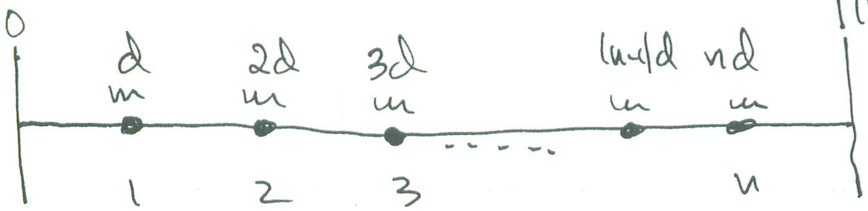
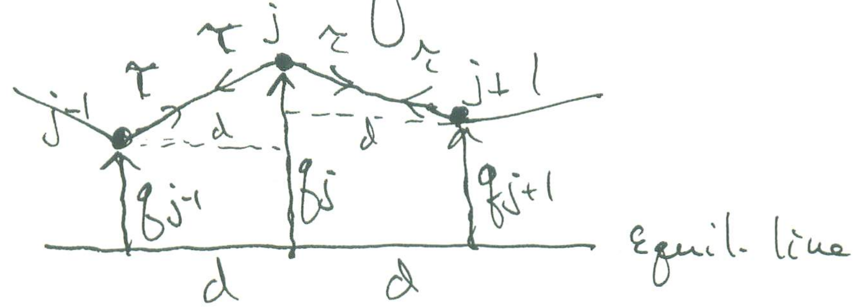
Continuum Mechanics

Next Consider the Vibrations of a String. First we will consider an elastic string with n-masses m attached at equal distances d at equilibrium.



The fixed ends string has length $L = (n+1)d$

We are interested in the small transverse oscillations about equilibrium as shown below



Let τ be the tension force in the string which we will assume is constant for ($\tau = \tau_0$ equil. tension) small oscillations. The "transverse" (vertical) displacements from the eq. line are denoted q_j. Hence the force on the j^{th} mass is

(i.e. $\sin\theta \simeq \tan\theta$)

$$F_j = -\tau \frac{(q_j - q_{j-1})}{d} - \tau \frac{(q_j - q_{j+1})}{d}$$

$q_0 = 0 = q_{n+1}$

which by N-2 $\Rightarrow$

$$m\ddot{q}_j = \frac{\tau}{d}\left[q_{j-1} - 2q_j + q_{j+1}\right]$$

i.e. nearest neighbor interactions - only adjacent particles

Could be otherwise – $\frac{1}{r^2}$ force or such, but usually screened.

(Note: same force equation for longitudinal vibrations with q_j long. displacements and $\frac{\tau}{d} \to K =$ spring constant.)

As well we could use Lagrangian techniques: The potential energy for stretching the springs = work done to stretch springs.

$U_j = -\int F_j \cdot dq_j$

$(U = \sum_j U_j$

$\frac{\tau}{d}\left[q_j^2 \tfrac{1}{2}\Big|_{q_{j-1}}^{q_j} - q_{j-1} q_j \Big|_{q_{j-1}}^{q_j}\right.$

$= \frac{1}{2}\frac{\tau}{d}(q_j - q_{j-1})^2 + \frac{\tau}{d}[\ ]$

$$U = \frac{1}{2}\frac{\tau}{d}\sum_{j=1}^{n+1}(q_{j-1} - q_j)^2$$

where $q_0 \equiv 0 \equiv q_{n+1}$ the fixed ends of the string.

Note

$$F_j = -\frac{\partial}{\partial q_j} U = -\frac{\tau}{2d}\frac{\partial}{\partial q_j}\left[(q_{j-1} - q_j)^2 + (q_j - q_{j+1})^2\right]$$

$$= +\frac{\tau}{d}(q_{j-1} - 2q_j + q_{j+1}) \checkmark$$

The kinetic energy is just

$$T = \frac{1}{2} m \sum_{j=1}^{n} \dot{q}_j^2 = \frac{1}{2} m \sum_{j=1}^{n+1} \dot{q}_j^2$$

since $q_{n+1} \equiv 0$. Thus

$$L = \frac{1}{2}\sum_{j=1}^{n+1}\left[m\dot{q}_j^2 - \frac{\tau}{d}(q_{j-1} - q_j)^2\right]$$

Hence E-L equations $\Leftrightarrow$ N-2

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \boxed{= 0 = +\frac{\tau}{d}(q_{j-1} - 2q_j + q_{j+1}) - m\ddot{q}_j}$$

Again we look for oscillatory solutions:

$$q_j(t) = a_j e^{i\omega t} \quad ; \quad a_j \in \mathbb{C} \text{ here}$$

N-2 $\Rightarrow$

$$-\frac{\tau}{d} a_{j-1} + \underbrace{\left(2\frac{\tau}{d} - m\omega^2\right)}_{\equiv \lambda} a_j - \frac{\tau}{d} a_{j+1} = 0$$

where $j = 1, \ldots, n$ and $a_0 \equiv 0 \equiv a_{n+1}$, the fixed ends.

