Now that the fundamental principles of Newtonian Mechanics are understood we will next study the consequences of various specific force laws in more detail.

To begin let us assume single particle motion which is constrained to move along the x-axis only; so $y(t) = 0 = z(t)$.

In general we consider the case where the particle is in equilibrium at the origin of our system and subject to a restorative, conservative force $F = F(x)$ when displaced from equilibrium: hence

$$F = F(x) = F(0) + x\frac{dF}{dx}(0) + \frac{1}{2!}x^2\frac{d^2F}{dx^2}(0) + \dots$$

Since $x = 0$ is the equilibrium point $\Rightarrow \boxed{F(0) =}$ $(F(0) = 0)$. Further for small displacements from equilibrium, we may neglect the higher powers of x other than linear. Since the force is restorative $F(x)$ points towards the origin: Thus

$$\frac{dF}{dx}(0) = -k \quad \text{with } k > 0 .$$

Hence we have $\boxed{F(x) \simeq -kx}$

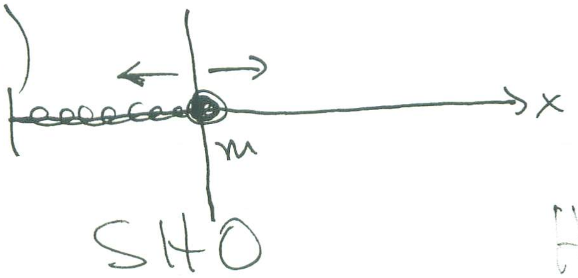
Thus we see for small displacements about stable equilibrium, the simple harmonic oscillator force law, $F=-kx$ is a reasonable approximation for the arbitrary force law.

Newton's 2^{nd} law then results in an equation of motion for a particle of mass m subject to $F=-kx$:

1) SHO

$$\boxed{m\frac{d^2x(t)}{dt^2}+kx(t)=0}$$

(no friction)



linear spring force $F=-kx$
Hooke's Law

Of course we would like to consider the addition of a frictional force opposing the motion of the particle and linearly proportional to the velocity

2) Damped SHO : Frictional Force $F_f=-bv$

$$\boxed{m\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+kx(t)=0}$$

Further we can imagine applying a driving force to this damped harmonic oscillator in order to keep the particle's motion continuing. This external time varying force will be denoted $F(t)$.

Hence Newton's 2^{nd} law becomes

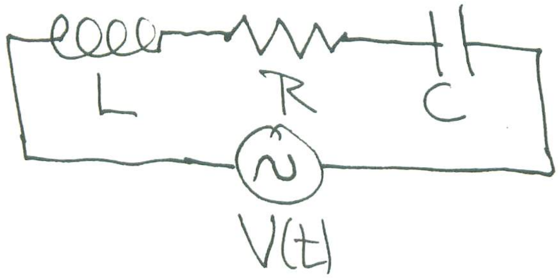
3) Driven, damped SHO

$$\boxed{m\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+kx(t)=F(t)}$$

Once again, the SHO is important since for small oscillations $F(x)=-kx$ is a reasonable approximation for any force. (i.e. system in equilibrium with restoring force)

As well in other branches of physics the HO equation appears. For example the DE for the charge on a capacitor C in an electric circuit containing an inductance L, capacitance C, and resistance R in series, subject to an external emf $V(t)$:

L-R-C circuit:



is

$$L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{q(t)}{C}=V(t)$$

with $q(t)$ = the charge on C.

(Recall $L=R=0$ $\frac{q}{C}=V \Rightarrow C=q/V$ the definition of C
if only $R\neq 0$ $R\frac{dq}{dt}=RI=V$ Ohm's Law.)

Also in quantum mechanics the SHO is equally important, since any potential energy

$$U(x)=U(0)+xU'(0)+\frac{1}{2}x^2U''(0)+\cdots$$

close to origin

$U \approx \frac{1}{2}U''(0)x^2$

Close to the origin the $O(x^3)$ terms may be neglected and $U(0)=0$ by choice of the zero of energy. Since the system is *stable* at the *minimum* $U'(0)=0$ and $U''(0)>0$.

So

$$U(x) \approx \frac{1}{2}U''(0)\,x^2 \equiv \frac{1}{2}kx^2 \, .$$

(wavefunctions)

The eigenstates of the SHO approximate those of the physical system, for example molecules in vibration. (Recall *classically* $F(x) = -\frac{dU}{dx} = -kx$: SHO)

In Quantum Field Theory, matter and radiation fields appear as an infinite collection of quantum mechanical simple harmonic oscillators located at each point in space which become quantum mechanically perturbed by their interactions.

