5 Coupled Oscillations

In this section let us consider a system of particles experiencing small oscillations about their equilibrium positions. In general the motion will be given by a superposition of harmonic oscillations. We will unravel this coupled motion into normal modes; that is we will find the generalized coordinates each of which oscillates at its own characteristic or eigen frequency. So consider a conservative system with n-degrees of freedom so that we can describe the configuration of the particles in the system by means of n-generalized coordinates q^a, with $a = 1, 2, \ldots, n$. Let us assume the relation of the generalized coordinates to the Cartesian coordinates $x_{\alpha\ i}$ are independent of time explicitly. That is

$$x_{\alpha\ i} = x_{\alpha\ i}(q^a). \tag{5.1}$$

We will assume that in the *stable equilibrium* configuration the values of the generalized coordinates are

$$q^a = q_0^a \tag{5.2}$$

and since the particles are in equilibrium, we also have that

$$\dot{q}^a = 0 = \ddot{q}^a \tag{5.3}$$

for $q^a = q_0^a$.

The dynamics of the system is given by the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0. \tag{5.4}$$

However, at equilibrium

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\Big|_{q^a=q_0^a} = 0 \tag{5.5}$$

since it always involves a $\dot{q}^a$ or a $\ddot{q}^a$ which vanish when the particles are at their equilibrium positions. Hence the Euler-Lagrange equations imply that

$$\left.\frac{\partial L}{\partial q^a}\right|_{q^a=q^a_0} = 0 = \left.\frac{\partial T}{\partial q^a}\right|_{q^a=q^a_0} - \left.\frac{\partial U}{\partial q^a}\right|_{q^a=q^a_0}. \tag{5.6}$$

Now if $x_{\alpha\ i} = x_{\alpha\ i}(q^a)$, we find that

$$\dot{x}_{\alpha\ i} = \sum_{a=1}^{n} \frac{\partial x_{\alpha\ i}}{\partial q^a}\dot{q}^a \tag{5.7}$$

only. Hence

$$\begin{aligned} T &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{1}{2}m_\alpha\left(\dot{x}_{\alpha\ i}\right)^2 = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha \frac{\partial x_{\alpha\ i}}{\partial q^a}\frac{\partial x_{\alpha\ i}}{\partial q^b}\dot{q}^a\dot{q}^b \\ &= \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} m_{ab}\dot{q}^a\dot{q}^b, \end{aligned} \tag{5.8}$$

where

$$m_{ab} \equiv \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q^a}\cdot\frac{\partial \vec{r}_\alpha}{\partial q^b} = m_{ba}. \tag{5.9}$$

So

$$\left.\frac{\partial T}{\partial q^a}\right|_0 = 0 \tag{5.10}$$

since $\dot{q}^a = 0$ at equilibrium (where the subscript 0 is shorthand for evaluating the expression at equilibrium $q^a = q^a_0$). Thus we find from the Euler-Lagrange equations of motion that

$$\left.\frac{\partial U}{\partial q^a}\right|_0 = 0 \tag{5.11}$$

that is the generalized forces

$$Q_a = -\left.\frac{\partial U}{\partial q^a}\right|_0 = 0 \tag{5.12}$$

at equilibrium.

We can consider small oscillations about q_0^a and Taylor expand the Lagrangian L about these positions

$$U(q^a) = U(q_0^a) + \sum_{a=1}^{n}(q^a - q_0^a)\frac{\partial U}{\partial q^a}\Big|_0 + \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}(q^a - q_0^a)(q^b - q_0^b)\frac{\partial^2 U}{\partial q^a \partial q^b}\Big|_0 + \cdots . \tag{5.13}$$

Since $\partial U/\partial q^a\Big|_0 = 0$ and for simplicity we can choose the arbitrary zero of potential energy to be at the equilibrium position q_0^a so that $U(q_0^a) = 0$, the Taylor expansion starts with the second order terms and we ignore the higher order terms for small oscillations

$$\begin{aligned} U(q^a) &= \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left(\frac{\partial^2 U}{\partial q^a \partial q^b}\Big|_0\right)(q^a - q_0^a)(q^b - q_0^b) \\ &\equiv \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} k_{ab}\,(q^a - q_0^a)(q^b - q_0^b). \end{aligned} \tag{5.14}$$

The second derivatives of the potential energy are defined as

$$k_{ab} \equiv \frac{\partial^2 U}{\partial q^a \partial q^b}\Big|_0 = k_{ba} \tag{5.15}$$

for U with continuous second partial derivatives. Hence k_{ab} is a $n \times n$ matrix of numbers, independent of the generalized coordinates q^a.

Since we are interested in small oscillations about equilibrium, the velocities will be small also so that

$$T = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} m_{ab}\dot{q}^a\dot{q}^b = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left[m_{ab}(q_0^c) + \sum_{c=1}^{n}(q^c - q_0^c)\frac{\partial m_{ab}}{\partial q^c}\Big|_0 + \cdots\right]\dot{q}^a\dot{q}^b. \tag{5.16}$$

However the terms beyond $m_{ab}(q_0^c)$ when multilplied by the velocities $\dot{q}^a\dot{q}^b$ are third order in smallness and so we will neglect them. The to the same order in small oscillations as the potential energy was expanded the kinetic energy becomes

$$T = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} m_{ab}(q_0^c)\dot{q}^a\dot{q}^b. \tag{5.17}$$

From now on we will drop the argument of $m_{ab}(q_0^c)$ and understand it to be just a $n \times n$ matrix of numbers independent of the generalized coordinates q^a.