This is a linear difference eq. we solve for eigenfreq. ω_r as before:

$$\begin{bmatrix} \lambda & -\frac{\tau}{d} & 0 & 0 & \cdots & & \\ -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & 0 & 0 & \cdots & \\ 0 & -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & 0 & 0 \cdots & \vdots \\ 0 & 0 & -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & 0 \cdots & \vdots \\ \vdots & \vdots & & \ddots & \lambda & \ddots & 0 \\ 0 & \cdots & & 0 & -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} \\ 0 & 0 & \cdots & & & -\frac{\tau}{d} & \lambda \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = 0$$

with $\boxed{\lambda \equiv 2\frac{\tau}{d} - m\omega^2}$

So a sol. exists if the secular det. vanishes

$$\det \begin{vmatrix} \lambda & -\frac{\tau}{d} & 0 & \cdots & \\ -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & 0 & \cdots \\ & & \ddots & \ddots & \\ \cdots & & & & \end{vmatrix} = 0$$

To find this solution consider cases:

$n=1$ 

$$\left(2\frac{\tau}{d} - m\omega^2\right)a_1 = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{2\tau}{md}}\,.$$

(just like springs with $\frac{\tau}{d} \to \kappa$; $\omega = \sqrt{\frac{2\kappa}{m}}$.

$n=2$  we worked this out earlier

$$\begin{vmatrix} \lambda & -\frac{\tau}{d} \\ -\frac{\tau}{d} & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\tau}{d}\right)^2 = 0$$

$$\Rightarrow 0 = m^2(\omega^2)^2 - 4\frac{m\tau}{d}(\omega^2) + 3\frac{\tau^2}{d^2}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{2\frac{\tau}{d} \pm \frac{\tau}{d}}{m}} \qquad \left(\kappa = \frac{\tau}{d}\right)$$

$$= \begin{cases} \sqrt{\frac{3\tau/d}{m}} \\ \sqrt{\frac{\tau}{dm}} \end{cases}$$

as we found ($\kappa_{12} = \kappa$) earlier

$$\sqrt{\frac{3\kappa}{m}}\,,\ \sqrt{\frac{\kappa}{m}}$$

$q_j(t) = a_j e^{i\omega t}$

For general n we need a more general method of solution.

Ansatz:
justified ex post facto

$$a_j = a e^{i(j\gamma - \delta)} \qquad \delta, \gamma, a \in \mathbb{R}$$

$$N2 \Rightarrow \quad -\frac{\tau}{d} e^{-i\gamma} + \left(2\frac{\tau}{d} - m\omega^2\right) - \frac{\tau}{d} e^{i\gamma} = 0$$

$\Rightarrow$

$$\omega^2 = \frac{2\tau}{md} - \frac{\tau}{md}\left(e^{i\gamma} + e^{-i\gamma}\right)$$

$$= \frac{2\tau}{md}(1 - \cos\gamma)$$

$$\boxed{\omega^2 = \frac{4\tau}{md}\sin^2\frac{\gamma}{2}}$$

Now det $= 0$ is order n poly. for ω^2; hence we have n-solutions

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\,\sin\frac{\gamma_r}{2} \quad ; r = 1, 2, \ldots, n$$

and we must evaluate γ_r & δ_r in a_{jr} by apply BC that ends of string are fixed.

So $\quad a_{jr} = a_r e^{i(j\gamma_r - \delta_r)}$

or Real parts $\quad \boxed{a_{jr} = a_r \cos(j\gamma_r - \delta_r)}$

Now

$$a_{0r} = a_{(n+1)r} \equiv 0$$

$a_{0r} = 0 \Rightarrow \cos\delta_r = 0 \Rightarrow \boxed{\delta_r = \frac{\pi}{2}}$

hence $\quad a_{jr} = a_r \cos(j\gamma_r - \frac{\pi}{2})$

$$= a_r \sin j\gamma_r$$

and for $j = n+1$

$$a_{n+1\, r} \equiv 0 = a_r \sin(n+1)\gamma_r$$

$\Rightarrow$

$$(n+1)\gamma_r = s\pi \qquad s = 1, 2, \ldots$$

$\Rightarrow \quad \boxed{\gamma_r = \frac{s\pi}{n+1} \quad ; \; s = 1, 2, \ldots}$