Before determining the solutions for the various cases above of the SHO, let's recall some general properties of linear DE of which the SHO are 2nd order.

(linear ⇒ independent variable $x(t)$ (or $q(t)$) appears only linearly in the DE)

Solutions of the general n^{th} order linear differential equation

$$a_n(t)\frac{d^n x(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{dx(t)}{dt} + a_0(t)x(t)$$

$$= b(t)$$

(If $b=0$, this is the homogeneous equation
If $b\neq 0$, this is the inhomogeneous equation)

Theorem:
The most general solution of the n^{th} order equation has n arbitrary constants (of integration) which specify the boundary (initial) conditions

$$\boxed{x = x(t; c_1, c_2, \ldots, c_n)} \quad (c_i = \text{real constants})$$

Theorem 1: If $x(t) = x_1(t)$ is a solution of the homogeneous equation then so is $C x_1(t)$ where C is a constant.

Theorem 2: If $x_1(t)$ and $x_2(t)$ are solutions of the homogeneous equation, then so is $x_1(t) + x_2(t)$.

The proof of these 2 theorems is straightforward by substituting into the DE and taking derivatives.

Hence, if $x_1(t), \ldots,$ homogeneous $x_n(t)$ are linearly independent solutions to the DE, then $x(t) = c_1 x_1(t) + \cdots + c_n x_n(t)$ is the general solution.

Recall $x_1(t), \ldots, x_n(t)$ are linearly independent if and only if the equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$$ is

satisfied by $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ only. Otherwise $x_1, \ldots, x_n$ are linearly dependent. Further the necessary and sufficient condition for $x_1, \ldots, x_n$ to be linearly dependent is that the Wronskian determinant vanishes:

.e.

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ \dot{x}_1 & \ldots & & \dot{x}_n \\ x_1^{(n-1)} & \ldots & & x_n^{(n-1)} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = 0$$

$$W \equiv \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_n \\ \ddot{x}_1 & \ddot{x}_2 & \cdots & \ddot{x}_n \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)} \end{vmatrix} = 0$$

where $x_i^{(n)} \equiv \dfrac{d^n x_i(t)}{dt^n}$

These Thms. apply only to the homogeneous DE.

Thus the solution to the 2^{nd} order homogeneous differential equation boils down to find 2 independent solutions $x_1(t)$ and $x_2(t)$. Then since $c_1 x_1(t)$ and $c_2 x_2(t)$ are also solutions by thm. 1 and by thm. 2 $x(t) = c_1 x_1(t) + c_2 x_2(t)$ is also a solution. Since it contains 2 arbitrary parameters (constants $c_{1,2}$) it is also the most general solution to the homogeneous DE.

So let's proceed to solve the homogeneous damped SHO equation with constant coefficients $m\ddot{x} + b\dot{x} + kx = 0$.

Consider $x = e^{rt}$ so $\dot{x} = re^{rt}$; $\ddot{x} = r^2 e^{rt}$

Hence substituting into $m\ddot{x} + b\dot{x} + kx = 0$

$$\Rightarrow \quad [mr^2 + br + k]e^{rt} = 0$$

$$\Rightarrow \quad [mr^2 + br + k] = 0$$

$$\Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} \equiv r_\pm$$

Hence $x_1(t) = e^{r_+ t}$; $x_2(t) = e^{r_- t}$ are two independent solutions:

$$W_2 = \begin{vmatrix} e^{r_+t} & e^{r_-t} \\ r_+e^{r_+t} & r_-e^{r_-t} \end{vmatrix} = r_-e^{(r_++r_-)t} - r_+e^{(r_++r_-)t}$$

$$= e^{-\frac{bt}{m}}(r_- - r_+) = \left(e^{-\frac{bt}{m}}\right)\left(\frac{-\sqrt{b^2-4km}}{m}\right) \neq 0$$

So as long as $b^2 \neq 4km$, x_1, x_2 are linearly independent.