The Lagrangian for small oscillations about the equilibrium positions is given by

$$L = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left[m_{ab}\dot{q}^a\dot{q}^b - k_{ab}\,(q^a - q_0^a)(q^b - q_0^b)\right]. \tag{5.18}$$

Since we are at a stable equilibrium as the minimum, then $k_{ab} \geq 0$. The kinetic energy $T = \frac{1}{2}\sum_{\alpha\ i} m_{\alpha\ i}\dot{x}^2_{\alpha\ i} \geq 0$, so both U and T are non-negative and equal to 0 only when $q^a = q_0^a$ for U or $\dot{q}^a = 0$ for T. The Euler-Lagrange equations of motion now become, calling the coordinates Q^a now (not to be confused with the symbol for the generalized forces which we are not using here)

$$Q^a \equiv (q^a - q_0^a) \tag{5.19}$$

and

$$\dot{Q}^a = \dot{q}^a, \tag{5.20}$$

$$\frac{\partial L}{\partial Q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}^a}\right) = 0 \tag{5.21}$$

which yields

$$\sum_{b=1}^{n}\left(m_{ab}\ddot{Q}^b + k_{ab}Q^b\right) = 0, \tag{5.22}$$

for $a = 1, 2, \ldots, n$.

This is just a system of coupled harmonic oscillators where m_{ab} are the generalized masses and k_{ab} are the generalized spring constants. We might as well try the complex exponential solution for the coordinates (we only want the real part of the complex coordinate, but we use the complex exponential for simplicity in solving the linear equation of motion)

$$Q^b(t) = a^b e^{i(\omega t - \delta)}, \tag{5.23}$$

where a^b and δ are real constants determined by the initial conditions. Substituting this guess into the differential equations renders them to be algebraic equations

$$\sum_{b=1}^{n} \left(-\omega^2 m_{ab} + k_{ab}\right) a^b = 0. \tag{5.24}$$

Thus we have an equation of the form

$$\sum_{b=1}^{n} C_{ab} y^b = 0, \tag{5.25}$$

that is n-equations for n-variables y^b. If there is to be a non-trivial solution the $\det C = 0$ so that C^{-1} does not exist. If $\det C \neq 0$ then we could multiply the equation by C^{-1} implying $C^{-1}Cy = 0 = y$ and hence $y = 0$ is trivial, that is $a^b = 0$ and so $Q^a = 0$. So we require

$$\begin{aligned} \left|k_{ab} - \omega^2 m_{ab}\right| &= 0 \\ \begin{vmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & \cdots \\ k_{12} - \omega^2 m_{12} & k_{22} - \omega^2 m_{22} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} &= 0, \end{aligned} \tag{5.26}$$

where we have used $k = k^T$ and $m = m^T$. This yields an n^{th}-order equation for ω^2 called the characteristic equation or secular equation of the system. It has in general n-roots which we label ω_r^2. The ω_r are called the *characteristic frequencies* or *eigenfrequencies* of the system. It can occur that two or more ω_r are equal, this is called *degeneracy*; we will discuss this later.

For each eigenfrequency ω_r the Euler-Lagrange equations of motion are

$$\sum_{b=1}^{n} \left(+k_{ab} - \omega_r^2 m_{ab}\right) a_r^b = 0, \tag{5.27}$$

with no sum over r and where now we label the coefficient a^b with the associated r: thus $\vec{a}_r$ is the n-dimensional *eigenvector* with eigenfrequencies ω_r^2

of the operator

$$\left(m^{-1}\,k\right)_{ab}, \tag{5.28}$$

that is

$$\sum_{b=1}^{n}\left(m^{-1}\,k\right)_{ab} a_r^b = \omega_r^2 a_r^a. \tag{5.29}$$

(Note that Thornton and Marion use the notation with both postscripts downstairs a_{br} for our a_r^b.) Since the principle of superpostion applies to this system of linear differential equations, we can write the Q^b as a linear combination of the n eigenvectors

$$Q^b(t) = \sum_{r-1}^{n} a_r^b e^{i(\omega_r t-\delta_r)}, \tag{5.30}$$

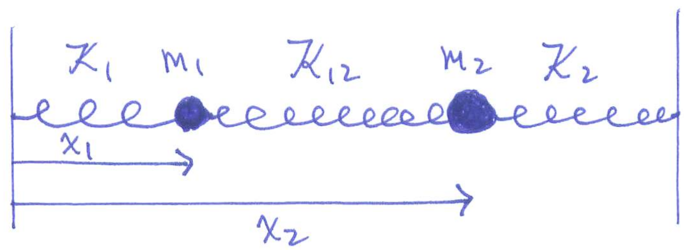
where the a_r^b are determined for each r except for an overall factor. So we have the n factors in a_r^b and n δ_r for the $2n$ needed initial conditions. We really only need the real part for the physically meaningful generalized coordinate, so

$$q^b(t) = q_0^b + \sum_{r=1}^{n} a_r^b \cos\left(\omega_r t - \delta_r\right). \tag{5.31}$$