Since we must have n-distinct ω_r's $s = 1, \ldots, n$. Since then there is a 1-1 relation between r & s let $r = s$ in $\gamma_r \Rightarrow$

$$\boxed{\gamma_r = \frac{r\pi}{n+1} \qquad r = 1, \ldots, n}$$

$\Rightarrow$

$$\boxed{a_{jr} = a_r \sin\left(j \frac{r\pi}{n+1}\right)}$$

Now q_j is a superposition of these n- ~~sol~~ eigensolutions

$$q_j = \sum_{r=1}^{n} \beta_r' a_{jr} e^{i\omega_r t}$$

$$= \sum_r \underbrace{\beta_r' a_r}_{\equiv \beta_r} \sin\left(j\frac{r\pi}{n+1}\right) e^{i\omega_r t}$$

$$\boxed{q_j = \sum_{r=1}^{n} \beta_r \sin\left(j\frac{r\pi}{n+1}\right) e^{i\omega_r t} \quad \text{with} \quad \omega_r = 2\sqrt{\frac{\tau}{md}} \sin\left[\frac{r\pi}{2(n+1)}\right]}$$

Agrees with $n=1,2$ case above. ✓ $\left[\begin{array}{l} n=2 \\ \sin\frac{\pi}{6} = \frac{1}{2}, \ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{array}\right]$

Note: a) $r = 0, n+1 \Rightarrow a_{jr} = 0$ <u>null</u> <u>modes</u>

b) $r = n+2, n+3, \ldots, 2n+1$ $\{a_{jr}\}$ same times (-1) as $r = 1 \ldots, n$ (opposite order)

c) $r = 2n+2$ $a_{jr} = 0$ null mode

d) $r = 2n+3, 2n+4, \ldots 2n+(2+n)$ $\{a_{jr}\}$ same as $\{a_{jr}\}$ $r = 1, \ldots, n$

So repeat after <u>nullmodes</u>

e) same if $r < 0$.

So indeed <u>$r = 1, \ldots, n$ unique n-modes</u>

Now we can define the normal modes:

$$\eta_r(t) \equiv \beta_r e^{i\omega_r t}$$

so that

$$q_j(t) = \sum_{r=1}^{n} \eta_r(t) \underbrace{\sin\left[j \frac{r\pi}{n+1}\right]}_{}$$

(replaces a_{jr} as previously)

Again the β_r are complex so take real part

real $$q_j(t) = \sum_r \sin\left[j \frac{r\pi}{n+1}\right] (\mu_r \cos\omega_r t - \nu_r \sin\omega_r t)$$

where

$$\beta_r \equiv \mu_r + i\nu_r .$$

Now we can relate μ, ν to initial conditions as before:

$$q_j(0) = \sum_r \mu_r \sin\left(j \frac{r\pi}{n+1}\right)$$

$$\dot{q}_j(0) = -\sum_r \omega_r \nu_r \sin\left(j \frac{r\pi}{n+1}\right)$$

Now we can exploit "orthogonality" of the a_{jr}

Recall the math identity (trig)

$$\sum_{j=1}^{n} \sin\left(j \frac{r\pi}{n+1}\right) \sin\left(j \frac{s\pi}{n+1}\right) = \frac{n+1}{2} \delta_{rs}$$

$r, s = 1, \ldots n.$

So mult. i.c. by $\sin\left(j \frac{s\pi}{n+1}\right)$ & $\sum_j$

$$\sum_{j=1}^{n} q_j(0) \sin\left(j \frac{s\pi}{n+1}\right) = \sum_{j,r} \mu_r \sin\left(j \frac{r\pi}{n+1}\right) \sin\left(j \frac{s\pi}{n+1}\right)$$

$$= \sum_{r} \frac{n+1}{2} \delta_{rs} \mu_r$$

$$= \frac{n+1}{2} \mu_s$$

$\Rightarrow$

$$\boxed{\begin{aligned} \mu_s &= \left(\frac{2}{n+1}\right) \sum_{j=1}^{n} q_j(0) \sin\left(j \frac{s\pi}{n+1}\right) \\ \nu_s &= -\left(\frac{2}{\omega_s(n+1)}\right) \sum_{j=1}^{n} \dot{q}_j(0) \sin\left(j \frac{s\pi}{n+1}\right) \end{aligned}}$$

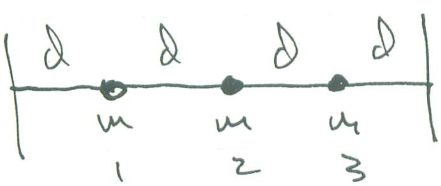
& likewise

Thus the complete description of motion of the loaded string ~~...~~ is accomplished.

