3 Lagrangian and Hamiltonian Dynamics

Next we desire to re-formulate the laws of Newtonian dynamics in a form that deals with scalar quantities like the energy or potential energy rather than vector quantities like the force and acceleration. This will be particularly useful when dealing with constrained motion. Remember the particle constrained to move on a surface of a sphere, Newton's law was somewhat messy to write out in spherical coordinates where the constraint $r = R$ was simple. We would like to avoid such complications, at least initially. Besides utility, this re-formulation will make the relationship between conservation laws and symmetries of space and time very transparant. Also the more general formulation in terms of Lagrangian dynamics has generalizations when special relativity becomes important. As well it is a useful starting point for quantum mechanical and field and string theoretical dynamics.

To repeat then, as we have seen, constraints are a cumbersome restriction to apply to Newton's law directly; in fact we never really know the forces of constraint, $\vec{F}_c$. Hence we would like a formulation of mechanics which allows us to describe the dynamics without explicit knowledge of the forces of constraint, i.e. the tension in a rope or the force holding a particle on a surface. For simple cases we know what to do, we just explicitly eliminate the resulting constrained coordinates from the problem. Then consider the equation of motion in non-constrained directions we already "know" (given) the solutions in the constrained directions. For example $z = 0$ implies $x - y$ motion. This is the case of holonomic constraints in which we have algebraic relations amongst the coordinates themselves. This allows us to eliminate them from the motion. This is not always possible; that is the case of non-holonomic constraints. We shall proceed in steps. First considering unconstrained motion, then holonomic constrained motion and finally non-holonomic constraints. We shall work out examples of each type of motion as we go along.

3.1 Unconstrained Motion and Generalized Coordinates

Consider a system of N particles of mass m_α, with $\alpha = 1, 2, \ldots, N$ and position vectors $\vec{r}_\alpha$ with respect to some inertial frame of reference. Express the position vectors in terms of Cartesian (rectangular) coordinates

$$\begin{aligned} \vec{r}_\alpha &= x_\alpha \hat{i} + y_\alpha \hat{j} + z_\alpha \hat{k} \\ &= x_{\alpha\ 1}\hat{e}_1 + x_{\alpha\ 2}\hat{e}_2 + x_{\alpha\ 3}\hat{e}_3 \\ &= \sum_{i=1}^{3} x_{\alpha\ i}\hat{e}_i. \end{aligned} \tag{3.1}$$

If the motion is unconstrained, we will need all $3N$ coordinates $(x_\alpha, y_\alpha, z_\alpha)$, $\alpha = 1, 2, \ldots, N$ to specify the state of the system at any time t. That is the configuration of the system at any time t can be given by no less than $3N$ coordinates.

Newton's 2^{nd} law can be written as

$$m_\alpha \ddot{\vec{r}}_\alpha = \vec{F}_\alpha, \tag{3.2}$$

for each $\alpha = 1, 2, \ldots, N$ where the $\vec{F}_\alpha$ are the total force on the α^{th} particle. We can simply write the acceleration as

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\partial}{\partial \dot{x}_{\alpha\ i}}\left[\dot{x}_{\beta\ j}\right]^2\right) = \ddot{x}_{\beta\ j}\ \delta_{\alpha\beta}\delta_{ij} \qquad \text{(no sum on } \beta\text{)}, \tag{3.3}$$

where we have used the fact that the velocities in each direction for each particle are independent

$$\frac{\partial}{\partial \dot{x}_{\alpha\ i}}\dot{x}_{\beta\ j} = \delta_{\alpha\beta}\delta_{ij} \ . \tag{3.4}$$

For example $\frac{\partial}{\partial v_{7\ x}} v_{8\ y} = 0$ but $\frac{\partial}{\partial v_{7\ x}} v_{7\ x} = 1$. This identity holds for each $\beta = 1, 2, \ldots, N$ and each component $j = 1, 2, 3$. So multiply the equation by m_β and sum over all β and components j

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_{\alpha\ i}} \sum_{\beta=1}^{N} \sum_{j=1}^{3} \frac{1}{2} m_\beta \left[\dot{x}_{\beta\ j}\right]^2 \right) &= \sum_{\beta=1}^{N} \sum_{j=1}^{3} m_\beta \ddot{x}_{\beta\ j}\ \delta_{\alpha\beta}\delta_{ij} \\ &= m_\alpha \ddot{x}_{\alpha\ i}. \end{aligned} \tag{3.5}$$

Hence we find Newton's 2^{nd} law becomes

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_{\alpha\ i}} T \right) = m_\alpha \ddot{x}_{\alpha\ i} = F_{\alpha\ i}, \tag{3.6}$$

where the total kinetic energy of the system of N particles is

$$\begin{aligned} T &= \sum_{\beta=1}^{N} \frac{1}{2} m_\beta v_\beta^2 \\ &= \sum_{\beta=1}^{N} \sum_{j=1}^{3} \frac{1}{2} m_\beta \left(\dot{x}_{\beta\ j}\right)^2. \end{aligned} \tag{3.7}$$

Equations (3.6) are called Lagrange's equations. Since $T = T(\{\dot{x}_{\alpha\ i}\})$, we have that

$$\frac{\partial T}{\partial x_{\alpha\ i}} = 0. \tag{3.8}$$

So we can write Lagrange's equations, (3.6), as

$$\left(\frac{\partial T}{\partial x_{\alpha\ i}} - \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_{\alpha\ i}} \right) = -F_{\alpha\ i}. \tag{3.9}$$

If $\vec{F}_\alpha$ consists partly of forces derivable from potentials

$$F_{\alpha\ i} = -\frac{\partial}{\partial x_{\alpha\ i}} U(\{x_{\beta\ j}\}, t) + \hat{F}_{\alpha\ i}, \tag{3.10}$$

then we find that

$$\frac{\partial (T-U)}{\partial x_{\alpha\ i}} - \frac{d}{dt} \frac{\partial (T-U)}{\partial \dot{x}_{\alpha\ i}} = -\hat{F}_{\alpha\ i}, \tag{3.11}$$

where we have used the fact that the potential energy is independent of the particles' velocities

$$\frac{\partial U}{\partial \dot{x}_{\alpha\ i}} = 0. \tag{3.12}$$

The combination $T - U \equiv L$ is called the **Lagrangian** and it is a function of the coordinates and velocities of the particles and perhaps the time

$$L = L(x_{\alpha\ i}, \dot{x}_{\alpha\ i}; t), \tag{3.13}$$

where we know leave off the { } symbols so that it is understood that we mean a function of all the $x_{\alpha\ i}$ coordinates and $\dot{x}_{\alpha\ i}$ velocities, with $\alpha = 1, 2, \ldots, N$ and $i = 1, 2, 3$. Lagrange's equations (3.11) become

$$\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right) = -\hat{F}_{\alpha\ i}. \tag{3.14}$$

The definition of the Lagrangian can be generalized to include, besides forces derivable from a potential energy function U, forces not derivable from a potential but that can be written as the Euler-Lagrange derivative of a scalar function. The Euler-Lagrange derivative is just the combination of derivatives appearing in Lagrange's equations (3.14) (also known as the Euler-Lagrange equations)

$$\left(\frac{\partial}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_{\alpha\ i}}\right). \tag{3.15}$$

So if $F_{\alpha\ i}$ can be written as

$$F_{\alpha\ i} = -\frac{\partial}{\partial x_{\alpha\ i}} U(\{x_{\beta\ j}\}, t) - \left(\frac{\partial}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_{\alpha\ i}}\right) M(\{x_{\beta\ j}\}, \{\dot{x}_{\beta\ j}\}, t) + \hat{F}_{\alpha\ i}, \tag{3.16}$$

then the Lagrangian is given by

$$L = T - U - M \tag{3.17}$$

and Newton's 2^{nd} can be written again as the Euler-Lagrange equations

$$\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right) = -\hat{F}_{\alpha\ i}. \tag{3.18}$$

