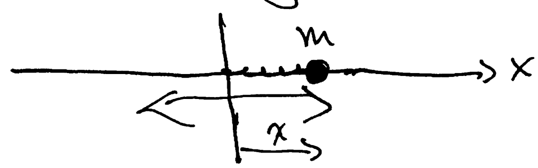
We can use conservation laws to determine the motion of particles

<u>Example</u>: Consider a single particle in one-dimensional motion subject to a Hooke's law force (spring force) about the origin



$$\vec{F} = -kx\,\hat{i} = -m\omega^2 x\,\hat{i}$$

This is the case of a simple <u>harmonic oscillator</u>. The potential energy of the particle is given by

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 x^2$$

$$\left(\text{check: } \vec{F} = -\hat{i}\,\frac{dU}{dx} = -m\omega^2 x\,\hat{i}\ \checkmark\right)$$

Since the force is time independent as well, the total energy of the particle is a constant of motion

$$\underset{\text{constant}}{\underset{\shortparallel}{E}} = T + U = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega^2 x^2$$

We can solve this for $\dot{x}$

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m} - \omega^2 x^2}\ , \text{ hence}$$

we only have to integrate one time to find $x = x(t)$

$$\int \frac{dx}{\sqrt{\frac{2E}{m} - \omega^2 x^2}} = \int_{t_0}^{t} dt = (t - t_0)$$

From integral tables: indefinite integral

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left(- \frac{b + 2cx}{\sqrt{-\Delta}} \right)$$

where $\Delta = 4ac - b^2$; $c < 0$
< 0

Applying this to our case $b = 0$, $c = -\omega^2$, $a = \frac{2E}{m}$

$\Delta = -\frac{8E\omega^2}{m}$

$$\int \frac{dx}{\sqrt{\frac{2E}{m} - \omega^2 x^2}} = \frac{1}{\omega} \cos^{-1}\left(\frac{2\omega^2 x}{\sqrt{\frac{8E\omega^2}{m}}} \right) = (t - t_0)$$

$$\Rightarrow \quad \boxed{x = \sqrt{\frac{2E}{m\omega^2}} \cos\omega(t - t_0)}$$

Simple harmonic motion

Check: $\dot{x} = -\omega\sqrt{\frac{2E}{m\omega^2}} \sin\omega(t - t_0)$

$$T = \tfrac{1}{2} m\dot{x}^2 = E \sin^2\omega(t - t_0)$$
$$U = \tfrac{1}{2} m\omega^2 x^2 = E \cos^2\omega(t - t_0)$$

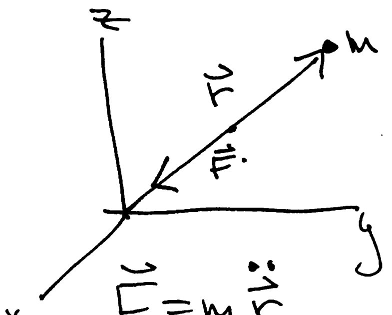
$$T + U = E = \text{constant}.$$

Consider the inverse square force law as another example.

<u>Example</u>: Suppose a particle of mass m moves under the influence of an inverse square force

$$\vec{F} = -k\frac{\hat{r}}{r^2} = -k\frac{\vec{r}}{r^3}$$

(ex. Newtonian gravity $k = GMm$
Coulomb's law $k = \frac{Qq}{4\pi\epsilon_0}$)



$\vec{F} = m\ddot{\vec{r}}$

Newton's 2nd Law involves 3 second order coupled differential equations

First note that $\vec{F}$ is conservative
1) it is time independent
2) $\vec{\nabla}\times\vec{F} = 0$

i.e. $\vec{\nabla}\times\frac{\vec{r}}{r^3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}$ (Use $\partial_i \frac{1}{r^3} = \frac{-3x_i}{r^5}$)

$$= \hat{i}\left[\frac{-3yz}{r^5} + \frac{3yz}{r^5}\right] - \hat{j}\left[\frac{-3xz}{r^5} + \frac{3xz}{r^5}\right] + \hat{k}\left[\frac{-3xy}{r^5} + \frac{3xy}{r^5}\right]$$

$$= 0$$

$$\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F} = -\vec{\nabla}U$$

$$U = -\frac{k}{r}$$

(Check $\vec{F} = -\vec{\nabla}U = +\vec{\nabla}\frac{k}{r} = -k\frac{\hat{r}}{r^2}$ ✓)