Recall:

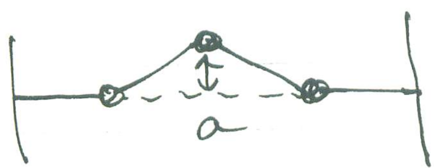
$$\boxed{q_j(t) = \sum_{r=1}^{n} \sin\left[j\frac{r\pi}{n+1}\right]\left[\left(\frac{2}{n+1}\right)\sum_{l=1}^{n} q_l(0)\sin\left(l\frac{r\pi}{n+1}\right)\cos\omega_r t + \left(\frac{2}{\omega_r(n+1)}\right)\sum_{l=1}^{n} \dot{q}_l(0)\sin\left(l\frac{r\pi}{n+1}\right)\sin\omega_r t\right]}$$

Ex.:



$n=3$

i.e.



$$\boxed{\begin{gathered} q_1(0) = 0 = q_3(0) \\ q_2(0) = a \\ \dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0 \end{gathered}}$$

$$\Rightarrow \quad \boxed{\nu_1 = \nu_2 = \nu_3 = 0}$$

&

$$\mu_r = \left(\frac{2}{n+1}\right)\sum_{j=1}^{n} q_j(0)\sin\left(j\frac{r\pi}{n+1}\right)$$

$n=3$

$$\boxed{\mu_r = \frac{1}{2}a\sin\left(\frac{r\pi}{2}\right) \quad ; \quad r = 1, 2, 3}$$

$$\Rightarrow \quad \boxed{\mu_1 = \frac{a}{2} \quad ; \mu_2 = 0 \quad ; \mu_3 = -\frac{a}{2}}$$

Now we need $\sin\left(j\,\frac{r\pi}{n+1}\right) = \sin\left(\frac{j}{4}\,r\pi\right)$

$$= \begin{array}{c|ccc} j \backslash r & 1 & 2 & 3 \\ \hline 1 & \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \\ 2 & 1 & 0 & -1 \\ 3 & \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \end{array}$$

$\Rightarrow$

$$q_j(t) = \sum_{r=1}^{3} \sin\left(j\,\frac{r\pi}{4}\right) \mu_r \cos\omega_r t$$

$\Rightarrow$

$$q_1(t) = \frac{\sqrt{2}}{4}\,a\left(\cos\omega_1 t - \cos\omega_3 t\right)$$

$$q_2(t) = \frac{a}{2}\left(\cos\omega_1 t + \cos\omega_3 t\right)$$

$$q_3(t) = \frac{\sqrt{2}}{4}\,a\left(\cos\omega_1 t - \cos\omega_3 t\right)$$

&

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\,\sin\left[\frac{r\pi}{8}\right] \qquad r = 1, 2, 3$$

Although the loaded string is of interest per se, it is more useful in its limiting case to provide the dynamics for the <u>Continuous string</u>

Suppose we consider $n \to \infty$ along with $d, m \to 0$ such that $(n+1)d = L$ = constant length

$$\frac{m}{d} = \rho = \text{constant density}$$

(linear mass density)

Then let $jd = x$ = the distance along the continuous string

hence

$$q_j(t) = \sum_r^1 \eta_r(t) \sin\left(j \frac{r\pi}{n+1}\right)$$

$$= \sum_r^1 \eta_r(t) \sin\left(r\pi \frac{x}{L}\right) \qquad \begin{array}{l} x = j\rho \\ L = (n+1)d \end{array}$$

hence $q_j(t) = q(x,t)$ becomes a function of x

$$q(x,t) = \sum_r^1 \eta_r(t) \sin\left(\frac{r\pi x}{L}\right)$$

$$= \sum_r^1 \beta_r e^{i\omega_r t} \sin\left(\frac{r\pi x}{L}\right)$$

Now recall there are n-particles, hence n normal mode ($r=1,\dots,n$) and n-eigenfrequencies ω_r ~ hence the sums over for $r=1,\dots,n=\infty$ there are ∞'ly many constants (i.c.) needed then Re & Im parts of β_r to specify the motion of a continuous string — this is nothing but specifying the initial functions $q(x,0)$; $\dot{q}(x,0)$. So

once again $\beta_r = \mu_r + i\nu_r$

$$q(x,0) = \sum_r \mu_r \sin\left(\frac{r\pi x}{L}\right)$$

$$\dot{q}(x,0) = -\sum_r \omega_r \nu_r \sin\left(\frac{r\pi x}{L}\right)$$

So multiply by $\sin\left(\frac{s\pi x}{L}\right)$ and $\int_{x=0}^{L} dx$

and use the trig. identity

$$\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)dx = \frac{L}{2}\delta_{rs}$$

to obtain

$$\mu_r = \frac{2}{L}\int_0^L dx\, q(x,0)\sin\left(\frac{r\pi x}{L}\right)$$

$$\nu_r = -\frac{2}{\omega_r L}\int_0^L dx\, \dot{q}(x,0)\sin\left(\frac{r\pi x}{L}\right).$$