An important example of such a force is that due to electromagnetic fields. Forces due to electric, $\vec{E}$, and magnetic, $\vec{B}$, fields on a charge q particle are given by the Lorentz force law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Recall that the electric and magnetic fields obey Maxwell's equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0}\rho && \text{Gauss's Law} \\
\vec{\nabla} \cdot \vec{B} &= 0 && \text{No Magnetic Monopoles} \\
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 && \text{Faraday'sLaw} \\
\vec{\nabla} \times \vec{B} - (\epsilon_0\mu_0)\frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J} && \text{Ampere's Law} \quad ,
\end{aligned} \tag{3.19}$$

where ρ and $\vec{J}$ are the charge density and charge current density, respectively, and the speed of light $c = 1/\sqrt{(\epsilon_0\mu_0)}$. We can introduce the electric scalar potential, ϕ, and the magnetic vector potential, $\vec{A}$, in order to solve 2 of the 4 Maxwell's equations. First since there are no magnetic monopoles the divergence of $\vec{B} = 0$ can be solved by

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{3.20}$$

Plugging this into Faraday's Law and interchanging space and time derivatives leads to

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{3.21}$$

Since the curl of the left hand side is zero it can be written in terms of the gradient of a scalar, hence we have

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{3.22}$$

The remaining two Maxwell equations become the wave equations for ϕ and $\vec{A}$, which can then be solved as in Physics 430/431.

As far as Newton's 2^{nd} Law goes, the Lorentz force on each charged particle can now be written in terms of the electric scalar potential and magnetic vector potential

$$\begin{aligned} \vec{F}_\alpha &= q_\alpha\left(\vec{E}+\vec{v}_\alpha\times\vec{B}\right) \\ &= q_\alpha\left(-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}+\vec{v}_\alpha\times(\vec{\nabla}\times\vec{A})\right), \end{aligned} \tag{3.23}$$

with q_α the electric charge of the α^{th} particle and where the electric and magnetic fields, and their potentials, are evaluated at the position of the charge q_α. Writing this out in Cartesian components yields

$$\begin{aligned} F_{\alpha\ i} &= -q_\alpha\left(\frac{\partial\phi}{\partial x_{\alpha\ i}}+\frac{\partial A_i}{\partial t}-\epsilon_{ijk}\epsilon_{klm}\dot{x}_{\alpha\ j}\frac{\partial A_m}{\partial x_{\alpha\ l}}\right) \\ &= -q_\alpha\left(\frac{\partial\phi}{\partial x_{\alpha\ i}}+\frac{\partial A_i}{\partial t}-\dot{x}_{\alpha\ j}\frac{\partial A_j}{\partial x_{\alpha\ i}}+\dot{x}_{\alpha\ j}\frac{\partial A_i}{\partial x_{\alpha\ j}}\right) \\ &= -q_\alpha\left(\frac{\partial}{\partial x_{\alpha\ i}}-\frac{d}{dt}\frac{\partial}{\partial\dot{x}_{\alpha\ i}}\right)(\phi-\dot{x}_{\alpha\ i}A_i), \end{aligned} \tag{3.24}$$

where we have used the fact that the potentials are functions of position and time only (they are not functions of the velocities of the particles), so for example

$$\frac{d}{dt}A_i(\vec{x}_\alpha,t)=\frac{\partial A_i}{\partial x_{\alpha\ j}}\frac{dx_{\alpha\ j}}{dt}+\frac{\partial A_i}{\partial t}. \tag{3.25}$$

Thus the Lorentz force on each particle can be written as

$$F_{\alpha\ i}=\left(\frac{\partial}{\partial x_{\alpha\ i}}-\frac{d}{dt}\frac{\partial}{\partial\dot{x}_{\alpha\ i}}\right)M_\alpha, \tag{3.26}$$

where the scalar function M_α is given by

$$M_\alpha=-q_\alpha\phi(\vec{x}_\alpha,t)+q_\alpha\dot{\vec{x}}_\alpha\cdot\vec{A}(\vec{x}_\alpha,t). \tag{3.27}$$

Hence for our system of N particles interacting with electromagnetic fields the Lagrangian becomes

$$L = T - U - \sum_{\alpha=1}^{N} M_\alpha \tag{3.28}$$

and the corresponding equations of motion are given by the Euler-Lagrange equations (3.18)

$$\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \right) = -\hat{F}_{\alpha\ i}. \tag{3.29}$$

If all the forces are derivable from a potential (or can be written as an Euler-Lagrange derivative) then $\hat{F}_{\alpha\ i} = 0$ and the Euler-Lagrange equations of motion for the particles becomes simply

$$\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \right) = 0. \tag{3.30}$$

Hence Newton's 2^{nd} Law is equivalent to Hamilton's Principle: Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval the actual path followed is that which minimizes the time integral of the difference between the kinetic energy and the potential energy. That is

$$\delta \int_{t_1}^{t_2} dt L(x, \dot{x}; t) = 0, \tag{3.31}$$

where $L = T - U$ is the Lagrangian and $\Gamma(t_1, t_2) \equiv \int_{t_1}^{t_2} L dt$ is the *action*. This new principle replaces the N vector equations of Newton as the starting dynamical principle. We need only calculate the kinetic energy and potential energy of the system and form the action. Of course $\delta\Gamma = 0$ implies, by the calculus of variations, the $3N$ Euler-Lagrange equations (3.30) which are equivalent to Newton's 2^{nd} law. Now T and U are the same whether we use rectangular coordinates or any other coordinate system, for example spherical polar coordinates, where some of the coordinates are not spatial distances but could be angles or some other measure of location, hence this dynamical principle is the same independent of the coordinates we use.

In general we can use any $3N$ independent quantities to describe the state of our system. We will call these *generalized coordinates* and denote them by q^A, where $A = 1, 2, \ldots, 3N$. Also the corresponding $3N$ generalized velocities are given by $\dot{q}^A$. We can express the configuration of the system in terms of the q^A or our original Cartesian coordinates $x_{\alpha\ i}$. That is there is a transformation between them

$$x_{\alpha\ i} = x_{\alpha\ i}(q^1, q^2, \ldots, q^{3N}; t), \tag{3.32}$$

where it is possible to have moving coordinates so we need a relation for each time t as indicated by the transformation's dependence on t. Likewise we have that

$$q^A = q^A(x_{11}, x_{12}, \ldots, x_{N3}; t). \tag{3.33}$$

Hence we want the transformation to be invertible, except perhaps at isolated points, so we require the Jacobian of the transformation to be non-zero except at isolated points

$$\left| \frac{\partial x_{\alpha\ i}}{\partial q^A} \right| \neq 0. \tag{3.34}$$

For example consider a single particle moving in the $x - y$ plane. Besides Cartesian coordinates x and y we can use polar coordinates r and θ to describe the location of the particle in the plane. The transformation between the coordinates is given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{3.35}$$

and the inverse

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1} y/x. \end{aligned} \tag{3.36}$$

The transformation is invertible except at the one point, the origin

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0(\text{ except at the origin}). \tag{3.37}$$

So we can convert from one set of coordinates to another. The Lagrangian then can be written in terms of the KE and PE in Cartesian coordinates or the KE and PE can be written in terms of the generalized coordinates. The Lagrangian is still $L = T - U$. For shorthand we will simple write $L = L(x, \dot{x}; t) = L(q, \dot{q}; t)$ to express the Lagrangian written in different coordinates. So let's consider the Euler-Lagrange derivative of the Lagrangian with respect to the generalized coordinates

$$\begin{aligned} \frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left\{\frac{\partial L}{\partial x_{\alpha\, i}}\frac{\partial x_{\alpha\, i}}{\partial q^A} + \frac{\partial L}{\partial \dot{x}_{\alpha\, i}}\frac{\partial \dot{x}_{\alpha\, i}}{\partial q^A}\right. \\ &\quad \left. -\frac{d}{dt}\left[\frac{\partial L}{\partial x_{\alpha\, i}}\frac{\partial x_{\alpha\, i}}{\partial \dot{q}^A} + \frac{\partial L}{\partial \dot{x}_{\alpha\, i}}\frac{\partial \dot{x}_{\alpha\, i}}{\partial \dot{q}^A}\right]\right\}. \end{aligned} \tag{3.38}$$