So the total energy is conserved $E = T + U = \text{constant}$

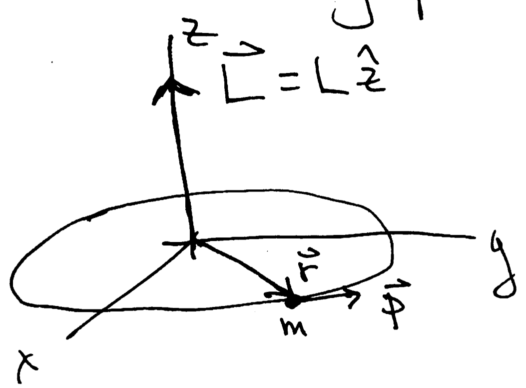
Also the force acts along the radius from the particle to the origin — it produces no torque on the particle $\vec{N} = \vec{r} \times \vec{F}$

$$= -k \frac{\vec{r} \times \vec{r}}{r^3} = 0$$

So the ∡ momentum is conserved

$$\vec{L} = \vec{r} \times \vec{p} = \text{constant vector} \quad \left(\frac{d\vec{L}}{dt} = \vec{N} = 0 \right)$$

Now $\vec{L}$ is a constant vector $\perp$ to both $\vec{r}$ & $\vec{p}$ let $\vec{L} = L\hat{z}$ $L = \text{constant}$ then the particle moves in the x-y plane i.e. $\vec{r}$ & $\vec{p}$ are in the x-y plane $\perp$ ~~to~~ $\vec{L}$



Hence we can use cylindrical coordinates to describe the motion of the particle (r, θ, z) where

$$r = \sqrt{x^2+y^2} \;;\; \tan\theta = y/x$$

So the particle's position is $\vec{r} = r\hat{e}_r$ and its velocity $\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$

$$\Rightarrow \vec{p} = m\dot{r}\,\hat{e}_r + mr\dot{\theta}\,\hat{e}_\theta$$

So $\vec{L} = \vec{r} \times \vec{p} = mr^2\dot{\theta} \underbrace{\hat{e}_r \times \hat{e}_\theta}_{=\hat{z}} = L\hat{z}$

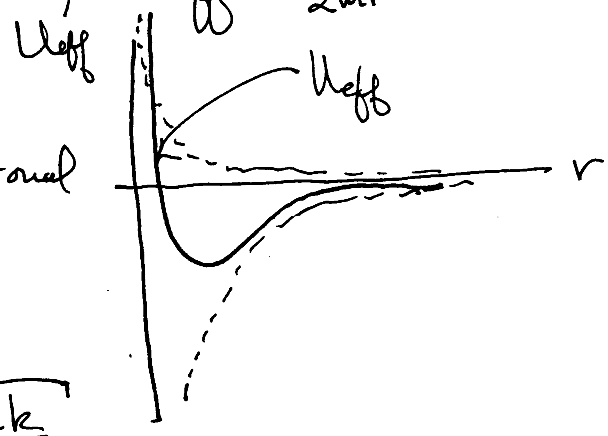
So $\boxed{L = mr^2\dot{\theta}}$ = constant

Now the total energy becomes

$$E = T + U = \tfrac{1}{2}m\dot{r}^2 + \underbrace{\tfrac{1}{2}mr^2\dot{\theta}^2}_{= \frac{L^2}{2mr^2}} - \frac{k}{r} = \text{constant}$$

$$E = \tfrac{1}{2}m\dot{r}^2 + U_{eff} \quad ; \quad U_{eff} = \frac{L^2}{2mr^2} + U(r)$$