(Note: if $b^2 = 4km$ $r_+ = r_- = r = -\frac{b}{2m}$ and $x_1 = e^{rt}$

$x_1 = e^{-\frac{b}{2m}t}$ is one solution while

$x_2 = te^{rt} = te^{-\frac{b}{2m}t}$ is the other linearly indep. solution:

$$\dot{x}_2 = e^{rt} + rte^{rt}$$
$$\ddot{x}_2 = 2re^{rt} + r^2te^{rt}.$$

$$\Rightarrow \quad m\ddot{x}_2 + b\dot{x}_2 + kx_2 = (mr^2+br+k)te^{rt} + (\underbrace{\underbrace{2rm}_{-b}+b}_{=0})e^{rt} = 0$$

and $mr^2+br+k = 0$

Further
$$W = \begin{vmatrix} e^{+rt} & te^{rt} \\ re^{rt} & e^{rt}+rte^{rt} \end{vmatrix}$$

$$= e^{2rt} + \cancel{rte^{2rt}} - \cancel{rte^{2rt}} = e^{2rt} \neq 0.$$

Hence $X(t) = c_1e^{rt} + c_2te^{rt}$ is the general solution of $b^2=4km$.)

In general then ($b^2 \neq 4km$) the most general solution to the Damped SHO equation

$m\ddot{x}+b\dot{x}+kx=0$ is

$$\boxed{\begin{aligned} x(t) &= C_1 e^{r_+ t} + C_2 e^{r_- t} \\ \text{where} \quad r_\pm &= -\frac{b}{2m} \pm \frac{\sqrt{b^2-4km}}{2m} \end{aligned}}$$

and $r_\pm \in \mathbb{C}$ (Note: $r_+^* = r_-$ if $b^2<4km$ and since $x=x^* \Rightarrow C_1^*=C_2$)

Defining $\beta \equiv \frac{b}{2m}$ = damping parameter

and $\omega_0 \equiv \sqrt{\frac{k}{m}}$ = HO angular frequency

we have $\boxed{r_\pm = -\beta \pm \sqrt{\beta^2-\omega_0^2}}$

So

$$\boxed{x(t) = e^{-\beta t}\left[C_1 e^{+\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]}$$

1) SHO: $\beta=0$ $\quad m\ddot{x}+kx=0$

$$x(t)=C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

Since x is the position of the

particle it is a Real number $\Rightarrow$

$$\boxed{C_1 = C_2^*}$$

(if C_1, C_2 were arbitrary complex numbers then we really have 4 real constants, which would be too many for a 2nd order Diff. eq.; Then x = complex and we would have 2 2nd order DE; one for Re x another for Im x).

So For $\beta = 0$ $C_1 = C_2^*$ so that $x = x^*$.

Let $C_1 = \rho e^{i\theta} = c + id$ then $C_2 = \rho e^{-i\theta} = c - id$

Hence $\rho = \sqrt{c^2+d^2}$, $\tan\theta = \frac{d}{c}$

Thus

$$x(t) = \rho e^{i\theta} e^{i\omega_0 t} + \rho e^{-i\theta} e^{-i\omega_0 t}$$
$$= \rho\left[e^{i(\omega_0 t+\theta)} + e^{-i(\omega_0 t+\theta)}\right]$$

And we secure

$$\boxed{x(t) = 2\rho\cos(\omega_0 t+\theta) \equiv A\cos(\omega_0 t+\theta)}$$

(i.e. $A \equiv 2\rho$)

Likewise

$$\sin(\omega_0 t + \theta + \tfrac{\pi}{2}) = \sin(\omega_0 t + \theta)\cos(\tfrac{\pi}{2}) + \cos(\omega_0 t + \theta)\sin(\tfrac{\pi}{2})$$

$$= \cos(\omega_0 t + \theta)$$

Letting $\varphi \equiv \theta + \frac{\pi}{2}$

$\Rightarrow$ we can write $x(t)$ as

$$x(t) = A\sin(\omega_0 t + \varphi)$$

or

$$x(t) = A\cos(\omega_0 t + \theta)$$

with $A, \theta (\varphi)$ real arbitrary constants

Consider the KE of the particle for the SHO:

$$T = \tfrac{1}{2} m \dot{x}^2 = \tfrac{1}{2} m \omega_0^2 A^2 \sin^2(\omega_0 t + \theta)$$

$$\left(\omega_0 \equiv \sqrt{\tfrac{k}{m}}\right) \quad = \tfrac{1}{2} k A^2 \sin^2(\omega_0 t + \theta)$$

On the other hand the PE $U = \frac{1}{2} k x^2$ (this yields the conservative force $F = -\frac{d}{dx} U = -kx$ as it must by definition)

So $\qquad U = \frac{1}{2} k A^2 \cos^2(\omega_0 t + \theta)$

As expected for a conservative force, the total energy E is a constant

$$\boxed{E = T + U = \tfrac{1}{2}kA^2 = \text{constant}.}$$

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2) Damped SHO: $\beta \neq 0$. $m\ddot{x} + b\dot{x} + kx = 0$

then

$$x(t) = e^{-\beta t}\left[C_1 e^{+\sqrt{\beta^2 - \omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