However the transformation only depends on the coordinates and the time, so

$$\frac{\partial x_{\alpha\, i}}{\partial \dot{q}^A} = 0 \tag{3.39}$$

and

$$\frac{dx_{\alpha\, i}}{dt} = \frac{\partial x_{\alpha\, i}}{\partial q^A}\frac{dq^A}{dt} + \frac{\partial x_{\alpha\, i}}{\partial t}, \tag{3.40}$$

that is

$$\dot{x}_{\alpha\, i} = \frac{\partial x_{\alpha\, i}}{\partial q^A}\dot{q}^A + \frac{\partial x_{\alpha\, i}}{\partial t}. \tag{3.41}$$

So this implies

$$\frac{\partial \dot{x}_{\alpha\ i}}{\partial \dot{q}^A} = \frac{\partial x_{\alpha\ i}}{\partial q^A} = 0, \tag{3.42}$$

since $\partial x_{\alpha\ i}/\partial q^A$ and $\partial x_{\alpha\ i}/\partial t$ are independent of $\dot{q}^B$. Hence we can gather terms

$$\begin{aligned} \frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left\{\frac{\partial L}{\partial x_{\alpha\ i}}\frac{\partial x_{\alpha\ i}}{\partial q^A} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{\partial \dot{x}_{\alpha\ i}}{\partial q^A} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{\partial x_{\alpha\ i}}{\partial q^A}\right]\right\} \\ &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left\{\frac{\partial L}{\partial x_{\alpha\ i}}\frac{\partial x_{\alpha\ i}}{\partial q^A} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{\partial \dot{x}_{\alpha\ i}}{\partial q^A} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right)\frac{\partial x_{\alpha\ i}}{\partial q^A} \right. \\ &\qquad \left. - \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\left(\frac{d}{dt}\frac{\partial x_{\alpha\ i}}{\partial q^A}\right)\right\}. \end{aligned} \tag{3.43}$$

The second term and the last term on the right hand side of the last equation above cancel with eachother because

$$\left(\frac{d}{dt}\frac{\partial x_{\alpha\ i}}{\partial q^A}\right) = \frac{\partial \dot{x}_{\alpha\ i}}{\partial q^A}. \tag{3.44}$$

This follows from the fact that q^A and $\dot{q}^A$ are independent just as are $x_{\alpha\ i}$ and $\dot{x}_{\alpha\ i}$. Comparing the expressions for each quantitiy above we find equality

$$\begin{aligned} \frac{d}{dt}\frac{\partial x_{\alpha\ i}}{\partial q^A} &= \frac{\partial^2 x_{\alpha\ i}}{\partial q^A \partial q^B}\frac{dq^B}{dt} + \frac{\partial^2 x_{\alpha\ i}}{\partial t \partial q^A} \\ \frac{\partial \dot{x}_{\alpha\ i}}{\partial q^A} &= \frac{\partial}{\partial q^A}\left(\frac{\partial x_{\alpha\ i}}{\partial q^B}\dot{q}^B + \frac{\partial x_{\alpha\ i}}{\partial t}\right) \\ &= \frac{\partial^2 x_{\alpha\ i}}{\partial q^A \partial q^B}\dot{q}^B + \frac{\partial^2 x_{\alpha\ i}}{\partial q^A \partial t} \\ &= \frac{d}{dt}\frac{\partial x_{\alpha\ i}}{\partial q^A}. \end{aligned} \tag{3.45}$$

So we finally obtain

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right). \tag{3.46}$$

Now recall Lagrange's equation (3.14), substitute this into the right hand side above

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}\hat{F}_{\alpha\ i}. \tag{3.47}$$

Accordingly we can define the *generalized forces* as

$$\begin{aligned} Q_A &\equiv \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}F_{\alpha\ i} \\ \hat{Q}_A &\equiv \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}\hat{F}_{\alpha\ i}. \end{aligned} \tag{3.48}$$

If $F_{\alpha\ i}$ is derivable from a potential, $F_{\alpha\ i} = -\partial U(x,t)/\partial x_{\alpha\ i}$, then

$$Q_A = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}\frac{\partial U(x,t)}{\partial x_{\alpha\ i}} = -\frac{\partial U(q,t)}{\partial q^A}. \tag{3.49}$$

We can interpret the generalized forces by the work, δW, that they perform for small displacements $\delta x_{\alpha\ i}$

$$\begin{aligned} \delta W &= \sum_{\alpha=1}^{N}\vec{F}_\alpha\cdot\delta\vec{r}_\alpha = \sum_{\alpha=1}^{N}\sum_{i=1}^{3}F_{\alpha\ i}\ \delta x_{\alpha\ i} \\ &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}F_{\alpha\ i}\sum_{A=1}^{3N}\frac{\partial x_{\alpha\ i}}{\partial q^A}\delta q^A = \sum_{A=1}^{3N}\underbrace{\left(\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}F_{\alpha\ i}\right)}_{=Q_A}\delta q^A \\ &= \sum_{A=1}^{3N}Q_A\delta q^A. \end{aligned} \tag{3.50}$$

The work δW here is called *virtual work* and the displacements $\delta x_{\alpha\ i}$, and correspondingly δq^A, *virtual displacements* that is independent displacements in the coordinates that are not necessarily an allowed motion of the system because $\delta x_{\alpha\ i}$ occurs instantaneously. An allowed motion of the system $dx_{\alpha\ i}$ occurs during time dt. Hence δW is called vitual work. Again δq^A is a virtual displacement in the coordinate q^A which is a change in

q^A that occurs instanteously, $dt = 0$. Note that if $Q_A = -\partial U/\partial q^A$, then $\delta W = \sum_{A=1}^{3N} Q_A \delta q^A = -\delta U$, as expected. If we need to find the form of the generalized force, Q_A, then usually it most easily found by find the virtual work for virtual displacement δq^A.

To summarize, we find that no matter what coordinates we use to describe the state of our system, we have the Euler-Lagrange equations in terms of those coordinates

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = -\hat{Q}_A, \tag{3.51}$$

and for irrotational forces where $\hat{Q}_A = 0$

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = 0. \tag{3.52}$$

This follows directly from Hamilton's Principle in any coordinate system

$$\delta \int_{t_1}^{t_2} dt L(q, \dot{q}; t) = 0, \tag{3.53}$$

where $L = T - U$ and the kinetic energy now may depend on the coordinates as well as the velocities, $T = T(q, \dot{q})$, and $U = U(q, t)$. Note that the δq variation corresponds to any virtual displacement in the coordinate path when the calculus of variations is applied.

Once again as an example consider a particle moving in 2 dimensions under the influence of a conservative force with potential $U(x, y)$. Transforming to polar coordinates (r, θ) the kinetic energy has a radial velocity piece, $\frac{1}{2}m\dot{r}^2$, and an angular velocity piece, $\frac{1}{2}mr^2\dot{\theta}^2$,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2. \tag{3.54}$$

The potential energy is expressed in terms of the polar coordinates so it becomes a function of r and θ, $U = U(r, \theta)$. The Lagrangian is then given by

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r, \theta). \tag{3.55}$$

The two Euler-Lagrange equations are found to be

$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 = -m\ddot{r} + mr\dot{\theta}^2 - \frac{\partial U}{\partial r} \\
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 = -\frac{d}{dt}\left(mr^2\dot{\theta}\right) - \frac{\partial U}{\partial \theta}.
\end{aligned} \tag{3.56}$$

Thus we obtain Newton's 2^{nd} Law in polar coordinates

$$\begin{aligned}
m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial U}{\partial r} \\
m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= -\frac{1}{r}\frac{\partial U}{\partial \theta}.
\end{aligned} \tag{3.57}$$

This is just $m\ddot{\vec{r}} = -\vec{\nabla}U$ in polar coordinates!