So we have an effective 1-dimensional problem solve for $\dot{r}$



$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2k}{mr}}$$

we can integrate this to find the time evolution of the particles position — But instead consider the spatial trajectory

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{L}{mr^2} \quad \text{from above.}$$

$$\Rightarrow \frac{L}{mr^2}\frac{dr}{d\theta} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2k}{mr}}$$

So integrate.

$$\int \frac{\frac{L}{mr^2}\,dr}{\sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2k}{mr}}} = \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0$$

$$\shortparallel$$

$$L\int \frac{dr/r^2}{\sqrt{2m\left(E - \frac{L^2}{2mr^2} + \frac{k}{r}\right)}} \quad \left(= L\int \frac{dr/r^2}{\sqrt{2m(E - U_{eff})}} \right)$$

Let $u = 1/r \Rightarrow du = -dr/r^2$

So

$$\boxed{\theta(r) - \theta_0 = -L\int \frac{du}{\sqrt{2m\left(E + ku - \frac{L^2}{2m}u^2\right)}}}$$

Recall the indefinite integral

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left(-\frac{b + 2cx}{\sqrt{-\Delta}} \right)$$

where now $a = 2mE$, $b = 2mk$, $c = -L^2$

$\Delta = 4ac - b^2 = -(8mEL^2 + 4m^2k^2)$

So

$$\theta(r) - \theta_0 = -L\frac{1}{\sqrt{L^2}} \cos^{-1}\left[\frac{2L^2u - 2mk}{\sqrt{8mEL^2 + 4m^2k^2}} \right]$$

$$\Rightarrow \quad \cos(\theta - \theta_0) = \frac{2L^2u - 2mk}{2mk\sqrt{\frac{2EL^2}{mk^2} + 1}}$$

recalling that $u = \frac{1}{r}$ $\Rightarrow$

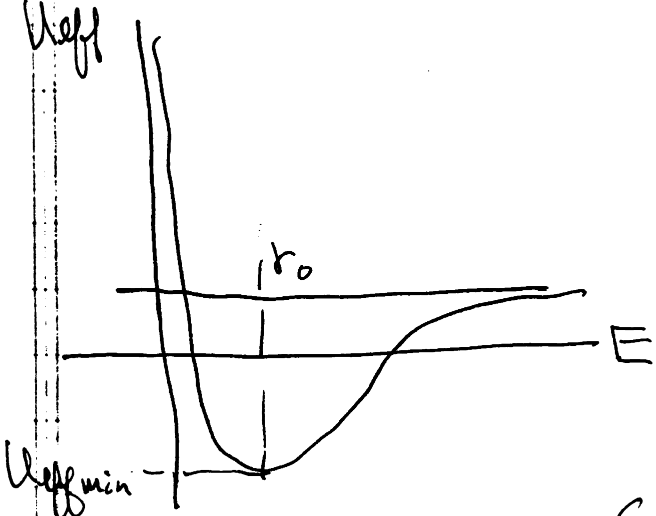
$$\boxed{\frac{L^2}{mk}\frac{1}{r} = 1 + \sqrt{\frac{2EL^2}{mk^2} + 1}\,\cos(\theta - \theta_0)}$$

This is the equation of a <u>conic section</u>

$$\frac{\alpha}{r} = 1 + \epsilon\cos(\theta - \theta_0)$$

$$\alpha = \frac{L^2}{mk} = \tfrac{1}{2}(\text{latus rectum})$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}} = \text{eccentricity}$$



$$U_{eff\,min} = -\frac{mk^2}{2L^2} = -\frac{1}{2}\frac{k}{r_0}$$

$$r_0 = \alpha$$

So $\epsilon = \sqrt{1 - E/U_{eff\,min}}$

$$0 \le \epsilon \le \infty$$

$\epsilon > 1$	$E > 0$	hyperbola
$\epsilon = 1$	$E = 0$	parabola
$0 < \epsilon < 1$	$U_{eff\,min} < E < 0$	Ellipse
$\epsilon = 0$	$E = U_{eff\,min}$	circle