There are 3 cases of interest corresponding to

1) Underdamping: $\omega_0^2 > \beta^2$

2) Critical damping: $\omega_0^2 = \beta^2$

3) Overdamping: $\omega_0^2 < \beta^2$

1) Underdamping: Define $\omega_1^2 \equiv \omega_0^2 - \beta^2$ and $\omega_1^2 > 0$

hence $x(t) = e^{-\beta t}\left[C_1 e^{+i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$

again x is real so $x(t) = x^*(t) \Rightarrow C_2 = C_1^*$ and as before we can write this as

$$\boxed{x(t) = A e^{-\beta t}\cos(\omega_1 t - \delta)}$$

Before considering the other cases, suppose we consider the SHO but with more degrees of freedom. For example consider a 3-dimensional SHO

$$m\ddot{x} = -k_x x$$
$$m\ddot{y} = -k_y y$$
$$m\ddot{z} = -k_z z$$

e.g. an atom in a crystal lattice

where in general the spring constants k_x, k_y, k_z are unequal. Since the motion in each direction decouples from that in the other directions, we can write the solutions for each coordinate separately; the actual motion being just a vector sum:

$$x(t) = A_x \cos(\omega_x t + \theta_x) \quad \text{with } \omega_x^2 = \frac{k_x}{m}$$
$$y(t) = A_y \cos(\omega_y t + \theta_y) \quad \text{with } \omega_y^2 = k_y/m$$
$$z(t) = A_z \cos(\omega_z t + \theta_z) \quad \text{with } \omega_z^2 = k_z/m.$$

6 initial conditions are needed in order to fix the six constants $A_x, \ldots, \theta_z$. These are the initial position and velocities in the x, y, z directions.

The motion is characterized as

1) Commensurable if $\exists$ n_x, n_y, n_z integers

$$\ni \quad \frac{\omega_x}{n_x} = \frac{\omega_y}{n_y} = \frac{\omega_z}{n_z} \equiv \omega \qquad (*)$$

(n_x, n_y, n_z are chosen to have no common integral factor $\tau = \frac{2\pi n_x}{\omega_x} = \frac{2\pi n_y}{\omega_y} = \frac{2\pi}{\omega}$

hence $(X(\overset{\frac{2\pi}{\omega}}{t}), y(t=\frac{2\pi}{\omega}), z(t=\frac{2\pi}{\omega}))$ have gone through (n_x, n_y, n_z) cylces, and so returned to the original $t=0$ $(x(0), y(0), z(0))$ point in the $(2A_x, 2A_y, 2A_z)$ solid.

Then

$$x(t) = A_x \cos(n_x \omega t + \theta_x)$$
$$y(t) = A_y \cos(n_y \omega t + \theta_y)$$
$$z(t) = A_z \cos(n_z \omega t + \theta_z)$$

After time $t \ni \omega t = 2\pi$ the "time" coordinate point has travelled $(2\pi n_x, 2\pi n_y, 2\pi n_z)$. That is (n_x, n_y, n_z) cycles and hence it has returned to its original position. Thus if (*) holds the motion is a closed curve which is periodic with period determined by the smallest $\omega_x, \omega_y, \omega_z$. (In 2 dimensions this closed curve is called a Lissajous curve)

2) Space-filling: If the frequencies are not commensurable then the motion is that of a space-filling curve — the curve is said to be open. The particle will never pass twice through the same point with the same velocity. Hence after a sufficiently long time has elapsed the curve will pass arbitrarily close to any given point lying in the volume $2A_x \times 2A_y \times 2A_z$. The curve will fill the rectangular parallelopiped.

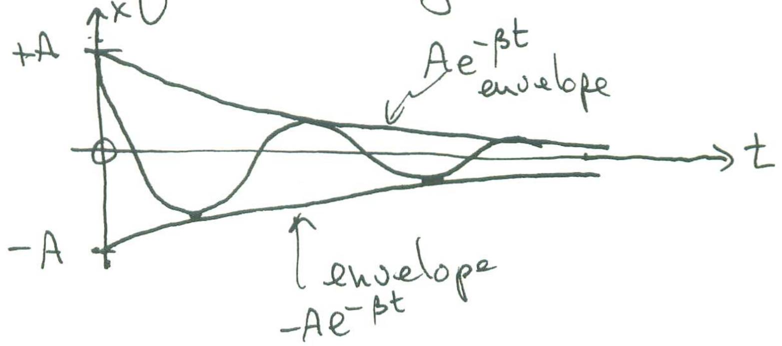
where $C_1 = c + id = \frac{1}{2} A e^{-i\delta}$