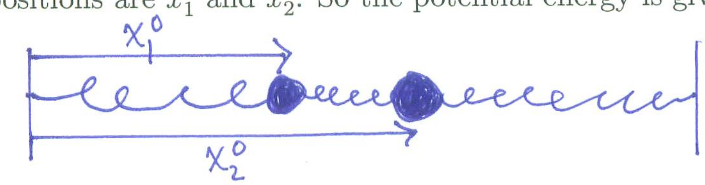
So each coordinate's motion is compounded of motions with each of the n frequencies. Thus, explicitly

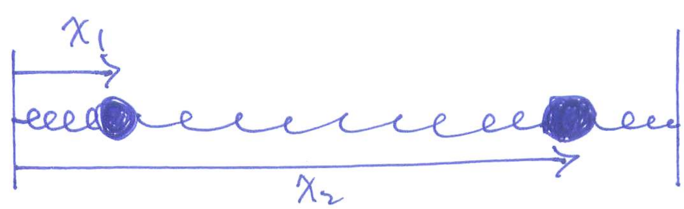
$$\sum_{b=1}^{n}\sum_{r=1}^{n}\left(-\omega_r^2 m_{ab} + k_{ab}\right) a_r^b e^{i(\omega_r t-\delta_r)} = \sum_{r=1}^{n}\left(\sum_{b=1}^{n}\left(k_{ab} - \omega_r^2 m_{ab}\right) a_r^b\right) e^{i(\omega_r t-\delta_r)} = 0. \tag{5.32}$$

The generalized coordinates $q^a(t)$ are **not** yet the normal coordinates. Example: Consider two coupled oscillators constrained to one dimensional motion Let the Cartesian coordinates be the generalized coordinates. The



equilibrium positions are x_1^0 and x_2^0. So the potential energy is given by





$$
\begin{aligned}
U &= \frac{1}{2}\kappa_1(x_1 - x_1^0)^2 + \frac{1}{2}\kappa_2(x_2 - x_2^0)^2\frac{1}{2}\kappa_{12}\underbrace{\left((x_2 - x_1) - (x_2^0 - x_1^0)\right)^2}_{\left[(x_2-x_2^0)-(x_1-x_1^0)\right]^2} \\
&= \frac{1}{2}(\kappa_1 + \kappa_{12})(x_1 - x_1^0)^2 + \frac{1}{2}(\kappa_2 + \kappa_{12})(x_2 - x_2^0)^2 - \kappa_{12}(x_1 - x_1^0)(x_2 - x_2^0) \qquad (5.33)
\end{aligned}
$$

The kinetic energy is

$$
T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2. \qquad (5.34)
$$

So we can read off the values for the generalized spring constant matrix k_{ab}

$$
\begin{aligned}
k_{11} &= \left.\frac{\partial^2 U}{\partial x_1 \partial x_1}\right|_0 = \kappa_1 + \kappa_{12} \\
k_{22} &= \left.\frac{\partial^2 U}{\partial x_2 \partial x_2}\right|_0 = \kappa_2 + \kappa_{12}
\end{aligned}
$$

$$k_{12} = k_{21} = \frac{\partial^2 U}{\partial x_1 \partial x_2}\bigg|_0 = -\kappa_{12}. \tag{5.35}$$

The generalized mass matrix m_{ab} is such that

$$\begin{aligned} m_{11} &= m_1 \\ m_{22} &= m_2 \\ m_{12} &= 0 = m_{21}. \end{aligned} \tag{5.36}$$

That is we have the matrices

$$k_{ab} = \begin{pmatrix} (\kappa_1 + \kappa_{12}) & -\kappa_{12} \\ -\kappa_{12} & (\kappa_2 + \kappa_{12}) \end{pmatrix} \tag{5.37}$$

and

$$m_{ab} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{5.38}$$

Thus the Lagrangian is given by

$$L = \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2} m_{ab}\dot{Q}^a\dot{Q}^b - \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2} k_{ab}Q^aQ^b. \tag{5.39}$$

where

$$\begin{aligned} Q^1 &= x_1 - x_1^0 \\ Q^2 &= x_2 - x_2^0. \end{aligned} \tag{5.40}$$

In matrix notation the Lagrangian can be written as

$$L = \frac{1}{2}\dot{Q}^T m \dot{Q} - \frac{1}{2}Q^T k Q. \tag{5.41}$$

The secular equation is

$$\left| k_{ab} - \omega^2 m_{ab} \right| = 0. \tag{5.42}$$

That is

$$
\begin{aligned}
0 &= \begin{vmatrix} \kappa_1 + \kappa_{12} - \omega^2 m_1 - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & \cdots \\ k_{12} - \omega^2 m_{12} & -\kappa_{12} & \\ -\kappa_{12} & \kappa_2 + \kappa_{12} - \omega^2 m_2 & \end{vmatrix} \\
&= (\kappa_1 + \kappa_{12})(\kappa_2 + \kappa_{12}) - \omega^2 \left(m_1(\kappa_2 + \kappa_{12}) + m_2(\kappa_1 + \kappa_{12})\right) + \omega^4 m_1 m_2 - \kappa_{12}^2 \\
&= (\omega^2)^2 m_1 m_2 - \omega^2 \left[m_1(\kappa_2 + \kappa_{12}) + m_2(\kappa_1 + \kappa_{12})\right] + ((\kappa_1\kappa_2 + (\kappa_1 + \kappa + 2)\kappa_{12})
\end{aligned}
\tag{5.43}
$$