Likewise

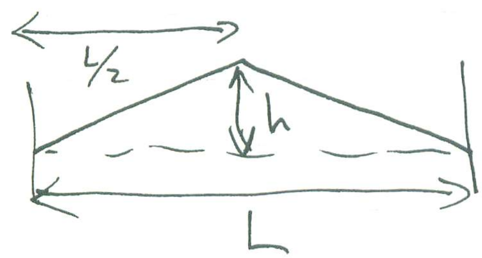
$$\omega_r = 2\sqrt{\frac{\tau}{md}}\,\sin\left[\frac{r\pi}{2(n+1)}\right]$$

$$= \frac{2}{d}\sqrt{\frac{\tau}{\rho}}\,\sin\left(\frac{r\pi d}{2L}\right)$$

as $d \to 0 \Rightarrow$

$$\boxed{\omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}}}$$

Ex.



i.c.

i.c.

$$q(x,0) = \begin{cases} \dfrac{2h}{L}x & 0 \le x \le L/2 \\ \dfrac{2h}{L}(L-x) & L/2 \le x \le L \end{cases}$$

$$\dot{q}(x,0) = 0 \quad \Rightarrow \quad \nu_r = 0$$

$\Rightarrow$

$$\mu_r = \frac{2}{L}\frac{2h}{L}\int_0^{L/2} dx\, x \sin\left(\frac{r\pi x}{L}\right)$$

$$+ \frac{2}{L}\frac{2h}{L}\int_{L/2}^{L} dx\,(L-x)\sin\left(\frac{r\pi x}{L}\right)$$

$\Rightarrow$

$$\boxed{\mu_r = \frac{8h}{r^2\pi^2}\sin\frac{r\pi}{2}}$$

So

$$\mu_r = \begin{cases} 0 & , r = \text{even} \\ \dfrac{8h}{r^2\pi^2}(-1)^{\frac{1}{2}(r-1)} & , r = \text{odd} \end{cases}$$

$\Rightarrow$

real

$$q(x,t) = \sum_{r=1}^{\infty} \Big[\mu_r \cos\omega_r t \sin\left(\frac{r\pi x}{L}\right) - \nu_r \sin\omega_r t \sin\left(\frac{r\pi x}{L}\right)\Big]$$

$$= \sum_{r=1,3,5,\dots}^{\infty} \frac{8h}{r^2\pi^2}(-1)^{\frac{1}{2}(r-1)} \cos\omega_r t \sin\left(\frac{r\pi x}{L}\right)$$

$$q(x,t) = \frac{8h}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\omega_1 t - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\omega_3 t + \dots\right]$$

&

$$\omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}}$$

Note: fundamental mode ω_1 & all odd harmonics are excited ($\omega_3, \omega_5, \dots$)

but no even harmonics are involved in the motion. This is because the initial displacement was symmetrical — so all subsequent motion must be symmetrical hence no even modes.

The string displacement is given in terms of the normal mode sum as

$$q(x,t) = \sum_r \eta_r(t) \sin\left(\frac{r\pi x}{L}\right)$$ (at any time Fourier series with fixed ends)

The normal coordinates are

$$\eta_r(t) = \beta_r e^{i\omega_r t}, \quad \beta_r \in \mathbb{C} \text{ here}$$

The physical modes being the Re part of η_r.

Now each "mass" particle ρdx at position $q(x,t)$ has KE

$$dT = \frac{1}{2}\rho \, d\cancel{x} \left(\dot{q}(x,t)\right)^2$$

$\Rightarrow$

$$T = \frac{1}{2}\rho \int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx$$ for the string's KE

In terms of normal modes we have

$$T = \frac{1}{2}\rho \int_0^L dx \left[\sum_r \dot{\eta}_r \sin\left(\frac{r\pi x}{L}\right)\right]\left[\sum_s \dot{\eta}_s \sin\left(\frac{s\pi x}{L}\right)\right]$$

but

$$\int_0^L dx \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right) = \frac{L}{2}\delta_{rs}$$

$\Rightarrow$

$$T = \frac{1}{2}\rho \sum_r \dot{\eta}_r \overbrace{\sum_s \dot{\eta}_s \delta_{rs}}^{=\dot{\eta}_r} \left(\frac{L}{2}\right)$$

$$\boxed{T = \sum_{r=1}^{\infty} \tfrac{1}{2}\rho\, \dot{\eta}_r(t)^2 \left(\tfrac{L}{2}\right)}$$