$C_2 = c - id = \frac{1}{2} A e^{+i\delta}$

and $x(t) = \frac{1}{2} A e^{-\beta t}\left[e^{+i(\omega_1 t - \delta)} + e^{-i(\omega_1 t - \delta)}\right]$

$$= A e^{-\beta t} \cos(\omega_1 t - \delta) \checkmark .$$

Graphically we find (wlog $\delta = 0$)



The "total energy" of the oscillator is not a constant due to the frictional damping, $\vec{F}_f = -bv$ cannot be derived from a potential

let $E \equiv T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

Then $$\frac{dE}{dt} = -b\dot{x}^2 = \vec{F}_f \cdot \vec{v} = -bv^2$$

as we found when considering conservation laws in general: $\vec{F} = -\vec{\nabla} U + \vec{F}_f$

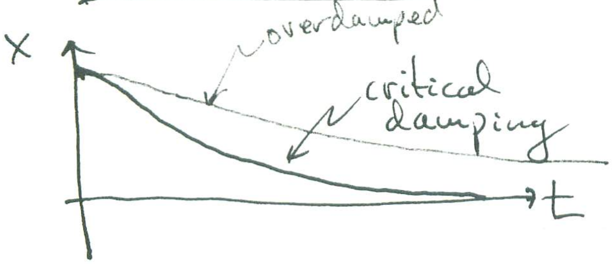
$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right)$$

$$= -\vec{\nabla} U \cdot \frac{d\vec{r}}{dt} + \vec{F}_f \cdot \vec{v} = -\frac{dU}{dt} + \vec{F}_f \cdot \vec{v}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v^2 + U\right) = \frac{d}{dt}(T + U) = -bv^2 .$$

2) Critical damping: $\omega_0^2 = \beta^2 \Rightarrow r_+ = r_- = -\beta$

hence $\boxed{x(t) = (C_1 + C_2 t)e^{-\beta t}}$ with C_1, C_2 real constants.



For a given set of initial conditions, a critically damped oscillator will approach equilibrium ($x=0$) at a rate more rapid than that for either an overdamped or an underdamped oscillator.

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3) Overdamped Oscillator: $\beta^2 > \omega_0^2$

$$x(t) = e^{-\beta t}\left[C_1 e^{+\omega_2 t} + C_2 e^{-\omega_2 t}\right]$$
$$= C_1 e^{-(\beta-\omega_2)t} + C_2 e^{-(\beta+\omega_2)t}, \quad C_1, C_2 \in \mathbb{R}$$

where now $\omega_2 \equiv \sqrt{\beta^2 - \omega_0^2}$, since $\beta - \omega_2 > 0$, both of these solutions correspond to damped motion with no oscillatory motion.

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3) (Forced) Driven, damped HO : $m\ddot{x}+b\dot{x}+kx = F(t)$
This will lead to the phenomenon of resonance.

Initially consider a sinusoidal driving force

$F(t) = F_0 \cos(\omega t+\theta_0)$ since we can always Fourier analyze an arbitrary force into sines and cosines. Hence we are interested in solving the equation

$$m\ddot{x}+b\dot{x}+kx = F_0\cos(\omega t+\theta_0) .$$

(Thm.) The solution to the inhomogeneous 2nd order linear DE will be of the form

$$x(t) = x_p(t) + x_h(t)$$

where $Lx_p = F(t)$ is the particular (or inhomogeneous solution

$Lx_h = 0$ is the homogeneous (or complementary solution.

with $L = m\frac{d^2}{dt^2}+b\frac{d}{dt}+k$ the linear differential operator.

As before the solution depends upon 2 arbitrary constants of integration (i.c.) in the homogeneous solution x_h: these will be transient solutions damped out in time

$$x_h(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right].$$

As we might guess, the particular solution will oscillate at the same frequency as the driving force, so

$$x_p(t) = A_s\cos(\omega t + \theta_s) \quad \text{(s for steady state)}$$

We can insert this into the DE to find A_s, θ_s as the text does – but let's take a different tack. We can simplify the algebra by letting $x(t)$ and $F(t)$ become complex and then take the real part of X, F to recover the problem at hand; this can be done for any linear D.E.

$$x \rightarrow X$$
$$F \rightarrow \mathcal{F}$$

with

$$\text{Re}\, X = x$$
$$\text{Re}\, \mathcal{F} = F.$$

Letting

$$\mathcal{F} = F_0 e^{i(\omega t+\theta_0)} = F_0\cos(\omega t+\theta_0) + iF_0\sin(\omega t+\theta_0)$$
$$= \mathcal{F}_0 e^{i\omega t} \quad (\text{i.e. } \mathcal{F}_0 = F_0 e^{i\theta_0})$$

Then $\text{Re}\,\mathcal{F} = F_0\cos(\omega t+\theta_0)$ (note the solution for a force $\sim \sin(\omega t+\theta_0)$ is simply $\text{Im}\, X$).