This is a quadratic equation for ω^2. For simplicity, let $\kappa_1 = \kappa = \kappa_2$ and $m_1 = m = m_2$. The secular equation then becomes

$$
0 = m^2(\omega^2)^2 - 2\omega^2 m(\kappa + \kappa_{12}) + \kappa(\kappa + 2\kappa_{12}). \tag{5.44}
$$

Solving the quadratic equation yields

$$
\begin{aligned}
\omega^2 &= \frac{2m(\kappa + \kappa_{12}) \pm \sqrt{4m^2(\kappa^2 + \kappa_{12}^2 + 2\kappa\kappa_{12}) - 4m^2(\kappa^2 + 2\kappa\kappa_{12})}}{2m^2} \\
&= \frac{\kappa + \kappa_{12} \pm \sqrt{\kappa_{12}^2}}{m} \\
&= \frac{\kappa + \kappa_{12} \pm \kappa_{12}}{m} = \begin{cases} \frac{\kappa}{m} \\ \frac{\kappa + \kappa_{12}}{m} \end{cases}.
\end{aligned}
\tag{5.45}
$$

These are the two eigenfrequencies

$$
\begin{aligned}
\omega_1 &\equiv \sqrt{\left(\frac{\kappa + 2\kappa_{12}}{m}\right)} \\
\omega_2 &\equiv \sqrt{\left(\frac{\kappa}{m}\right)}.
\end{aligned}
\tag{5.46}
$$

The oscillations are then found to be (using lower postscript notation here)

$$
x_1(t) = x_1^0 + a_{11}\cos(\omega_1 t - \delta_1) + a_{12}\cos(\omega_2 t - \delta_2)
$$

$$x_2(t) = x_2^0 + a_{21}\cos(\omega_1 t - \delta_1) + a_{22}\cos(\omega_2 t - \delta_2). \tag{5.47}$$

Notice, although we have written 4 $a'_{ab}s$ they are not independent. The eigenvector equation (that is just the Euler-Lagrange equation of motion) determines the ratios of the $a's$ for each eigenvalue. Since the eigenequation is linear the normalization of the $a's$ are arbitrary. The a coefficients must obey

$$\sum_{b=1}^{2} \left(k_{ab} - \omega_r^2 m_{ab}\right) a_{br} = 0. \tag{5.48}$$

That is

$$(k_{11} - \omega_r^2 m_{11})a_{1r} + (k_{12} - \omega_r^2 m_{12})a_{2r} = 0. \tag{5.49}$$

This has the solution

$$\begin{aligned} a_{2r} &= -\frac{(k_{11} - \omega_r^2 m_{11})}{(k_{12} - \omega_r^2 m_{12})} a_{1r} \\ &= +\frac{(\kappa + \kappa_{12} - \omega_r^2 m)}{\kappa_{12}} a_{1r}. \end{aligned} \tag{5.50}$$

For $r = 1, 2$ we find

$$\begin{aligned} a_{21} &= -a_{11} \\ a_{22} &= +a_{12}. \end{aligned} \tag{5.51}$$

Hence the Cartesian coordinates are given by

$$\begin{aligned} x_1(t) &= x_1^0 + a_{11}\cos(\omega_1 t - \delta_1) + a_{12}\cos(\omega_2 t - \delta_2) \\ x_2(t) &= x_2^0 - a_{11}\cos(\omega_1 t - \delta_1) + a_{12}\cos(\omega_2 t - \delta_2). \end{aligned} \tag{5.52}$$

We can look at the q^a as vectors in a n-dimensional vector space. Since we have n-eigenvectors and our motion can be written as a superposition of these, then the eigenvectors are seen form an orthogonal set of vectors in this n-dimensional space–they span the space. Returning to our eigenvector equation

$$\sum_{b=1}^{n} \left(k_{ab} - \omega_r^2 m_{ab}\right) a_r^b = 0 \tag{5.53}$$

for each r we can write this as a matrix equation. Let k and m be $n \times n$ matrices and a_r be a $n \times 1$ column vector. Then equation (5.53) becomes in matrix notation

$$k\ a_r = \omega_r^2 m\ a_r. \tag{5.54}$$

Multiply this by a_s^T, where the superscript T means transpose matrix, in this case $a_s^T$is a $1 \times n$ row vector, thus

$$a_s^T\ k\ a_r = \omega_r^2\ a_s^T\ m\ a_r. \tag{5.55}$$

Similarly, interchanging r and s

$$a_r^T\ k\ a_s = \omega_s^2\ a_r^T\ m\ a_s. \tag{5.56}$$

But we assumed that k and m are symmetric matrices so that

$$\begin{aligned} a_s^T\ k\ a_r &= \sum_{a,b} a_s^a k_{ab} a_r^b = \sum_{a,b} a_s^a k_{ba} a_r^b = \sum_{a,b} a_r^b k_{ba} a_s^a \\ &= a_r^T\ k\ a_s. \end{aligned} \tag{5.57}$$

