\alpha \dot{x}_{\alpha\ i} \\ &= p_{\alpha\ i}\end{aligned} \tag{6.43}$$

the linear momentum. Hence the homogeneity of space implies that

$$\sum_{\alpha=1}^{N} p_{\alpha\ i} = \text{constant}, \tag{6.44}$$

that is

$$\sum_{\alpha=1}^{N} \vec{p_i} = \vec{P} = \text{constant}, \tag{6.45}$$

the total linear momentum is conserved.

Finally consider the isotropy of space, that is the orientation of the axes of the frame of reference does not alter the dynamics. In this case we have from equation (6.7) that $\delta t = 0$ and from equations (6.11) and (6.10) that $\Delta x_{\alpha\ i} = \delta x_{\alpha\ i} = \epsilon_{ijk}\delta\theta_j x_{\alpha\ k}$ and so $\Delta \dot{x}_{\alpha\ i} = \epsilon_{ijk}\delta\theta_j \dot{x}_{\alpha\ k}$. The orientation of the space axes should not matter; so we require that $L$ be rotationally invariant

$$L(\vec{x}_\alpha + \delta\vec{\theta} \times \vec{x}_\alpha, \dot{\vec{x}}_\alpha + \delta\vec{\theta} \times \dot{\vec{x}}_\alpha; t) = L(\vec{x}_\alpha, \dot{\vec{x}}_{\alpha\ i}; t). \tag{6.46}$$

Hence, applying this to the first line of equation (6.18), we have

$$\Delta L = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \left( \frac{\partial L}{\partial x_{\alpha\ i}} \epsilon_{ijk} \delta\theta_j x_{\alpha\ k} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \epsilon_{ijk} \delta\theta_j \dot{x}_{\alpha\ k} \right) = 0. \tag{6.47}$$

As before the first form and second form of Noether's Theorem reduces to the same equation when $\delta t = 0$, so equation (6.19) becomes

$$\Delta L = \frac{d}{dt} \left( \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \Delta x_{\alpha\ i} \right) = 0, \tag{6.48}$$

since $\Delta L = 0$. So

$$\begin{aligned} \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \Delta x_{\alpha\ i} &= \text{constant} \\ &= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} p_{\alpha\ i} \epsilon_{ijk} \delta\theta_j x_{\alpha\ k} \\ &= \delta\theta_j \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \epsilon_{ijk} p_{\alpha\ i} x_{\alpha\ k} \\ &= \delta\theta_j \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \epsilon_{jki} x_{\alpha\ k} p_{\alpha\ i} \\ &= \delta\theta_j \left( \sum_{\alpha=1}^{N} \vec{x}_\alpha \times \vec{p}_\alpha \right)_j \\ &= \delta\theta_j L_j = \text{constant}. \end{aligned} \tag{6.49}$$

Since $\delta\theta_j$ is an arbitrary constant this implies

$$L_j = \sum_{\alpha=1}^{N} (\vec{x}_\alpha \times \vec{p}_\alpha)_j = \text{constant}, \tag{6.50}$$

the total angular momentum is conserved.

To summarize the relations between the properties of space and time, symmetries and conservation laws we have the table

| Inertial Frame Has | Symmetry Property of $L$ | Conserved Quantity |
|---|---|---|
| Time Homogeneous | $L(x,\dot{x};t+\epsilon)=L(x,\dot{x};t)\rightarrow\frac{\partial L}{\partial t}=0$ | $H(=E)$ |
| Space Homogeneous | $L(x+a,\dot{x};t)=L(x,\dot{x};t)\rightarrow\sum_{\alpha}\partial L/\partial x_{\alpha\ i}=0$ | $\vec{P}$ |
| Space Isotropic | $L(Rx,R\dot{x};t)=L(x,\dot{x};t)$ | $\vec{L}$ |

(6.51)

where $R$ stands for a rotation.