Before investigating Lagrangian dynamics when the motion is constrained, we will study a particle's motion for small oscillations about a stable equilibrium position. This will be described by the simple harmonic oscillator. We will include damping and driving forces as well. Since we will study this motion in one and two dimensions it can be said that we are already considering constrained motion in spaces lower than three dimensions. Essentially we are applying one of the techniques of handling holonomic constraints: we eliminate the constrained coordinates from the problem explicitly. For constrained motion along the x-axis we have set $y = 0 = z$ and only consider the x motion. After a brief introduction to the harmonic oscillator we will return to the holonomic constraint case to formalize these techniques. Then we will return again to the harmonic oscillator for a more detailed study of its motion.

Consider the motion of a single particle of mass m in one dimension along the x-axis. Suppose the potential energy of the particle is given by that of the harmonic oscillator:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2. \tag{3.58}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2, \tag{3.59}$$

and hence the Lagrangian becomes

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2. \tag{3.60}$$

The corresponding Euler-Lagrange equation governing the motion of the particle is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = -m\omega_0^2 x - \frac{d}{dt}(m\dot{x}), \tag{3.61}$$

cancelling the common mass factor yields the simple harmonic oscillator equation of motion

$$\ddot{x} + \omega_0^2 x = 0. \tag{3.62}$$

In addition the particle can experience a retarding frictional force $\hat{F} = -b\dot{x} = -2m\beta\dot{x}$ proportional to its velocity. Since $\hat{F}$ cannot be derived from a potential the Euler-Lagrange equation becomes that of equation (3.14)

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\hat{F} = 2m\beta\dot{x}, \tag{3.63}$$

which becomes upon cancelling the common mass factor

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{3.64}$$

Finally, a time dependent driving force can be applied to the particle in order to overcome the damping resistance to keep it oscillating. The driving force is given by $\vec{F}(t) = F(t)\hat{\imath}$ and is derivable from the time dependent potential function

$$U(x,t) = -xF(t). \tag{3.65}$$

The total potential is then given by

$$U = \frac{1}{2}m\omega_0^2 x^2 - xF(t) \tag{3.66}$$

and the Lagrangian takes the form

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 + xF(t). \tag{3.67}$$

The Euler-Lagrange equation again yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\hat{F} = 2m\beta\dot{x}. \tag{3.68}$$

This leads to the driven oscillator equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}. \tag{3.69}$$

The solution to these equations of motion in each case will now be taken up in turn. But first let's return to the question of treating holonomic constraints and applying our dynamic principle to the case of constrained motion.

3.2 Holonomic Constraints and Lagrange Multipliers

Suppose there are forces of constraint acting on our system so that in order to specify the configuration of our system at any time t it is only necessary to use n independent variables; let's denote these generalized coordinates again by q^a but now $a = 1, 2, \ldots, n < 3N$. That is there exists relations among the coordinates $x_{\alpha\ i}$ so that some can be eliminated. Said otherwise, any $x_{\alpha\ i}$ at time t can be given by the generalized coordinates q^a, there exists a relation

$$x_{\alpha\ i} = x_{\alpha\ i}(q^1, q^2, \ldots, q^n; t) = x_{\alpha\ i}(q^a; t). \tag{3.70}$$

Or in terms of the generalized coordinates the relation becomes

$$q^A = q^A(q^a; t). \tag{3.71}$$

That is we have *holonomic constraints*; every change in the generalized coordinates δq^a is a possible virtual displacement; the δq^a are independent but the corresponding changes in the $3N$ coordinates δq^A are not independent. There are $(3N - n)$ relations amongst them. These can be expressed as

$$n^r_A \delta q^A = 0, \tag{3.72}$$

where $r = 1, 2, \ldots, (3N - n)$. Or integrating (holonomic) these we have

$$g^r(q^A; t) = 0. \tag{3.73}$$

Thus for δq^A this yields

$$\frac{\partial g^r}{\partial q^A} \delta q^A = 0 \tag{3.74}$$

so that

$$n^r_A = \frac{\partial g^r}{\partial q^A}. \tag{3.75}$$

The number of independent virtual displacements, in this case the (number of δq^a) $= n$, is called the *number of degrees of freedom of the system* and is equal to n.

We have two ways to handle these constraints. To start, let's imagine we know not only the external or applied forces, denoted $F_{\alpha\ i}$ but also the forces of constraint, denoted $F^c_{\alpha\ i}$, needed to implement the constraints $g^r(q^A; t) = 0$. Since we know all the forces acting on the system we have $3N$ Lagrange's equations

$$\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} = -\left(\hat{F}_{\alpha\ i} + F^c_{\alpha\ i}\right), \tag{3.76}$$

or in terms of generalized coordinates

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_A} = -\left(\hat{Q}_A + Q^c_A\right), \tag{3.77}$$

with $A = 1, 2, \ldots, 3N$, where $\hat{F}$ and $\hat{Q}$ are that part of the external forces not derivable from a potential. We can proceed as before in the unconstrained case. Let

$$x_{\alpha\ i} = x_{\alpha\ i}(q^a; t) \tag{3.78}$$

so that the Euler-Lagrange derivative of L with respect to the q^a yields

$$\begin{aligned}\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left\{\frac{\partial L}{\partial x_{\alpha\ i}}\frac{\partial x_{\alpha\ i}}{\partial q^a} + \frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{\partial \dot{x}_{\alpha\ i}}{\partial q^a}\right. \\ &\qquad \left. - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\frac{\partial \dot{x}_{\alpha\ i}}{\partial \dot{q}^a}\right]\right\}. \end{aligned} \tag{3.79}$$

As previously

$$\frac{dx_{\alpha\ i}}{dt} = \frac{\partial x_{\alpha\ i}}{\partial q^a}\frac{dq^a}{dt} + \frac{\partial x_{\alpha\ i}}{\partial t}, \tag{3.80}$$

that is

$$\dot{x}_{\alpha\ i} = \frac{\partial x_{\alpha\ i}}{\partial q^a}\dot{q}^a + \frac{\partial x_{\alpha\ i}}{\partial t} \tag{3.81}$$

so that

$$\frac{\partial \dot{x}_{\alpha\ i}}{\partial \dot{q}^a} = \frac{\partial x_{\alpha\ i}}{\partial q^a}. \tag{3.82}$$

Hence the Euler-Lagrange derivative of the Lagrangian becomes

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^a}\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right). \tag{3.83}$$

Since the δq^a are independent, we can multiply by them and sum over $a = 1, 2, \ldots, n$. There must still be term by term equality in the sum

$$\begin{aligned}\sum_{a=1}^{n}\delta q^a\left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a}\right) &= \sum_{a=1}^{n}\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^a}\delta q^a\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right) \\ &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\sum_{a=1}^{n}\frac{\partial x_{\alpha\ i}}{\partial q^a}\delta q^a\right)\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right)\end{aligned} \tag{3.84}$$

Now recall for virtual displacements that there is no explicit change in t, so the virtual displacements $\delta x_{\alpha\ i}$ are found from just the δq^a virtual displacements

$$\sum_{a=1}^{n}\frac{\partial x_{\alpha\ i}}{\partial q^a}\delta q^a = \delta x_{\alpha\ i}. \tag{3.85}$$

So equation (3.84) becomes

$$\sum_{a=1}^{n} \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} \right) = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \delta x_{\alpha\ i} \left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} \right)$$
$$= \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \delta x_{\alpha\ i} \left(-\hat{F}_{\alpha\ i} - F^c_{\alpha\ i} \right). \qquad (3.86)$$

We now have two choices in how to proceed. The first method is to explicitly eliminate the constrained coordinates. The second method is to introduce "extra" coordinates as Lagrange multipliers. Exploiting the first method first, we can use the "principle of virtual work". The virtual work done by all the forces when the system undergoes a virtual displacement $\delta x_{\alpha\ i}$ is

$$\delta W = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \delta x_{\alpha\ i} \left(F_{\alpha\ i} + F^c_{\alpha\ i} \right). \qquad (3.87)$$