Likewise for

$$a_s^T\ m\ a_r = a_r^T\ m a_s. \tag{5.58}$$

That is in matrix notation

$$(a_s^T\ k\ a_r)^T = a_s^T\ k\ a_r \tag{5.59}$$

since this is just a 1×1 matrix it is equal to its transpose. But also

$$(a_s^T\ k\ a_r)^T \quad = \quad a_s^T\ k\ a_r$$

$$\begin{aligned} &= a_r^T\, k^T\, a_s^{TT} = a_r^T\, k^T\, a_s \\ &= a_r^T\, k\, a_s. \end{aligned} \tag{5.60}$$

So applying this to our equations we have

$$\begin{aligned} a_s^T\, k\, a_r &= \omega_r^2\, a_s^T\, m\, a_r \\ a_r^T\, k\, a_s &= a_s^T\, k\, a_r = \omega_s^2\, a_s^T\, m\, a_r. \end{aligned} \tag{5.61}$$

Subtracting these equations yields

$$\left(\omega_r^2 - \omega_s^2\right) a_s^T\, m\, a_r = 0. \tag{5.62}$$

Hence for $r \neq s$ in the non-degenerate case $\omega_r^2 - \omega_s^2 \neq 0$ means that

$$a_s^T\, m\, a_r = 0. \tag{5.63}$$

For $r = s$ we obtain no constraints.

However we also know that for each r the $(n-1)$ ratios of the components of the vectors are determined from the eigenvector (Euler-Lagrange) equations

$$\sum_{b=1}^{n} \left(k_{ab} - \omega^2 m_{ab}\right) a^b = 0. \tag{5.64}$$

Let the n^{th} term be on the right hand side

$$\sum_{b=1}^{(n-1)} \left(k_{ab} - \omega^2 m_{ab}\right) a^b = (\omega^2 m_{an} - k_{an}) a_n. \tag{5.65}$$

But

$$\det\left[k - \omega^2 m\right]_{(n-1)} \neq 0 \tag{5.66}$$

since ω^2 is the eigenvalue of the $n \times n$ system and there is no degeneracy. This implies that

$$\left[k - \omega^2 m\right]_{(n-1)}^{-1} \qquad \text{exists.} \tag{5.67}$$

Hence the $a^1, a^2, \ldots, a^{n-1}$ are all determined in terms of a^n. So the overall length of the vector $\vec{a}_r$ is undetermined. The eigenvector equation only determines the $\vec{a}_r$ vector's direction. Hence we are free to normalize the vector as we wish. We could choose

$$\sum_{r=1}^{n} a_r^T \; a_r = 1, \tag{5.68}$$

BUT it is more useful to use

$$\sum_{r=1}^{n} a_r^T \; m \; a_r = 1, \tag{5.69}$$

as long as m_{ab} is positive definite this is possible. And it is since

$$T = \frac{1}{2}\dot{q}^T \; m \; \dot{q} \tag{5.70}$$

and $T \geq 0$ but only zero if all $\dot{x}_{\alpha\ i} = 0$. That is iff all $\dot{q}^a = 0$. So m is positive definite

$$\sum_{r=1}^{n} a_r^T \; m \; a_r \geq 0 \tag{5.71}$$

and equal to 0 if and only if $a_r = 0$, which in NOT the case. ($m > 0$ allows a metric to be defined on the space by $a^T ma$.) So we have the normalization

$$a_s^T \; m \; a_r = \delta_{rs}. \tag{5.72}$$

In detail

$$\sum_{a=1}^{n} \sum_{b=1}^{n} m_{ab} a_s^a \; a_r^b = \delta_{rs}. \tag{5.73}$$

Thus, the $\vec{a}_r$ form an orthonormal set with respect to the metric m_{ab}.

We are now in a position to find the *normal coordinates*; those generalized coordinates whose motion is harmonic with only one of the eigenfrequencies. Recall that (complexifying the coordinate for now)

$$q^a(t) - q_0^a = \sum_{r=1}^{n} \alpha_r a_r^a e^{i(\omega_r t - \delta_r)}. \tag{5.74}$$

Writing

$$\beta_r \equiv \alpha_r e^{-i\delta_r} \tag{5.75}$$

so that

$$q^a(t) - q_0^a = \sum_{r=1}^{n} \beta_r e^{i\omega_r t} a_r^a. \tag{5.76}$$

Defining the normal coordinate as

$$\eta_r(t) \equiv \beta_r e^{i\omega_r t}, \tag{5.77}$$

we have that

$$q^a(t) = q_0^a + \sum_{r=1}^{n} a_r^a \, \eta_r(t). \tag{5.78}$$

From the definition of the normal coordinate $\eta_r(t)$, equation (5.77), we see that it obeys the uncoupled simple harmonic oscillator equation

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = 0 \tag{5.79}$$

for each $r = 1, 2, \ldots, n$. The η_r oscillate at frequencey ω_r only.