Then we try to find the complex solution to the equation

$$m\ddot{X} + b\dot{X} + kX = F = F_0 e^{i\omega t} \qquad (*)$$

Our ansatz for the steady state (particular) solution is of the form

$$X = X_0 e^{i\omega t} \quad ; \quad \dot{X} = i\omega X$$
$$\ddot{X} = -\omega^2 X \quad .$$

Plugging this into $(*)$, we find

$$[-m\omega^2 + ib\omega + k] X_0 e^{i\omega t} = F_0 e^{i\omega t}$$

$$\Rightarrow \quad X_0 = \frac{F_0}{[-m\omega^2 + ib\omega + k]} = \frac{F_0/m}{\left[k/m - \omega^2 + \frac{2i\omega b}{2m}\right]}$$

$$= \frac{F_0/m}{[\omega_0^2 - \omega^2 + 2i\beta\omega]}$$

where recall $\beta = b/2m$
$\omega_0^2 = k/m$.

Multiplying numerator and denominator by $[\omega_0^2 - \omega^2 - 2i\omega\beta] \Rightarrow$

$$X_0 = \frac{\frac{F_0}{m}\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right]}{\left[\omega_0^2 - \omega^2 + 2i\beta\omega\right]\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right]}$$

$$\boxed{\text{So} \quad X_0 = \frac{(F_0/m)\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right]}{\left[\omega_0^2 - \omega^2\right]^2 + 4\beta^2\omega^2}}$$

Hence we obtain the complex particular solution

$$X(t) = X_0 e^{i\omega t} = \frac{(F_0/m)\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right]}{\left[\omega_0^2 - \omega^2\right]^2 + 4\beta^2\omega^2} e^{i(\omega t + \theta_0)}$$

Expanding the numerator, we find

$$\text{Numerator} = \frac{F_0}{m}\left[\omega_0^2 - \omega^2 - 2i\beta\omega\right]\left[\cos(\omega t + \theta_0) + i\sin(\omega t + \theta_0)\right]$$

$$= \frac{F_0}{m}\left\{\left[(\omega_0^2 - \omega^2)\cos(\omega t + \theta_0) + 2\beta\omega\sin(\omega t + \theta_0)\right]\right.$$

$$\left. + i\left[(\omega_0^2 - \omega^2)\sin(\omega t + \theta_0) - 2\beta\omega\cos(\omega t + \theta_0)\right]\right\}$$

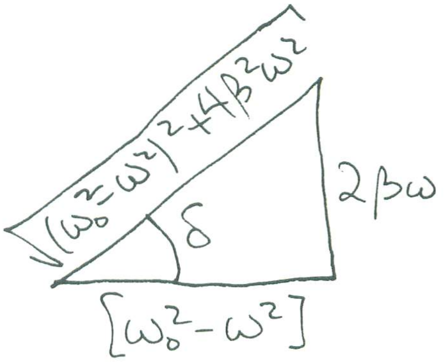
Taking the real part of $\mathbb{X}(t)$, we obtain the particular solution to our original equation and problem:

$$\boxed{\begin{aligned} x_p(t) &= \mathrm{Re}\,\mathbb{X}(t) \\ &= \frac{\frac{F_0}{m}\left[(\omega_0^2-\omega^2)\cos(\omega t+\theta_0)+2\beta\omega\sin(\omega t+\theta_0)\right]}{[\omega_0^2-\omega^2]^2+4\beta^2\omega^2} \end{aligned}}$$

This can be put in the form of the answer obtained in the text by recalling that

$$\cos(\omega t+\theta_0-\delta) = \cos\delta\cos(\omega t+\theta_0)+\sin\delta\sin(\omega t+\theta_0)$$

with δ given by



Thus

$$\boxed{\begin{aligned} x_p(t) &= \frac{\frac{F_0}{m}}{\sqrt{[\omega_0^2-\omega^2]^2+4\beta^2\omega^2}}\cos(\omega t+\theta_0-\delta) \\ \text{with}\quad \tan\delta &= \left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right) \end{aligned}}$$

Note: This steady state solution contains no arbitrary constants. This is reasonable since it does not depend on the initial conditions as the transient solution does.

