Physics 410 Classical Mechanics

# 1 Newton's Laws

Classical mechanics is perhaps the oldest of the physical sciences with its origins shrouded in antiquity. The first complete mathematical (quantitative) statement of its principles was given by Issac Newton in 1687 in his *Principia*. These principles have become to be known as:

**Newton's Three Laws of Classical Mechanics.**

I) **Law of Inertia:** A body remains at rest or in uniform motion unless acted upon by a force.

II) **Law of Motion:** A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

III) **Force Law:** If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

In order to express the consequences of these laws in a more mathematical form, we need to discuss the fundamental or primitive concepts that are assumed and that lie behind the laws. In that way we can fine-tune our understanding of Newton.

Mechanics deals with the motion of bodies in space over the course of time. In general we will deal with *point particles*, and we will build up the motion of macroscopic bodies from conglomerates of point particles. By point particle we mean our fundamental particles will not have internal structure but will be assigned only the location of a point in 3 dimensional space. That is as far as describing their motion is concerned, we can neglect their extent in space, that is their dimensions. Of course this idealization depends on the problem e.g. planets may be regarded as point particles when considering their motion around the sun but not when considering their rotation about their axes. Thus the fundamental particle's mass, electric charge, dipole moment or other property will be an intrinsic quantity given once and for all and beyond explanation. If these quantities vary in time the "particle" is more than fundamental. How good such a description is depends also on whether another more complex description of our particle leads to predictions of any of these previously intrinsic quantities.

When we speak of the position or the motion of a particle we mean the location relative to some other body suitable for that purpose. We might refer the motion of the planets to the center of gravity of the sun or the motion of a pulley relative to the earth. All motions are described as motions relative to some reference body. Ideally once we have chosen our reference body (point) we imagine setting up a framework of rods (axes) that extend into space in 3 mutually perpendicular directions. Hence by this Cartesian coordinate system we can characterize the spatial location of any event by 3 numbers; the Cartesian coordinates of that space point. Thusly we have constructed a **frame of reference**.

Some bodies are suitable reference bodies, others are not. Of course we can describe nature and formulate its laws in any frame of reference we want. However, there may exist a frame or frames in which the laws of nature are fundmentally simpler. They may contain fewer parameters than otherwise e.g. the laws of planetary motion have a simpler appearance in a heliocentric rather than a geocentric frame of reference. Thus we must determine experimentally the most suitable frame of reference in which to describe the laws of nature. This is what is summarized in Newton's first law; the laws of inertia. It was proposed by Galileo and included by Newton in his laws and summarizes experiments done on uniformly moving particles in the absence of forces (that is zero net force). So the First Law defines the **inertial frame of reference.**

Among all the frames of reference conceivable there exists a set of frames with respect to which the law of inertia takes its familiar form: In the absence of (net) forces, the space coordinates of a point particle are linear functions of time. These frames of reference are called **inertial frames of reference.**

Certainly a free particle moving in a straight line traverses a constant distance for each time interval (second) that passes. If a second now and a second at some later time varied in length, the distance covered by the particle would vary. Hence it would appear to accelerate or decelerate depending upon the variable time interval. So we conclude that Newton's First Law requires time to be homogeneous in an inertial frame of reference. The interval of time is fixed now and in the future. Likewise, for each fixed interval of time the free particle must traverse the same distance for the velocity to be constant. This is so wherever in space the particle travels. Thus the interval of distance (meter) between points in space must be the same throughout space and in whatever direction the particle moves. Thus space must be homogeneous and isotropic in an inertial frame of reference. The First Law states properties of space and time; in an inertial frame of reference time is homogeneous and space is homogeneous and isotropic.

In addition, it is found experimentally that all the laws of motion take the same form when stated in terms of any one of these inertial systems. In these frames the properties of space and time are the same, and the laws of mechanics are the same. Hence from the point of view of mechanics all inertial frames are equivalent. (We can determine if a particle is accelerated or not by comparing its motion to that of a particle not subject to forces. But whether a free particle is "at rest" or "in uniform motion" depends on the frame one is using—and has no absolute meaning. There is no way to determine who is "moving" and who is "at rest", it is all relative.) This fact that all inertial systems are equivalent for the description of Nature is called the **principle of** (Galilean) **relativity**. The laws of mechanics have the same form in all inertial frames—this is called the **covariance** (form-invariance) of the laws of nature.