Now take physical T we need $\mathrm{Re}\,\eta_r$

so

$$T = \frac{L}{2}\sum_{r=1}^{\infty} \tfrac{1}{2}\rho\left[\mathrm{Re}\,\dot{\eta}_r(t)\right]^2$$

But $\left(\mathrm{Re}\,\dot{\eta}_r\right)^2 = \left[\mathrm{Re}\,\frac{d}{dt}\left((\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\right)\right]^2$

$$= \left[-\omega_r \mu_r \sin\omega_r t - \omega_r \nu_r \cos\omega_r t\right]^2$$

So KE of string is

$$\boxed{T = \frac{\rho L}{4}\sum_{r=1}^{\infty} \omega_r^2\left[\mu_r \sin\omega_r t + \nu_r \cos\omega_r t\right]^2}$$

Now recall for the loaded string PE is

$$U = \frac{1}{2}\frac{\tau}{d}\sum_{j=1}^{n}(q_{j-1} - q_j)^2$$

Taking the limit to obtain the continuous string we find

$$U = \frac{1}{2}\tau \sum_{j=1}^{\infty} \left(\frac{q_{j-1} - q_j}{d}\right)^2 d \qquad \text{as } d \to 0$$

$$\Rightarrow \quad \boxed{U = \frac{1}{2}\tau \int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx}$$

Again in terms of normal modes

$$\frac{\partial q}{\partial x} = \sum_r \frac{r\pi}{L} \eta_r \cos\left(\frac{r\pi x}{L}\right)$$

$\Rightarrow$

$$U = \frac{\tau}{2}\int_0^L dx \left[\sum_r \frac{r\pi}{L}\eta_r \cos\left(\frac{r\pi x}{L}\right)\right]\left[\sum_s \frac{s\pi}{L}\eta_s \cos\left(\frac{s\pi x}{L}\right)\right]$$

but

$$\int_0^L dx \cos\left(\frac{r\pi x}{L}\right)\cos\left(\frac{s\pi x}{L}\right) = \frac{L}{2}\delta_{rs}$$

$\Rightarrow$

$$U = \frac{\tau}{2}\sum_r \frac{r\pi}{L}\eta_r \underbrace{\sum_s \frac{s\pi}{L}\eta_s \frac{L}{2}\delta_{rs}}_{=\left(\frac{L}{2}\right)\frac{r\pi}{L}\eta_r}$$

$$= \frac{\tau}{2}\sum_r \frac{r^2\pi^2}{L^2}\cdot\frac{L}{2}\eta_r\eta_r$$

but $\quad \omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}} \quad \Rightarrow \quad \omega_r^2 = \frac{r^2\pi^2}{L^2}\frac{\tau}{\rho}$

hence

$$\boxed{U = \frac{\rho L}{4} \sum_{r=1}^{\infty} \omega_r^2 \eta_r^2}$$

Again we need $(\mathrm{Re}\,\eta_r)^2$ in $U \Rightarrow$

$$\boxed{U = \frac{\rho L}{4} \sum_{r=1}^{\infty} \omega_r^2 \left[\mu_r \cos\omega_r t - \nu_r \sin\omega_r t\right]^2 .}$$

Since ~~the~~ string forces are conservative (springs) the total energy is a constant

$$E = T + U$$

$$= \frac{\rho L}{4} \sum_{r=1}^{\infty} \omega_r^2 (\mu_r^2 + \nu_r^2)$$

$$= \frac{\rho L}{4} \sum_{r=1}^{\infty} \omega_r^2 |\beta_r|^2 .$$

$E =$ const and a sum over contributions from each normal mode.

The Lagrangian in terms of Normal modes

yields

$$L = T - U = \frac{\rho L}{4}\sum_{r=1}^{\infty} \omega_r^2 \left[(\nu_r^2 - \mu_r^2)\cos 2\omega_r t + 2\mu_r\nu_r \sin 2\omega_r t\right]$$

$$\left(= -\frac{\rho L}{4}\sum_{r=1}^{\infty} \mathrm{Re}(\dot{\eta}_r)^2 \right).$$

$$\boxed{\begin{aligned} &= \tfrac{1}{2}\rho\int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx - \tfrac{1}{2}\tau\int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx \\ L &= \tfrac{1}{2}\int_0^L dx \left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \tau\left(\frac{\partial q}{\partial x}\right)^2\right] \end{aligned}}$$

As before we can consider the Euler-Lag. equations for the string.