The Euler-Lagrange equations have become separated as we desired. That is with

$$\begin{aligned} q^a(t) &= q_0^a + \sum_{r=1}^{n} a_r^a \eta_r(t) \\ \dot{q}^a(t) &= \sum_{r=1}^{n} a_r^a \dot{\eta}_r(t) \end{aligned} \tag{5.80}$$

we find

$$\begin{aligned} T &= \frac{1}{2}\dot{q}^T \, m \, \dot{q} = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} \dot{\eta}_r \, a_r^T \, m \, a_s \, \dot{\eta}_s \\ &= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} \dot{\eta}_r \, \delta_{rs} \, \dot{\eta}_s = \frac{1}{2}\sum_{r=1}^{n} \dot{\eta}_r \dot{\eta}_r. \end{aligned} \tag{5.81}$$

Likewise the potential energy becomes

$$\begin{aligned} U &= \frac{1}{2}\left(q^T - q_0^T\right)\ k\ (q - q_0) = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_r a_r^T\ k\ a_s\ \eta_s \\ &= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\omega_r^2\eta_r\ \eta_s\ \delta_{rs} = \frac{1}{2}\sum_{r=1}^{n}\omega_r^2\eta_r\ \eta_r, \end{aligned} \tag{5.82}$$

where in the second line the eigenvalue equation and orthonormality of the eigenvectros were used

$$a_r^T\ k\ a_s = \omega_s^2 a_r^T\ m\ a_s = \omega_s^2\delta_{rs}. \tag{5.83}$$

So the Lagrangian takes on the simple form of n uncoupled simple harmonic oscillators each with theirown eigenfrequency ω_r, $r = 1, 2, \ldots, n$

$$L = \frac{1}{2}\sum_{r=1}^{n}\left[\dot{\eta}_r^2 - \omega_r^2\eta_r^2\right]. \tag{5.84}$$

The corresponding Euler-Lagrange equations of motion for these normal coordinates are

$$\frac{\partial L}{\partial \eta_r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_r} = 0, \tag{5.85}$$

which yields

$$\ddot{\eta}_r + \omega_r^2\ \eta_r = 0. \tag{5.86}$$

Next we would like to relate the amplitude of the normal coordinates' oscillations, β_r, to the initial positions and velocities of the particles. Recall

$$\eta_r(t) = \beta_r e^{i\omega_r t} \tag{5.87}$$

and

$$q^a(t) = q_0^a + \sum_{r=1}^{n} a_r^a \eta_r(t). \tag{5.88}$$

Hence

$$q^a(0) = q_0^a + \sum_{r=1}^{n} a_r^a\ \beta_r$$

$$\dot{q}^a(0) = \sum_{r=1}^{n} a_r^a \, i\omega_r \, \beta_r. \tag{5.89}$$

Treating q^a and q_0^a as n-dimensional vectors, we find that

$$a_s^T \, m \, q(0) = a_s^T \, m q_0 + \sum_{r=1}^{n} \underbrace{a_s^T \, m \, a_r}_{=\delta_{sr}} \, \beta_r, \tag{5.90}$$

so that

$$\beta_r = a_r^T \, m \, \left(q(0) - q_0\right). \tag{5.91}$$

Also

$$a_s^T \, m \, \dot{q}(0) = \sum_{r=1}^{n} i\omega_r \, \beta_r \, \underbrace{a_s^T \, m \, a_r}_{=\delta_{sr}}, \tag{5.92}$$

so that

$$\beta_r = -\frac{i}{\omega_r} a_r^T \, m\dot{q}(0). \tag{5.93}$$

Now letting $q(0)$ and $\dot{q}(0)$ be real quantities we find

$$\begin{aligned} \text{Re } \beta_r &= a_r^T \, m \, \left(q(0) - q_0\right) \\ \text{Im } \beta_r &= -\frac{1}{\omega_r} a_r^T \, m \, \dot{q}(0). \end{aligned} \tag{5.94}$$

Hence we have for the normal coordinates

$$\eta_r(t) = a_r^T \, m \, \left[q(0) - q_0 - \frac{i}{\omega_r}\dot{q}(0)\right] e^{i\omega_r t}. \tag{5.95}$$

Since only the real part is physically relevant, we have that for Reη_r

$$\eta_r(t) = a_r^T \, m \, \left(q(0) - q_0\right) \cos\omega_r t + \frac{a_r^T \, m \, \dot{q}(0)}{\omega_r} \sin\omega_r t, \tag{5.96}$$

where η_r is just the real part of equation (5.95).

To summarize this, we have that the normal coordinates obey the Euler-Lagrange equations

$$\ddot{\eta}_r + \omega_r^2 \, \eta_r = 0 \tag{5.97}$$

along with initial conditions for $\eta_r(0)$ and $\dot{\eta}_r(0)$. This implies that

$$\eta_r(t) = \eta_r(0)\cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t. \tag{5.98}$$

But we have from the orthonormality of the a_r eigenvectors that

$$\begin{aligned} \eta_r(t) &= a_r^T\ m\ (q(t) - q_0) \\ \dot{\eta}_r(t) &= a_r^T\ m\dot{q}(t). \end{aligned} \tag{5.99}$$

Hence evaluating these at $t = 0$ yields

$$\begin{aligned} \eta_r(0) &= a_r^T\ m\ (q(0) - q_0) \\ \dot{\eta}_r(0) &= a_r^T\ m\dot{q}(0), \end{aligned} \tag{5.100}$$

and hence equation (5.96) once again

$$\eta_r(t) = a_r^T\ m\ (q(0) - q_0)\cos\omega_r t + \frac{a_r^T\ m\ \dot{q}(0)}{\omega_r}\sin\omega_r t. \tag{5.101}$$

Now back to our example. We found the eigenvectors

$$\begin{aligned} a_{21} &= -a_{11} \\ a_{22} &= +a_{12}. \end{aligned} \tag{5.102}$$