We can derive a further statistical "conservation" law for the ensemble of particles by considering time averages of the motion of the system. Suppose we know that the system has bounded motion in $x_{\alpha\ i}$ and $p_{\alpha\ i}$, we can define the scalar quantity

$$S\equiv\sum_{\alpha=1}^{N}\sum_{i=1}^{3}p_{\alpha\ i}x_{\alpha\ i}=\sum_{\alpha=1}^{N}\vec{p}_{\alpha}\cdot\vec{r}_{\alpha}. \tag{6.52}$$

Then consider the time derivative

$$\frac{dS}{dt}=\sum_{\alpha=1}^{N}\dot{\vec{p}}_{\alpha}\cdot\vec{r}_{\alpha}+\vec{p}_{\alpha}\cdot\dot{\vec{r}}_{\alpha}. \tag{6.53}$$

The time average of $dS/dt$ over the time interval $[0,T]$ is given by

$$\left\langle\frac{dS}{dt}\right\rangle=\frac{1}{T}\int_{0}^{T}\frac{dS}{dt}dt=\frac{S(T)-S(0)}{T}. \tag{6.54}$$

If the motion is periodic and $T$ is an integer multiple of the period, then $S(T)=S(0)$ so that

$$\left\langle\frac{dS}{dt}\right\rangle=0. \tag{6.55}$$

If the motion is not periodic but just bounded then $S$ is bounded and for $T\rightarrow\infty$

$$\left\langle\frac{dS}{dt}\right\rangle=0. \tag{6.56}$$

So we find that

$$\Big\langle \sum_{\alpha=1}^{N} \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \Big\rangle = -\Big\langle \sum_{\alpha=1}^{N} \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha \Big\rangle. \tag{6.57}$$

Now $\vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha = 2T_\alpha$, the kinetic energy for each particle, while $\dot{\vec{p}}_\alpha = \vec{F}_\alpha$, the force on the $\alpha^{th}$ particle. The above equation then becomes

$$\Big\langle T \Big\rangle = -\frac{1}{2}\Big\langle \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \vec{r}_\alpha \Big\rangle, \tag{6.58}$$

with the total kinetic energy $T = \sum_{\alpha=1}^{N} T_\alpha$. This is called the Clausius Virial Theorem. $[-1/2 < \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \vec{r}_\alpha >]$ is called the virial of the system.

Suppose $\vec{F}_\alpha$ is derived from a potential, then

$$\Big\langle T \Big\rangle = \frac{1}{2}\Big\langle \sum_{\alpha=1}^{N} \vec{r}_\alpha \cdot \vec{\nabla}_\alpha U_\alpha \Big\rangle. \tag{6.59}$$

For example, consider central forces between two bodies with

$$U = kr^{n+1}, \tag{6.60}$$

where $r$ is the distance between the bodies. Then

$$\begin{aligned} \sum_{\alpha=1}^{N} \vec{r}_\alpha \cdot \vec{\nabla}_\alpha U_\alpha &= \vec{r}_1 \cdot \vec{\nabla}_1 U(|\vec{r}_1 - \vec{r}_2|) + \vec{r}_2 \cdot \vec{\nabla}_2 U(|\vec{r}_1 - \vec{r}_2|) \\ &= (\vec{r}_1 - \vec{r}_2) \cdot \vec{\nabla}_1 U(|\vec{r}_1 - \vec{r}_2|) \\ &= \vec{r} \cdot \vec{\nabla} U(|\vec{r}|) \\ &= r\frac{dU}{dr} = (n+1)kr^{n+1} = (n+1)U. \end{aligned} \tag{6.61}$$

This yields

$$\Big\langle T \Big\rangle = \frac{(n+1)}{2}\Big\langle U \Big\rangle. \tag{6.62}$$

For the $1/r^2$ force $n = -2$ i.e. for the gravitational force, then we find

$$\Big\langle T \Big\rangle = -\frac{1}{2}\Big\langle U \Big\rangle, \tag{6.63}$$

the virial theorem for the $1/r^2$ force ($1/r$ potential).