The complete solution to the driven oscillator is the sum of x_p and the homogeneous solution

$$x(t) = \overbrace{Ae^{-\beta t}\cos(\omega_1 t+\theta)}^{\text{transiant solution (dies away)} = \text{negligible for } t \gg \frac{1}{\beta}}$$

$$+ \underbrace{\frac{F_0}{m}\frac{\cos(\omega t+\theta_0-\delta)}{\sqrt{[\omega_0^2-\omega^2]^2+4\beta^2\omega^2}}}_{\text{steady state solution}}$$

The two boundary (initial) conditions determine A & θ

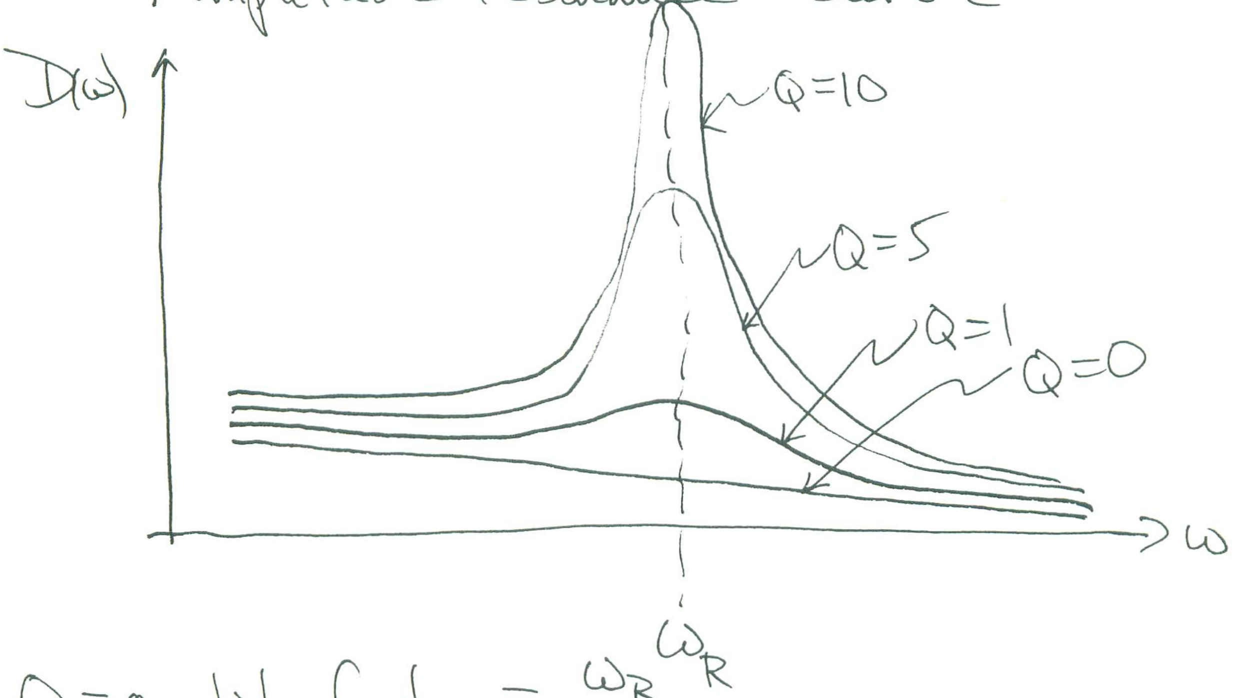
ex. at $t=0$ $\quad x(0) = x_0$

$\dot{x}(0) = v$

We can plot the maximum steady state displacement, $D(\omega)$, vs the driving frequency ω

$$D(\omega) = \frac{F_0}{m} \frac{1}{\sqrt{[\omega_0^2 - \omega^2]^2 + 4\beta^2\omega^2}}$$

Amplitude Resonance Curve



$$Q = \text{quality factor} \equiv \frac{\omega_R}{2\beta}$$

where the resonance frequency ω_R is the value of ω at which $D(\omega = \omega_R) =$ maximum

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0 = \frac{F_0}{m} \frac{2\omega_R[(\omega_0^2 - \omega_R^2) - 2\beta^2]}{[[\omega_0^2 - \omega_R^2]^2 + 4\beta^2\omega_R^2]}$$

$$\Rightarrow \boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

If there is little damping then Q is big and resonance curve $\longrightarrow$ the curve for the undamped oscillator

If Q is small, i.e. large damping, and if $2\beta^2 > \omega_0^2$, There is no resonance – just damping.