The normalization condition becomes

$$\begin{aligned} 1 = \sum_{a,b=1}^{2} a_{ar}\ m_{ab}\ a_{br} &= \begin{pmatrix} a_{1r} & a_{2r} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} a_{1r} \\ a_{2r} \end{pmatrix} \\ &= m\left(a_{1r}^2 + a_{2r}^2\right) \\ &= 2m\ a_{1r}^2. \end{aligned} \tag{5.103}$$

So this yields

$$a_{1r} = \frac{1}{\sqrt{2m}}, \tag{5.104}$$

for $r = 1, 2$. Hence in detail we find the eigenvectors

$$\begin{aligned} r = 1 \qquad a_{11} &= \frac{1}{\sqrt{2m}} \qquad ; \qquad a_{21} = -\frac{1}{\sqrt{2m}} \\ r = 2 \qquad a_{12} &= \frac{1}{\sqrt{2m}} \qquad ; \qquad a_{22} = +\frac{1}{\sqrt{2m}}, \end{aligned} \tag{5.105}$$

or in vector notation

$$\begin{aligned} \vec{a}_1 &= \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & , & -1 \end{pmatrix} \\ \vec{a}_2 &= \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & , & 1 \end{pmatrix}. \end{aligned} \tag{5.106}$$

The normal coordinates for the choice of **initial condition**

$$\dot{q}(0) = 0, \tag{5.107}$$

are

$$\begin{aligned} \eta_r(t) &= a_r^T \, m \, (q(0) - q_0) \cos \omega_r t \\ &= m \left(a_{1r} \left(q^1(0) - q_0^1 \right) + a_{2r} \left(q^2(0) - q_0^2 \right) \right) \cos \omega_r t \\ \eta_r(t) &= m \left[a_{1r} \left(x_1(0) - x_1^0 \right) + a_{2r} \left(x_2(0) - x_2^0 \right) \right] \cos \omega_r t. \end{aligned} \tag{5.108}$$

So for each normal coordinate we find

$$\begin{aligned} \eta_1(t) &= \sqrt{\frac{m}{2}} \left[(x_1(0) - x_2(0)) - \left(x_1^0 - x_2^0 \right) \right] \cos \omega_1 t \\ \eta_2(t) &= \sqrt{\frac{m}{2}} \left[(x_1(0) + x_2(0)) - \left(x_1^0 + x_2^0 \right) \right] \cos \omega_2 t. \end{aligned} \tag{5.109}$$

Now first, let

$$1) \qquad \left(x_1(0) - x_1^0 \right) = - \left(x_2(0) - x_2^0 \right) \equiv x_0 \tag{5.110}$$

which implies that

$$\eta_1(t) = x_0 \sqrt{2m} \cos \omega_1 t$$

$$\eta_2(t) = 0. \tag{5.111}$$

Second, let

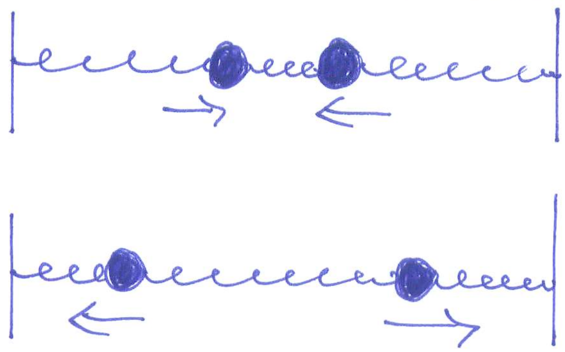
$$2) \qquad \left(x_1(0) - x_1^0\right) = + \left(x_2(0) - x_2^0\right) \equiv x_0 \tag{5.112}$$

which implies that

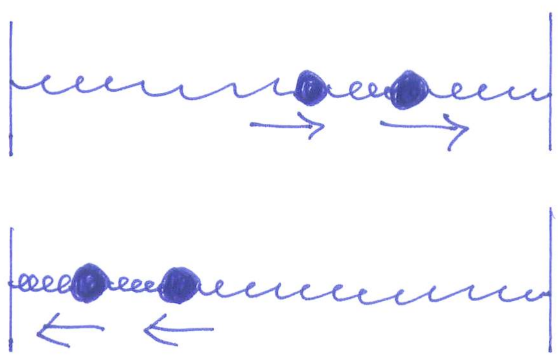
$$\begin{aligned} \eta_1(t) &= 0 \\ \eta_2(t) &= x_0\sqrt{2m}\cos\omega_2 t. \end{aligned} \tag{5.113}$$

Hence we have isolated the motion of the normal modes:

Mode 1: Oscillation out of phase with angular frequency $\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{m}}$



Mode 2: Oscillation in phase with angular frequency $\omega_2 = \sqrt{\frac{\kappa}{m}}$



This can be seen more clearly by considering the Cartesian coordinates

$$x_a(t) = x_a^0 + \sum_{r=1}^{2} a_{ar}\eta_r(t), \tag{5.114}$$

that is

$$\begin{aligned} x_1(t) &= x_1^0 + a_{11}\eta_1(t) + a_{12}\eta_2(t) = x_1^0 + \frac{1}{\sqrt{2m}}(\eta_2(t) + \eta_1(t)) \\ x_2(t) &= x_2^0 + a_{21}\eta_1(t) + a_{22}\eta_2(t) = x_2^0 + \frac{1}{\sqrt{2m}}(\eta_2(t) - \eta_1(t)) . \end{aligned} \tag{5.115}$$