However the forces of constraint are such that for any change in the generalized coordinates δq^a, only motions in $x_{\alpha\ i}$ consistent with the constraint equations are allowed. That is the $\delta x_{\alpha\ i}$ are constrained. In particular, by definition the allowed displacements in $x_{\alpha\ i}$ are orthogonal to the forces of constraint

$$\delta \vec{r}_\alpha \cdot \vec{F}^c_\alpha = 0, \qquad (3.88)$$

for each α. So the sum over particles of this equals zero also

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \delta x_{\alpha\ i} F^c_{\alpha\ i} = 0. \qquad (3.89)$$

The work done by the forces of constraint is zero! Consider an example in which two particles are held a fixed distance L apart

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = L^2 \qquad (3.90)$$

or written differentially

$$(x_1 - x_2)\delta(x_1 - x_2) + (y_1 - y_2)\delta(y_1 - y_2) + (z_1 - z_2)\delta(z_1 - z_2) = 0. \quad (3.91)$$

The force of the second particle on the first particle is $\vec{F}$, the force of the first particle on the second particle is $-\vec{F}$. The force is along the line joining the particles $\vec{F} = (\vec{r}_1 - \vec{r}_2)f$. So the total work done for displacements $\delta\vec{r}_1$ and $\delta\vec{r}_2$ is

$$\delta W = \vec{F} \cdot \delta\vec{r}_1 - \vec{F} \cdot \delta\vec{r}_2 = \vec{F} \cdot \delta(\vec{r}_1 - \vec{r}_2). \quad (3.92)$$

But the centrality of the force, $\vec{F} = (\vec{r}_1 - \vec{r}_2)f$, implies

$$\delta W = f(\vec{r}_1 - \vec{r}_2) \cdot \delta(\vec{r}_1 - \vec{r}_2) = 0, \quad (3.93)$$

by the constraint!

So we find the right hand side of equation (3.84) only involves the external or applied forces

$$\sum_{a=1}^{n} \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} \right) = - \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \delta x_{\alpha\ i} \hat{F}_{\alpha\ i}. \quad (3.94)$$

Now

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{3} \delta x_{\alpha\ i} \hat{F}_{\alpha\ i} = \sum_{a=1}^{n} \delta q^a \frac{\partial x_{\alpha\ i}}{\partial q^a} \hat{F}_{\alpha\ i} \equiv \sum_{a=1}^{n} \delta q^a \hat{Q}_a, \quad (3.95)$$

where $\hat{Q}_a$ are the generalized external forces. The virtual work now becomes

$$\delta W = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \delta x_{\alpha\ i} (F_{\alpha\ i} + F^c_{\alpha\ i}) = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \delta x_{\alpha\ i} F_{\alpha\ i} = \sum_{a=1}^{n} \delta q^a \hat{Q}_a. \quad (3.96)$$

and hence the Euler-Lagrange derivative takes the form

$$\sum_{a=1}^{n} \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} \right) = - \sum_{a=1}^{n} \delta q^a \hat{Q}_a. \quad (3.97)$$

Since the δq^a displacements are independent each term in the sum must be zero

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = -hatQ_a, \tag{3.98}$$

where $a = 1, 2, \ldots, n$.

Hence if all the applied forces are derivable from a potential ($\hat{Q} = 0$), we find the Euler-Lagrange equations for the system in terms of the generalized coordinates are derivable from Hamilton's principle

$$\delta \int_{t_1}^{t_2} L(q^a, \dot{q}^a; t) dt = 0, \tag{3.99}$$

in terms of the independent (degrees of freedom) coordinates only. the configuration of the system is then determined at any time from the degrees of freedom q^a and constraints $g^r = 0$, that is $x_{\alpha\ i} = x_{\alpha\ i}(q^a; t)$ for all $\alpha = 1, 2, \ldots, N$ and $i = 1, 2, 3$.

Instead of eliminating the constrained coordinates explicitly and finding equations of motion for the remaining independent generalized coordinates q^a we can introduce "extra" coordinates in the form of Lagrange multipliers as the second method to solve the motion of a system with constraints. Return to our original Euler-Lagrange derivative equations (3.76)

$$\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} = -\left(\hat{F}_{\alpha\ i} + F^c_{\alpha\ i}\right). \tag{3.100}$$

The constraint forces $F^c_{\alpha\ i}$, although undetermined explicitly, result in the coordinates being constrained by

$$g^r(x_{\alpha\ i}; t) = 0 \tag{3.101}$$

so that for virtual displacements $\delta x_{\alpha\ i}$ (they are "instantaneous" displacements) we have

$$\frac{\partial g^r}{\partial x_{\alpha\ i}} \delta x_{\alpha\ i} = 0. \tag{3.102}$$

That is we have $m = (3N - n)$ vectors denoted $\vec{n}^r$ in the $3N$ dimensional confuguration space with components $n^r_{\alpha\ i} = \frac{\partial g^r}{\partial x_{\alpha\ i}}$ so that

$$\vec{n}^r \cdot \delta\vec{x} = 0 = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} n^r_{\alpha\ i}\delta x_{\alpha\ i}. \tag{3.103}$$

But we know that the virtual work done by the constraint forces is zero

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{3} F^c_{\alpha\ i}\delta x_{\alpha\ i} = 0. \tag{3.104}$$

Hence the $3N$ dimensional vector $\vec{F}^c$ is also orthogonal to $\delta\vec{x}$. Since the $\vec{n}^r$ vectors span the m dimensional space orthogonal to $\delta\vec{x}$ we can expand $\vec{F}^c$ in terms of the $\vec{n}^r$ basis elements at any time t

$$\vec{F}^c = \sum_{r=1}^{m} \lambda_r(t)\vec{n}^r. \tag{3.105}$$

The $\lambda_r(t)$ are independent of $\delta x_{\alpha\ i}$. The $\lambda_r(t)$ are *Lagrange undetermined multipliers*. So we find

$$\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}} = -\hat{F}_{\alpha\ i} - \sum_{r=1}^{m} \lambda_r n^r_{\alpha\ i}, \tag{3.106}$$

where again $n^r_{\alpha\ i} = \frac{\partial g^r}{\partial x_{\alpha\ i}}$. In addition to the Euler-Lagrange equations we also have the m constraint equations

$$g^r(x_{\alpha\ i}; t) = 0. \tag{3.107}$$

Thus we have $(3N + m)$ equations for the $3N$ coordinates $x_{\alpha\ i}$ plus m Lagrange multipliers λ_r unknowns.

As previously we can multiply this by $\frac{\partial x_{\alpha\ i}}{\partial q^A}$ and sum over α and i to obtain

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial x_{\alpha\ i}}{\partial q^A}\left(\frac{\partial L}{\partial x_{\alpha\ i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha\ i}}\right) = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial x_{\alpha\ i}}{\partial q^A}\hat{F}_{\alpha\ i}$$

$$-\sum_{r=1}^{m}\lambda_r\sum_{\alpha=1}^{N}\sum_{i=1}^{3}n^r_{\alpha\ i}\frac{\partial x_{\alpha\ i}}{\partial q^A}. \tag{3.108}$$

This just yields

$$\frac{\partial L}{\partial q^A}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}=-\hat{Q}_A-\sum_{r=1}^{M}\lambda_r N^r_A, \tag{3.109}$$

where

$$\begin{aligned} N^r_A &= \sum_{\alpha=1}^{N}\sum_{i=1}^{3}n^r_{\alpha\ i}\frac{\partial x_{\alpha\ i}}{\partial q^A}=\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial g^r}{\partial x_{\alpha\ i}}\frac{\partial x_{\alpha\ i}}{\partial q^A} \\ &= \frac{\partial g^r}{\partial q^A}. \end{aligned} \tag{3.110}$$

So Lagrange's equations for the generalized coordinates and Lagrange multipliers plus the constraint become

$$\frac{\partial L}{\partial q^A}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}+\sum_{r=1}^{m}\lambda_r N^r_A=-\hat{Q}_A, \tag{3.111}$$

with

$$g^r(q^A;t)=0. \tag{3.112}$$

As before the generalized force of constraint

$$Q^c_A=\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{\partial x_{\alpha\ i}}{\partial q^A}F^c_{\alpha\ i}=\sum_{r=1}^{m}\lambda_r N^r_A. \tag{3.113}$$

So the Lagrange multipliers just represent the components of the generalized force when expanded in terms of the $\vec{N}^r$ generalized vectors.