Newtonian physics also requires that time intervals are the same for all inertial observers, especially those in uniform relative motion. If an event occurs in the inertial frame of observer $S$ at time $t$, then this same event appears to occur in the inertial frame of observer $S'$ at time $t'$ where

$$t' = t + \tau, \tag{1.1}$$

with the two observers' clocks differing in their zero of time by constant $\tau$. Consequently, the time intervals in the two frames are the same (assuming the same units for time in both frames, as above)

$$dt' = dt. \tag{1.2}$$

Hence we see that the law of inertia says something more significant since at first glance all the law states is that a particle not subject to forces is unacclerated. Mathematically if we locate the position of the particle with coordinates $x, y, z$ or $x_1, x_2, x_3$; to be unaccelerated simply means

$$\ddot{x} = \ddot{y} = \ddot{z} = 0 \tag{1.3}$$

or

$$\ddot{x}_i = 0 \quad , \quad \forall_i \tag{1.4}$$

with

$$\dot{x} = \frac{d}{dt}x(t) \qquad ; \qquad \ddot{x} = \frac{d^2}{dt^2}x(t), \tag{1.5}$$

and so on for higher derivatives. That is $\dot{x}_i = v_i =$ constant. Of course there always exists a frame of reference in which the particle is unaccelerated–the rest frame of the particle. However, the real power of the First Law is that there exists a set of frames of reference with respect to which all bodies not subjected to forces are unaccelerated—the inertial frames of reference.

Now we would like to determine all inertial frames given one. That is, suppose a particle is in uniform motion in the inertial frame of reference of observer $S$. We can ask what types of coordinate transformations leave the form of the law of inertia unchanged (and further exhibit the covariance of all the laws of mechanics). The law states

$$\ddot{x}_1 = 0, \ddot{x}_2 = 0, \ddot{x}_3 = 0. \tag{1.6}$$

Hence in the inertial frame of observer $S$ the particle's trajectory is described by a straight line parameterized by time

$$x_i(t) = x_{0i} + v_i t, \tag{1.7}$$

where $\vec{x}_0$ is the particle's initial position at $t = 0$ and $\vec{v}$ is the particle's initial velocity; both initial position and velocity vectors are constants and the trajectory is linear in time. Likewise, observer $S'$ sets up an inertial frame using coordinates $x'_i(t')$. According to observer $S'$ the same free particle will have the straight line trajectory

$$x'_i(t') = x'_{0i} + v'_i t'. \tag{1.8}$$

We desire to determine the transformations of the coordinates between the two frames of reference. In general the coordinates used by $S'$ are an arbitrary function of the space and time coordinates used by $S$

$$x_i'(t') = a_i(x,t). \tag{1.9}$$

The velocity and acceleration of the particle can then be found using the chain rule

$$\begin{aligned}
\frac{dx_i'(t')}{dt'} &= \frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial x_j}\frac{dx_j}{dt} \\
\frac{d^2x_i'(t')}{dt'^{\,2}} &= \frac{\partial^2 a_i}{\partial t^2} + 2\frac{\partial^2 a_i}{\partial t \partial x_j}\frac{dx_j}{dt} + \frac{\partial^2 a_i}{\partial x_j \partial x_k}\frac{dx_j}{dt}\frac{dx_k}{dt} + \frac{\partial a_i}{\partial x_j}\frac{d^2x_j}{dt^2}.
\end{aligned} \tag{1.10}$$