Mode 1:

$$\begin{aligned} x_1(t) - x_1^0 &= x_0 \cos\omega_1 t \\ x_2(t) - x_2^0 &= -x_0 \cos\omega_1 t. \end{aligned} \tag{5.116}$$

Hence

$$\begin{aligned} \dot{x}_1(t) &= -\omega_1 x_0 \sin\omega_1 t \\ \dot{x}_2(t) &= +\omega_1 x_0 \sin\omega_1 t. \end{aligned} \tag{5.117}$$

So we see that the velocities of particles 1 and 2 are opposite.

Mode 2:

$$x_1(t) - x_1^0 = x_0 \cos\omega_2 t = x_2(t) - x_2^0. \tag{5.118}$$

Hence the distance between the particles is constant

$$x_2(t) - x_1(t) = x_2^0 - x_1^0 = \text{constant}. \tag{5.119}$$

Further

$$\dot{x}_1(t) = \dot{x}_2(t) = -\omega_2 x_0 \sin\omega_2 t. \tag{5.120}$$

So we see that the velocities of particles 1 and 2 are the same.

Finally let us summarize coupled oscillations and normal modes from a slightly different point of view. For small oscillations about equilibrium, $\partial U/\partial q^a\Big|_{q=q_0} = 0$, the Lagrangian has the form

$$\frac{1}{2}\sum_{a,b=1}^{n}\left[m_{ab}\dot{q}^a\dot{q}^b - k_{ab}\left(q^a - q_0^a\right)\left(q^b - q_0^b\right)\right], \tag{5.121}$$

where

$$m_{ab} = m_{ab}(q_0^c) = \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q^a} \cdot \frac{\partial \vec{r}_\alpha}{\partial q^b}\bigg|_{q^c=q_0^c} = m_{ba} \tag{5.122}$$

and

$$k_{ab} \equiv \frac{\partial^2 U}{\partial q^a \partial q^b}\bigg|_0 = k_{ba}. \tag{5.123}$$

Since $T = \dot{q}^T \ m \ \dot{q} \geq 0$ and $= 0$ iff all $\dot{q} = 0$, m_{ab} is positive definite and symmetric. Hence we can find its inverse m_{ab}^{-1}, moreover m_{ab} has no zero eigenvalues, so we can define $(m^{\pm 1/2})$.

Now let's transform the q^a coordinates to new generalized coordinates called the normal coordinates

$$(q^a(t) - q_0^a) \equiv \sum_{r=1}^{n} a_r^a \ \eta_r(t), \tag{5.124}$$

which implies that

$$\dot{q}^a(t) = \sum_{r=1}^{n} a_r^a \ \dot{\eta}_r(t), \tag{5.125}$$

where the a_r^a are such that they are 1) eigenvectors and 2) normalized

$$1) \qquad k_{ab} a_s^b = \omega_s^2 \ m_{ab} \ a_s^b \qquad (\text{ no sum on } s), \tag{5.126}$$

and

$$2) \qquad a_r^a \ m_{ab} \ a_s^b = \delta_{rs} \qquad . \tag{5.127}$$

This implies that

$$a_r^a \ k_{ab} \ a_s^b = \omega_s^2 \ a_r^a \ m_{ab} \ a_s^b = \omega_r^2 \delta_{rs}, \tag{5.128}$$

Further we can invert the equations (5.124) and (5.125)

$$\begin{aligned} a_s^T \ m \ (q(t) - q_0) &= \sum_{r=1}^{n} a_s^T \ m \ a_r \ \eta_r(t) = \eta_s(t) \\ \dot{\eta}_s(t) &= a_s^T \ m \ \dot{q}(t). \end{aligned} \tag{5.129}$$

Substituting this into the Lagrangian yields

$$\begin{aligned} L &= \frac{1}{2}\sum_{a,b=1}^{n} m_{ab} \sum_{r,s=1}^{n} a_r^a\, a_s^b\, \dot{\eta}_r\, \dot{\eta}_s - \frac{1}{2}\sum_{a,b=1}^{n} k_{ab} \sum_{r,s=1}^{n} a_r^a\, a_s^b\, \eta_r\, \eta_s \\ &= \frac{1}{2}\sum_{r,s=1}^{n}\left(\sum_{a,b=1}^{n} m_{ab} a_r^a\, m_{ab}\, a_s^b\right)\dot{\eta}_r\, \dot{\eta}_s - \frac{1}{2}\sum_{r,s=1}^{n}\left(\sum_{a,b=1}^{n} k_{ab} a_r^a\, m_{ab}\, a_s^b\right)\eta_r\, \eta_s \\ L &= \frac{1}{2}\sum_{r=1}^{n}\left(\dot{\eta}_r^2 \;-\; \omega_r^2\, \eta_r^2\right). \end{aligned} \tag{5.130}$$

The η_r are called *normal coordinates* and obey the equation of motion of the simple harmonic oscillator with frequency ω_r:

$$\ddot{\eta}_r + \omega_r^2\, \eta_r = 0. \tag{5.131}$$