Hence in the holonomic constraint case the Lagrange multiplier method leads to equations of motion for the $3N$ generalized coordinates as well as the m constraint equations themselves by treating the Lagrange multipliers

λ_r as an additional m coordinates. Then we obtain $(3N+m)$ equations of motion as Euler-Lagrange derivatives of the auxiliary Lagrangian

$$L(q^A,\dot{q}^A;t)+\sum_{r=1}^{m}\lambda_r(t)g^r(q^A;t) \tag{3.114}$$

with respect to the $(3N+m)$ coordinates (q^A,λ_r) and recalling that $N^r_A = \partial g^r/\partial q^A$ and the only λ_r dependence is its explicit appearance in the auxiliary Lagrangian.

If $\hat{Q}_A = 0$, all forces (besides the constraint) are derivable from a potential and we find

$$\frac{\partial L}{\partial q^A}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}+\sum_{r=1}^{m}\lambda_r N^r_A = -\hat{Q}_A \tag{3.115}$$

and

$$g^r(q^A;t)=0. \tag{3.116}$$

These again result from the Hamilton action principle. The path the system moves along when subject to constraints $g^r = 0$ is given by the minimum of the action with Lagrange multipliers

$$\Gamma(t_1,y_2)\equiv\int_{t_1}^{t_2}dt\left[L(q^A,\dot{q}^A;t)+\sum_{r=1}^{m}\lambda_r g^r(q^A;t)\right], \tag{3.117}$$

where $\{q^A,\lambda_r\}$ are treated independently. That is $\delta\Gamma = 0$ implies

$$\begin{aligned}\frac{\partial(L+\lambda_r g^r)}{\partial q^A}-\frac{d}{dt}\frac{\partial(L+\lambda_r g^r)}{\partial \dot{q}^A} &= 0\\ \frac{\partial(L+\lambda_r g^r)}{\partial \lambda_r}-\frac{d}{dt}\frac{\partial(L+\lambda_r g^r)}{\partial \dot{\lambda}_r} &= 0.\end{aligned} \tag{3.118}$$

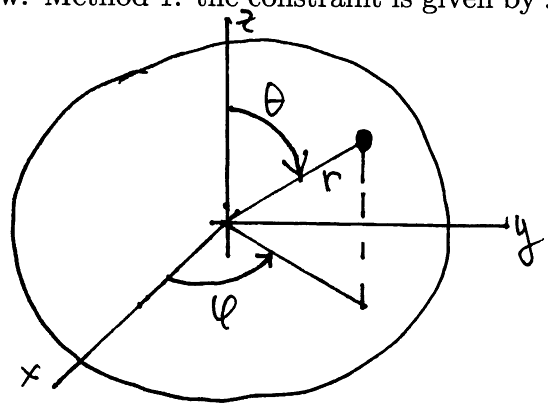
These just become

$$\frac{\partial L}{\partial q^A}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}+\sum_{r=1}^{M}\lambda_r\frac{\partial g^r}{\partial q^A} = -\hat{Q}_A, \qquad A=1,2,\ldots,3N$$

$$g^r = 0 \quad , \qquad r = 1, 2, \ldots, m, \quad (3.119)$$

having used $\partial g^r / \partial \dot{q}^A = 0$. Note: g^r is not set to zero until after Γ is varied!

Example 1: Consider a particle constrained to move on the surface of a sphere subject to conservative external forces. Method 1, we can descrive the motion of the particle using 2 spherical polar coordinates with the constrained radial coordinate fixed. Method 2, we can use the 3 spherical polar coordinates of the particle plus the Lagrange multiplier for the constraint. Finally, we could use method 2 in Cartesian coordinates for a direct relation to Newton's 2^{nd} Law. Method 1: the constraint is given by $x^2 + y^2 + z^2 = a^2$



or $x\delta x + y\delta y + z\delta z = 0$ and can be most easily implemented in spherical polar coordinates in which the constraint is $r = a$ or $\delta r = 0$. So choosing the generalized coordinates as

$$\begin{aligned} q^1 &= \theta \\ q^2 &= \varphi \end{aligned} \quad (3.120)$$

the kinetic energy is

$$T = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right) \quad (3.121)$$

and the potential energy has the form

$$U = U(a, \theta, \varphi). \quad (3.122)$$

The Euler-Lagrange equations for θ and φ are given by

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = 0, \tag{3.123}$$

so that

$$\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 = ma^2 \sin\theta\cos\theta\dot{\varphi}^2 - \frac{\partial U}{\partial \theta} - ma^2\ddot{\theta} \\
\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} &= 0 = -\frac{\partial U}{\partial \varphi} - \frac{d}{dt}\left(ma^2 \sin^2\theta\dot{\varphi}\right).
\end{aligned} \tag{3.124}$$

The equations of motion are

$$\begin{aligned}
ma^2\ddot{\theta} - ma^2\sin\theta\cos\theta\dot{\varphi}^2 &= -\frac{\partial U}{\partial \theta} \\
\frac{d}{dt}\left(ma^2\sin^2\theta\dot{\varphi}\right) &= -\frac{\partial U}{\partial \varphi},
\end{aligned} \tag{3.125}$$

with $r = a$ the constraint.

Method 2: Choose the generalized coordinates as all three spherical polar coordinates

$$\begin{aligned}
q^1 &= r \\
q^2 &= \theta \\
q^3 &= \varphi,
\end{aligned} \tag{3.126}$$

and the constraint equation

$$g = r - a = 0 \tag{3.127}$$

used as a Lagrange multiplier "additional coordinate". The kinetic energy is now

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) \tag{3.128}$$

and the potential energy has the form

$$U = U(r, \theta, \varphi). \tag{3.129}$$

Using the Lagrange multiplier augmented Lagrangian $(L+\lambda g)$, and the corresponding action $\Gamma = \int_{t_1}^{t_2} dt\,(L + \lambda g)$, the Euler-Lagrange equations are given by

$$
\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial g}{\partial r} &= 0 \\
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial g}{\partial \theta} &= 0 \\
\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} + \lambda\frac{\partial g}{\partial \varphi} &= 0 \\
\frac{\partial L}{\partial \lambda} &= 0.
\end{aligned}
\tag{3.130}
$$

Using $g = r - a$ so that $\partial g/\partial r = 1$, these become

$$
\begin{aligned}
mr\dot{\theta}^2 + mr\sin^2\theta\dot{\varphi}^2 - m\ddot{r} + \lambda &= \frac{\partial U}{\partial r} \\
mr^2\sin\theta\cos\theta\dot{\varphi}^2 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) &= \frac{\partial U}{\partial \theta} \\
\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\varphi}\right) &= -\frac{\partial U}{\partial \varphi} \\
g &= 0 = r - a.
\end{aligned}
\tag{3.131}
$$

Working in reverse order, the equations of motion imply

$$
\begin{aligned}
r &= a \\
\frac{d}{dt}\left(ma^2\sin^2\theta\dot{\varphi}\right) &= -\frac{\partial U}{\partial \varphi}|_{r=a} \\
ma^2\ddot{\theta} = ma^2\sin\theta\cos\theta\dot{\varphi}^2 &= -\frac{\partial U}{\partial \theta}|_{r=a} \\
\lambda &= -ma\dot{\theta}^2 - ma\sin^2\theta\dot{\varphi}^2 + -\frac{\partial U}{\partial r}|_{r=a}.
\end{aligned}
\tag{3.132}
$$

Note that λ is the force of constraint in the $\hat{r}$ direction. It just cancels $F_r = -\partial U/\partial r$ plus the centrifugal force!

$$
F_r^c = \lambda N_r = \lambda\frac{\partial g}{\partial r} = \lambda. \tag{3.133}
$$

So we can find forces of constraint this way if desired, i.e. tension in a pulley string, stresses in beams in buildings.