The above frequency ω_R is the amplitude resonance frequency.
Instead we can calculate the steady state rate at which energy is pumped into the oscillator to find the resonance frequency for energy absorption

Recall the power that is $\frac{\text{work}}{\text{time}}$, done in driving the oscillator is given by the driving force $\cdot \vec{v}$

$$\frac{dW_D}{dt} = Fv = F_0 \cos(\omega t + \theta_0)\,\dot{x}$$

$$= -\frac{F_0^2}{m}\,\frac{\omega \cos(\omega t+\theta_0)\sin(\omega t+\theta_0-\delta)}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

Note: $\sin(\omega t+\theta_0-\delta) = \sin(\omega t+\theta_0)\cos\delta - \cos(\omega t+\theta_0)\sin\delta$

So $$\frac{dW_D}{dt} = \frac{F_0^2}{m}\,\frac{\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\Big[\sin\delta\cos^2(\omega t+\theta_0)$$
$$-\cos\delta\sin(\omega t+\theta_0)\cos(\omega t+\theta_0$$

and $\sin(\omega t+\theta_0)\cos(\omega t+\theta_0) = \frac{1}{2}\sin 2(\omega t+\theta_0)$

Since the $\sin 2(\omega t+\theta_0)$ term is equally positive as negative over each cycle, its time average is <u>zero</u>

$$\int_{t=0}^{\frac{2\pi}{\omega}} \cos(\omega t+\theta_0)\sin(\omega t+\theta_0)\,dt$$

$$= \frac{1}{\omega}\int_{t=0}^{\frac{2\pi}{\omega}} \sin(\omega t+\theta_0)\, d(\sin(\omega t+\theta_0))$$

$$= \frac{1}{2\omega}\sin^2(\omega t+\theta_0)\Big|_0^{\frac{2\pi}{\omega}}$$

$$= \frac{1}{2\omega}\left[\sin^2(\theta_0+2\pi)-\sin^2\theta_0\right] = 0 \checkmark$$

On the other hand $\cos^2$ averages to $\frac{1}{2}$

$$\frac{\pi}{\omega} = \int_{t=0}^{\frac{2\pi}{\omega}} \cos^2(\omega t+\theta_0)\,dt = \int_{t=0}^{\frac{2\pi}{\omega}}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos 2(\omega t+\theta_0)\right]dt$$

$$= \frac{\pi}{\omega} + \underbrace{\frac{1}{2}\int_0^{\frac{2\pi}{\omega}} \cos 2(\omega t+\theta_0)\,dt}$$

$$= \frac{1}{4\omega}\sin 2(\omega t+\theta_0)\Big|_0^{\frac{2\pi}{\omega}}$$

$$= \frac{\pi}{\omega}$$

$$= \frac{1}{4\omega}\left[\sin 2(\theta_0+2\pi)-\sin 2\theta_0\right]$$

$$= 0.$$

Hence, the time average of the power over a cycle is the work done by the oscillator per cycle period

$$\left\langle \frac{dW_D}{dt} \right\rangle = \frac{1}{2\pi/\omega} \int_{t=0}^{\frac{2\pi}{\omega}} dt \; \frac{dW_D}{dt}$$

$$= \frac{F_0^2}{2m} \frac{\omega \sin\delta}{\sqrt{[\omega_0^2 - \omega^2]^2 + 4\beta^2\omega^2}}$$

Recalling that $\sin\delta = \dfrac{2\beta\omega}{\sqrt{[\omega_0^2 - \omega^2]^2 + 4\beta^2\omega^2}}$

we find

$$\boxed{\begin{aligned}\left\langle \frac{dW_D}{dt} \right\rangle &= \frac{F_0^2}{m} \frac{\beta\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]} \\ &= \frac{1}{2} F_0 \dot{x}_{max} \sin\delta\end{aligned}}$$

Thus $\left\langle \frac{dW_D}{dt} \right\rangle_{max}$ occurs at $\boxed{\omega = \omega_0}$

This is different from the amplitude Resonance frequency ω_R, This is typical of non-conservative systems.

Note: This is average power supplied by driving force equal to the average power dissipated by friction $\left\langle \frac{dW_f}{dt} \right\rangle = \langle -F_f v \rangle = \langle b v^2 \rangle$.