Of course we can express the Lagrangian in terms of normal coordinates $\eta_r(t)$ (real now) thus

$$L = \sum_{r=1}^{\infty}\left(\frac{L}{4}\rho\dot{\eta}_r^2 - \frac{L}{4}\rho\omega_r^2\eta_r^2\right)$$

Hence E-L eq.'s ⇒

$$-\frac{L\rho}{2}\left[\omega_r^2\eta_r + \ddot{\eta}_r\right] = 0$$

Thus $\boxed{\ddot{\eta}_r + \omega_r^2 \eta_r = 0}$ for each mode r, with $\omega_r^2 = \frac{r^2\pi^2}{L^2}\cdot\frac{\tau}{\rho}$

This result can be more compactly expressed as a PDE: the wave equation for the continuous coordinate $q(x,t)$.

Recall (Real) $q(x,t) = \sum_r' \eta_r(t)\sin\left(\frac{r\pi x}{L}\right)$

So

$$\frac{\partial^2 q(x,t)}{\partial t^2} = \sum_r' \ddot{\eta}_r(t)\sin\left(\frac{r\pi x}{L}\right)$$

$$= -\sum_r' \omega_r^2 \eta_r(t)\sin\left(\frac{r\pi x}{L}\right)$$

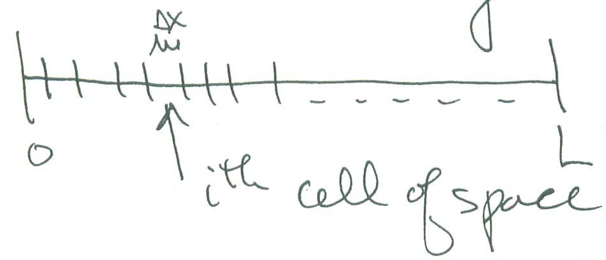
$$= -\frac{\tau}{\rho}\sum_r' \eta_r(t)\frac{r^2\pi^2}{L^2}\sin\left(\frac{r\pi x}{L}\right)$$

$$\frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 q(x,t)}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\rho}{\tau}\frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\partial^2 q(x,t)}{\partial x^2}}$$

Wave eq. in one dimension

we could obtain this same result by expanding the $q(x,t)$ in terms of discrete x_i : i.e. As in the beginning



$$X = i\Delta x \quad ; \quad q(x,t) \simeq q(i\Delta x, t) \equiv q_i(t) \quad \text{for } x \in i^{th} \text{ cell of space}$$

Then
$$L = \frac{1}{2}\int_0^L dx\left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \tau\left(\frac{\partial q}{\partial x}\right)^2\right] = \int_0^L dx\, \mathcal{L}(q, \dot{q}, q')$$
$$= \frac{1}{2}\sum_i \Delta x\left[\rho\, \dot{q}_i(t)^2 - \tau\left[\frac{q_i - q_{i-1}}{\Delta x}\right]^2\right]$$

E-Lag ⇒
$$\Delta x\left(-\rho\,\ddot{q}_i(t) - \tau\frac{[q_i - q_{i-1}]}{\Delta x^2} + \tau\frac{[q_{i+1} - q_i]}{\Delta x^2}\right) = 0$$

⇒
$$\boxed{(\rho\Delta x)\,\ddot{q}_i(t) - \left(\frac{\tau}{\Delta x}\right)(q_{i-1} - 2q_i + q_{i+1}) = 0}$$

Just our discrete equation:

Now
$$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{q(x+\Delta x, t) - q(x,t)}{\Delta x}\right)$$
$$= \frac{q(x+2\Delta x) - q(x+\Delta x) - q(x+\Delta x) + q(x)}{\Delta x^2}$$

So $$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$$

$\Rightarrow$ in the continuum

$$\boxed{\frac{\rho}{\tau}\ddot{q}(x,t) = +\frac{\partial^2 q(x,t)}{\partial \cancel{x}^2}}$$

Likewise we know that

$$p_i(t) \equiv \frac{\partial L}{\partial \dot{q}_i(t)} = (\rho \Delta x)\,\dot{q}_i(t) = \Delta x \frac{\partial \mathcal{L}}{\partial \dot{q}_i(t)}$$

Define the momentum density "field"

$x \in i^{\text{th}}$ cell $$\Pi(x,t) \equiv \Pi(x_i,t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i(t)} = \frac{1}{\Delta x} p_i(t)$$
$$= \frac{\partial \mathcal{L}}{\partial \dot{q}(x,t)} = \rho \dot{q}(x,t)$$

The Hamiltonian is

$$H = \sum_i [p_i \dot{q}_i] - L$$
$$= \sum_i \Delta x \left[\frac{1}{\Delta x} p_i \dot{q}_i - \mathcal{L}_i\right] = \sum_i \Delta x\left[\Pi(x_i,t)\dot{q}(x_i,t) - \mathcal{L}\right]$$
$$= \int dx\, \mathcal{H}$$

$$\boxed{\mathcal{H} = \Pi(x,t)\,\dot{q}(x,t) - \mathcal{L}(q,\dot{q},q')} = \rho\dot{q}^2 - \tfrac{1}{2}\rho\dot{q}^2 + \tfrac{1}{2}\tau q'^2$$