More about "orbits" later

There is an additional <u>conserved</u> <u>vector</u> discovered by Laplace-Runge-Lenz (vector)

Define (Runge-Lenz Vector)

$$\boxed{\vec{A} \equiv \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}} = \vec{p}\times\vec{L} + m\vec{r}\,U(r)$$

$$\frac{d\vec{A}}{dt} = \underbrace{\dot{\vec{p}}}_{=\vec{F}} \times \vec{L} + \vec{p}\times\dot{\vec{L}} + \underbrace{m\dot{\vec{r}}}_{=\vec{p}}\,U(r) + m\vec{r}\,\frac{dU}{dt}$$

But $\vec{L}$ is conserved So $\dot{\vec{L}} = 0$; $\frac{dU}{dt} = \vec{\nabla}U \cdot \frac{d\vec{r}}{dt} = -\vec{F}\cdot\vec{v}$

So

$$\frac{d\vec{A}}{dt} = \vec{F}\times\vec{L} + \vec{p}\,U(r) - \vec{r}(\vec{F}\cdot\vec{p})$$

Now

$$\vec{F}\times\vec{L} = -\frac{k}{r^3}\,\vec{r}\times(\vec{r}\times\vec{p})$$

$$= -\frac{k}{r^3}\left[\vec{r}(\vec{r}\cdot\vec{p}) - \vec{p}(\vec{r}\cdot\vec{r})\right]$$

$$= \vec{r}(\vec{F}\cdot\vec{p}) - \vec{p}\,U(r)$$

Putting this together

$$\frac{d\vec{A}}{dt} = \vec{r}(\cancel{\vec{F}\cdot\vec{p}}) - \cancel{\vec{p}\,U(r)} + \cancel{\vec{p}\,U(r)} - \cancel{\vec{r}(\vec{F}\cdot\vec{p})} = 0$$

So $\boxed{\vec{A} = \text{constant vector}}$

$\vec{A}$ is conserved

Note
$$\vec{L}\cdot\vec{A} = \vec{L}\cdot\left[\vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}\right]$$

but $\vec{L}\cdot(\vec{p}\times\vec{L}) = 0$ & $\vec{L}\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{r} = 0$

So $\boxed{\vec{L}\cdot\vec{A} = 0}$ $\boxed{\vec{A} \text{ is } \perp \text{ to the constant vector } \vec{L}}$

Again let $\vec{L} = L\hat{z}$ So $\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta$

it lies in plane of orbit the x-y plane

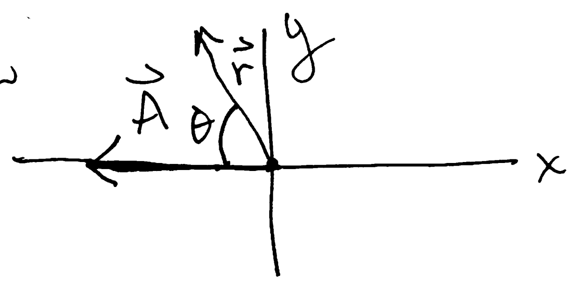
Consider

$$\vec{r}\cdot\vec{A} = \vec{r}\cdot\left[\vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}\right]$$

$$= \vec{L}\cdot(\vec{r}\times\vec{p}) - mkr = \vec{L}^2 - mkr$$

So $\boxed{\vec{r}\cdot\vec{A} = L^2 - mkr}$

Now



$\vec{r}\cdot\vec{A} = rA\cos\theta$

So $rA\cos\theta = L^2 - mkr$

$\Rightarrow$

$$\boxed{\frac{L^2}{mk}\frac{1}{r} = 1 + \frac{A}{mk}\cos\theta}$$

Again we find the conic section spatial trajectory

$A = mk\,\epsilon$ in this case and θ is the angle between $\vec{r}$ & $\vec{A}$

$\alpha = L^2/mk$ and the origin is at the "left" focus