Finally we could use the method of Lagrange multipliers in Cartesian coordinates. The Lagrangian is simply

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z) \tag{3.134}$$

and the constraint equation is

$$g = \frac{1}{2}\left(x^2 + y^2 + z^2 - a^2\right) = 0. \tag{3.135}$$

The Euler-Lagrange equations become

$$\begin{aligned} m\ddot{x} &= -\frac{\partial U}{\partial x} - \lambda x \\ m\ddot{y} &= -\frac{\partial U}{\partial y} - \lambda y \\ m\ddot{z} &= -\frac{\partial U}{\partial z} - \lambda z, \end{aligned} \tag{3.136}$$

or in vector notation

$$m\ddot{\vec{r}} = vec\nabla U - \lambda\vec{r}. \tag{3.137}$$

But the constraint implies $g = 0$, hence

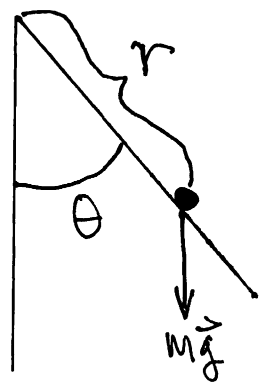
$$x^2 + y^2 + z^2 = a^2. \tag{3.138}$$

Thus

$$m\ddot{\vec{r}} = vec\nabla U - \lambda a\hat{r}. \tag{3.139}$$

This is just Newton's 2^{nd} Law with $\vec{F}^c = -\lambda a\hat{r}$.

Example 2: A light (massless) rod swings freely from one end in a vertical plane. An ant of mass m crawls along the rod towards the pivot end with



uniform speed v (relative to the rod). Find the equation of motion of the coordinate of the ant.

Since $r = r_0 - vt$, with r_0 the position of the ant at $t = 0$, we know the position of the ant on the rod. Hence we have a moving constraint. The angle θ the rod makes with the vertical in the plane is all that is needed. This is the one degree of freedom and θ is the generalized coordinate. The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}m\left(v^2 + (r_0 - vt)^2\dot{\theta}^2\right) \tag{3.140}$$

and the potential energy of the ant is

$$U = -mgr\cos\theta = -mg(r_0 - vt)\cos\theta. \tag{3.141}$$

With the constraint equation built into the coordinates, we have

$$L = \frac{1}{2}m\left(v^2 + (r_0 - vt)^2\dot{\theta}^2\right) + mg(r_0 - vt)\cos\theta. \tag{3.142}$$

The equation of motion for θ follows from Lagrange's equation

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0, \tag{3.143}$$

yielding

$$(r_0 - vt)\ddot{\theta} - 2v\dot{\theta} + g\sin\theta = 0. \tag{3.144}$$

The method of Lagrange multipliers may be used to solve this problem also, however it is messier. Using the Lagrange multiplier augmented Lagrangian

$$L + \lambda(r - r_0 + vt), \tag{3.145}$$

the Euler-Lagrange equations become

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda &= 0 = -m\ddot{r} + mr\dot{\theta}^2 + mg\cos\theta + \lambda \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 = -mgr\sin\theta - \frac{d}{dt}(mr^2\dot{\theta}) \\ \frac{\partial L}{\partial \lambda} &= 0 = r - r_0 + vt. \end{aligned} \tag{3.146}$$

Once again utilizing the equations in opposite order yields

$$\begin{aligned} r &= r_0 - vt \\ (r_0 - vt)\ddot{\theta} - 2v\dot{\theta} + g\sin\theta &= 0 \\ \underbrace{\lambda}_{\text{force of constraint}} &= \underbrace{-mg\cos\theta}_{\text{cancels gravity}} \qquad \underbrace{-m(r_o - vt)\dot{\theta}^2}_{\text{cancels centrifugal force}} \end{aligned} \tag{3.147}$$

3.3 Non-Holonomic Constraints

Lastly, we study systems with non-holonomic constraints, that is constraints that cannot be integrated to yield algebraic relations amongst coordinates. Consider a system which requires n generalized coordinates to describe its configuration at any time t

$$x_{\alpha\ i} = x_{\alpha\ i}(q^1, q^2, \ldots, q^n; t), \tag{3.148}$$

but the system's motions are further constrained so that there exists relations amongst the generalized velocities and coordinates: $f^r(q, \dot{q}; t) = 0$ with $r = 1, 2, \ldots, m < n$. There are even more general constraints than non-holonomic, constraints, for example, that confine a particle's motion to a region of space. These can be written as inequalities $f^r(q,\dot{q};t) < 0$. We will not deal with all possible non-holonomic constraints but only consider the type that are expressible as a relation among coordinate differentials of the form

$$\sum_{a=1}^{n} n_a^r(q;t)dq^a + m^r(q;t)dt = 0, \tag{3.149}$$

where $r = 1, 2, \ldots, m < n$. That is a constraints of the form

$$\sum_{a=1}^{n} n_a^r(q;t)\dot{q}^a + m^r(q;t) = 0. \tag{3.150}$$

Note, if $m^r = \partial f^r/\partial t$ and $n_a^r = \partial f^r/\partial q^a$ then the constraint equation becomes

$$\sum_{a=1}^{n} n_a^r(q;t)\dot{q}^a + m^r(q;t) = \sum_{a=1}^{n} \frac{\partial f^r}{\partial q^a}\frac{dq^a}{dt} + \frac{\partial f^r}{\partial t} = \frac{df^r}{dt} = 0, \tag{3.151}$$

so $f^r =$ constant, and the constraint is again holonomic (integrable).

We also notice the other aspect of non-holonomic constraints, the number of degrees of freedom is less than the number of generalized coordinates. The number of degrees of freedom being $(n - m)$ while we need n generalized coordinates to specify the configuration of the system. The number of degrees of freedom is the number of independent directions in which the system can move (i.e. the number of independent δq^a, there are n q^a necessary to specify the configuration of the system at any time, but there are m constraint relations among them, hence $(n-m)$ independent δq^a, the number of degrees of freedom). For holonomic constraints the number of degrees of freedom are always equal to $(3N - m)$ which was equal to the number of generalized coordinates necessary to describe the configuration of the system.

We can handle the non-holonomic constraints of the form of equation (3.149) as we handled holonomic constraints with Lagrange multipliers since for the virtual displacements δq^a the time does not vary $dt = 0$ and so the constraint equation (3.149) reduces to

$$\sum_{a=1}^{n} n_a^r(q;t)\delta q^a = 0, \tag{3.152}$$

just like the holonomic case. That is the forces of constraint Q_a^c do not contribute to the virtual work

$$\delta W = \sum_{a=1}^{n} Q_a^c \delta q^a = 0 \tag{3.153}$$

where the δq^a obey the instantaneous constraint equations

$$\sum_{a=1}^{n} n_a^r(q;t)\delta q^a = 0. \tag{3.154}$$

Hence we have that

$$Q_a^c = \sum_{r=1}^{m} \lambda_r(t) n_a^r. \tag{3.155}$$

Lagrange's equations are then given by

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = -\hat{Q}_a - \sum_{r=1}^{m} \lambda_r(t) n_a^r, \tag{3.156}$$

where the $\hat{Q}_a$ are forces not derivable from a potential; forces derivable from a potential are in L. In addition we have the constraint equations (3.149)

$$\sum_{a=1}^{n} n_a^r dq^a + m^r dt = 0, \tag{3.157}$$

that is the differential equations of constraint

$$\sum_{a=1}^{n} n_a^r \dot{q}^a + m^r = 0. \tag{3.158}$$

There are $(n+m)$ unknown coordinates $\{q^a, \lambda_r\}$ but we have $(n+m)$ differential equations (3.156) and (3.158) in order to determine them.