So
$$\mathcal{H} = \frac{1}{2}\left(\rho\left(\frac{\partial q}{\partial t}\right)^2 + \tau\left(\frac{\partial q}{\partial x}\right)^2\right)$$
$$H = \frac{1}{2}\int_0^L dx\left[\rho\left(\frac{\partial q}{\partial t}\right)^2 + \tau\left(\frac{\partial q}{\partial x}\right)^2\right] = E.$$

Now consider the action
$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L} = \int_\Omega d^2x\, \mathcal{L}$$
The calculus of variations implies:
$$0 = \delta S = \int_\Omega d^2x\, \delta\mathcal{L} = \int dt \sum_i{}' \Delta x\, \frac{1}{2}\left(\rho\, \delta \dot{q}_i(t)^2 - \tau\, \delta\left(\frac{q_i - q_{i-1}}{\Delta x}\right)^2\right)$$

$$= \int dt \sum_i{}' \Delta x\left[\rho \dot{q}_i\, \delta \dot{q}_i - \frac{\tau}{\Delta x^2}\left((q_i - q_{i-1}) - q_{i+1} + q_i\right) \times \delta q_i\right]$$

$$= \int dt \sum_i{}' \left[-(\rho \Delta x)\ddot{q}_i + \frac{\tau}{\Delta x}(q_{i-1} - 2q_i + q_{i+1})\right]\delta q_i = 0$$
$$\Rightarrow \boxed{(\rho\Delta x)\ddot{q}_i - \left(\frac{\tau}{\Delta x}\right)(q_{i-1} - 2q_i + q_{i+1}) = 0}$$

Just E-L eq. again

In general the space derivatives become say

$$\frac{1}{2}\delta\left(\frac{q_i - q_{i-1}}{\Delta x}\right)^2 = \left(\frac{q_i - q_{i-1}}{\Delta x}\right)\delta\left(\frac{q_i - q_{i-1}}{\Delta x}\right)$$

$$\rightarrow \frac{\partial q}{\partial x}\cdot\delta\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial q}{\partial x}\delta q\right) - \frac{\partial^2 q}{\partial x^2}\delta q$$

as we found above

$$\delta S = \int dt \sum_i \Delta x\left[-\rho \ddot{q}_i + \frac{\tau}{\Delta x^2}(q_{i+1} - 2q_i + q_{i-1})\right]\delta q_i$$

$$\rightarrow = \int dt \int dx\left[-\rho\ddot{q}(x,t) + \tau\, q''(x,t)\right]\delta q(x,t)$$

$$= \int d^2x\left[-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial q'}\right]\delta q = 0$$

$\Rightarrow$ The E-L eq.

$$\boxed{\frac{\partial \mathcal{L}}{\partial q(x,t)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}(x,t)} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial q'(x,t)} = 0}.$$

Thus in general we view the Calculus of Variations for functions of 2-variables

$$S = \int_\Omega dt\,dx\,\mathcal{L}(q(x,t), \dot{q}(x,t), q'(x,t))$$

Then

$$\delta S = \int_\Omega d^2x\,\delta\mathcal{L}$$

$$\delta S = \int_\Omega d^2x \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \partial_\mu q} \delta \partial_\mu q \right)$$

$$= \int_\Omega d^2x \left[\left(\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu q} \right) \delta q + \cancel{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu q} \delta q \right)} \right]$$

$\to 0$

at boundary $\delta q = 0$
t_1, t_2
$x = 0, L$

$$= \int_\Omega d^2x \left[\left(\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu q} \right) \delta q \right] = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu q} = 0} \quad \text{E-L eq.}$$

<u>String</u> $\quad \mathcal{L} = \frac{1}{2} \left[\rho \left(\frac{\partial q}{\partial t} \right)^2 - \tau \left(\frac{\partial q}{\partial x} \right)^2 \right]$

So $\frac{\partial \mathcal{L}}{\partial q} = 0$ &

$$\frac{\partial \mathcal{L}}{\partial \partial_t q} = \rho \frac{\partial q}{\partial t} \quad ; \quad \frac{\partial \mathcal{L}}{\partial \partial_x q} = -\tau \frac{\partial q}{\partial x}$$

$$\Rightarrow \quad \partial_t \frac{\partial \mathcal{L}}{\partial \partial_t q} + \partial_x \frac{\partial \mathcal{L}}{\partial \partial_x q} = 0$$

$$\Rightarrow \quad \boxed{\rho \ddot{q} - \tau q'' = 0} \quad \text{the wave eq.}$$