If $\hat{Q}_a = 0$ the equations of motion are derivable from a variational principle

$$\delta \int_{t_1}^{t_2} dt L = 0, \tag{3.159}$$

but before we set the integrand to zero we eliminate the constrained variations

$$\delta \int_{t_1}^{t_2} dt L = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} \right) \delta q^a \tag{3.160}$$

however the δq^a are not independent but

$$\sum_{a=1}^{n} n_a^r \delta q^a = 0 \tag{3.161}$$

as does

$$\int_{t_1}^{t_2} dt \sum_{r=1}^{m} \sum_{a=1}^{n} \lambda_r \; n_a^r \delta q^a = 0. \tag{3.162}$$

Hence we add this to $\delta\Gamma$ above

$$0 = \int_{t_1}^{t_2} dt \sum_{a=1}^{n} \left[\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r \; n_a^r \right] \delta q^a. \tag{3.163}$$

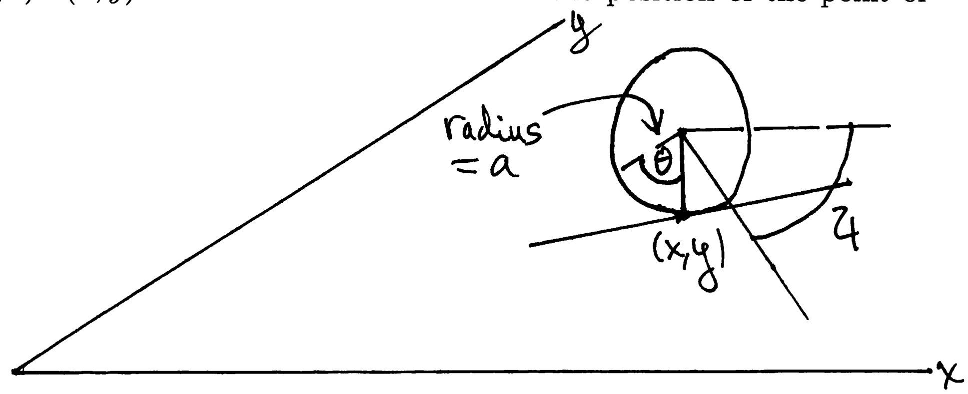
Now $(n - m)$ of the δq^a are independent, so we use m of the arbitrary λ_r to cancel the integral. The remaining $(n - m)$ of the δq^a are independent. So the integral must vanish and we obtain the Euler-Lagrange equations of motion for the system

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r \; n_a^r = 0 \tag{3.164}$$

along with the equations of constraint

$$\sum_{a=1}^{n} n_a^r dq^a + m^r dt = 0. \tag{3.165}$$

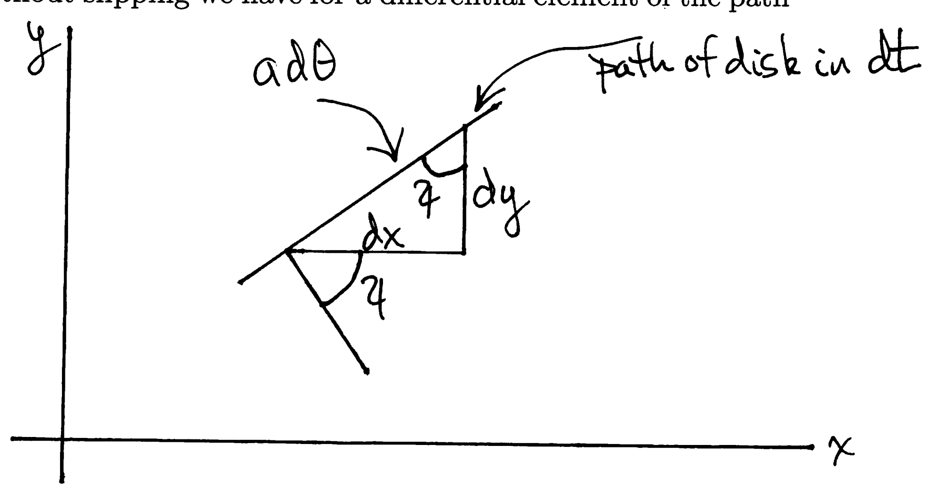
For example any time rolling "without slipping" occurs we have non-holonomic constraints. Consider a flat uniform disk which rolls upright without slipping on a horizontal plane. The coordinates needed to describe the configuration of the disk at any time are given by 4 generalized coordinates (x, y, ψ, θ). (x, y) Cartesian coordinates locate the position of the point of



constact of the disk with the plane. ψ specifies the angle between the axis of the disk and the x-axis. θ is the angle between a fixed radius of the disk and the vertical direction. The Lagrangian is

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\psi}^2, \tag{3.166}$$

where the moments of inertia of the disk are $I_1 = \frac{1}{2} ma^2$ and $I_2 = \frac{1}{4} ma^2$. Now if the disk rolls without slipping we have for a differential element of the path in time dt So



$$\begin{aligned} dx &= a d\theta \sin \psi \\ dy &= a d\theta \cos \psi . \end{aligned} \tag{3.167}$$

So we have the non-holonomic constraints

$$\begin{aligned} dx - a\sin\psi d\theta &= 0 = n^1_a dq^a \\ dy - a\cos\psi d\theta &= 0 = n^2_a dq^a, \end{aligned} \tag{3.168}$$

which cannot be integrated. Hence, reading off from the above constraint equations, the components of n^r_a are

$$\begin{aligned} n^1_x = 1 \quad , \quad n^1_y = 0 \quad , \quad n^1_\theta &= -a\sin\psi \quad , \quad n^1_\psi = 0 \\ n^2_x = 0 \quad , \quad n^2_y = 1 \quad , \quad n^2_\theta &= -a\cos\psi \quad , \quad n^2_\psi = 0. \end{aligned} \tag{3.169}$$

There are 2 constraint equations, hence we need 2 Lagrange multipliers λ_1 and λ_2. The Euler-Lagrange equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r \, n^r_a = 0 \tag{3.170}$$

become

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda_1 \, n^1_x + \lambda_2 \, n^2_x &= 0 \\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda_1 \, n^1_y + \lambda_2 \, n^2_y &= 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda_1 \, n^1_\theta + \lambda_2 \, n^2_\theta &= 0 \\ \frac{\partial L}{\partial \psi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} + \lambda_1 \, n^1_\psi + \lambda_2 \, n^2_\psi &= 0. \end{aligned} \tag{3.171}$$

Taking the derivatives of L these equations become

$$\begin{aligned} -m\ddot{x} + \lambda_1 &= 0 \\ -m\ddot{y} + \lambda_2 &= 0 \\ -I_1\ddot{\theta} - \lambda_1 a\sin\psi - \lambda_2 a\cos\psi &= 0 \end{aligned}$$

$$-I_2\ddot{\psi} = 0. \tag{3.172}$$

Thus we find 6 equations for the six unknowns $x, y, \theta, \psi, \lambda_1, \lambda_2$

$$\begin{aligned} m\ddot{x} &= \lambda_1 \\ m\ddot{y} &= \lambda_2 \\ I_1\ddot{\theta} + \lambda_1 a \sin\psi + \lambda_2 a \cos\psi &= 0 \\ I_2\ddot{\psi} &= 0. \end{aligned} \tag{3.173}$$

Note that we need 4 generalized coordinates to specify the location and orientation of the disk but the velocity constraints relate 2 of the variations of the coordinates

$$\begin{aligned} dx &= ad\theta \sin\psi \\ dy &= ad\theta \cos\psi. \end{aligned} \tag{3.174}$$

Hence there are only 2 degrees of freedom. This is a characteristic of non-holonomic constraints.