In general even if $\ddot{x}_i = 0$, the acceleration in $S'$'s frame is not necessarily zero, and hence $S'$ would not be using an inertial frame. First both observers can choose to use a Cartesian coordinate system in their respective frames of reference. Hence $a_i(x,t)$ can only be a linear function of $\vec{x}$

$$a_i(x,t) = a_{ij}(t)x_j(t) + b_i(t). \tag{1.11}$$

This simplifies the expressions for the velocity and acceleration in $S'$'s frame

$$\begin{aligned}
\frac{dx_i'(t')}{dt'} &= \frac{da_{ij}}{dt}x_j + a_{ij}\frac{dx_j}{dt} + \frac{db_i}{dt} \\
\frac{d^2x_i'(t')}{dt'^{\,2}} &= \frac{d^2b_i}{dt^2} + \frac{d^2a_{ij}}{dt^2}x_j + 2\frac{da_{ij}}{dt}\frac{dx_j}{dt} + a_{ij}\frac{d^2x_j}{dt^2}.
\end{aligned} \tag{1.12}$$

For the free particle $S$ observes that $\dot{x}_i = v_i$ and $S'$ must also observe $\frac{dx_i'}{dt'} = v_i'$ and consequently it must be that the acceleration of the particle in $S^prime$ is zero: $\frac{d^2x_i'}{dt'^{\,2}} = 0$. This can only be achieved if

$$b_i(t) = a_i - V_i t, \tag{1.13}$$

with $\vec{V}$ and $\vec{a}$ both constants and the matrix $a_{ij} = \lambda_{ij}$ also a constant matrix. Now if both observers use the same unit of distance we require, considering that $\vec{V}$ and $\vec{a}$ are zero, that $\vec{x}\,'\cdot\vec{x}\,' = \vec{x}\cdot\vec{x}$. This implies that the $\lambda_{ij}$ matrices are $3\times 3$ orthogonal matrices

$$(\lambda^{-1})_{ij} = \lambda_{ji}, \tag{1.14}$$

that is $\lambda^{-1} = \lambda^T$.

Hence all inertial frames of reference have coordinates that are related by these transformations, a differing zero of time, a translation of the origin, a rotation of the axes and uniform motion of the axes relative to each other. The transformations between coordinates are summarized by the equations

$$\begin{aligned} t' &= t + \tau \\ x_i' &= \lambda_{ij}x_j + a_i - V_i t \end{aligned} \tag{1.15}$$

and are known as the Galilean Transformations. The acceleration of a particle as described in each frame is related $\frac{d^2x_i'(t')}{dt'^{\,2}} = \lambda_{ij}\frac{d^2x_j(t)}{dt^2}$. So if one vanishes the other vanishes. The square of the differential length interval $ds^2 = dx_i dx_i = dx_i\delta_{ij}dx_j$ (where the Kronecker $\delta$ is the Euclidean space metric (see below)) remains unchanged under Galilean transformations:

$$\begin{aligned} t' &= t + \tau \\ x_i' &= \lambda_{ij}x_j + a_i - V_i t, \end{aligned} \tag{1.16}$$

hence (recall that time and space are independent variables as are their differentials)

$$\begin{aligned} dt' &= dt \\ dx_i' &= \lambda_{ij}dx_j \end{aligned} \tag{1.17}$$

and the interval is invariant implying $\lambda$ is orthogonal

$$ds'^2 = ds^2 \Rightarrow \lambda^{-1} = \lambda^T \tag{1.18}$$

That is $(\lambda^{-1})_{ij} = \lambda_{ji}$. So now to show that $\lambda^{-1} = \lambda^T$ implies that the transformation is a rotation consider the special case where $x_3' = x_3$ and ignore $\tau = 0$, $a^i = 0$ and $V^i = 0$ transformations

$$\begin{pmatrix} x_i' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\equiv \lambda} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.19}$$

Now we require $\lambda$ to be a proper rotation that is $\det \lambda = 1 \Rightarrow \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 1$. Further

$$\lambda^{-1} = \frac{1}{\det \lambda} \begin{bmatrix} \lambda_{22} & -\lambda_{12} & 0 \\ -\lambda_{21} & \lambda_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.20}$$

(Check this by multiplying the matrices

$$\lambda^{-1}\lambda = \frac{1}{\det \lambda} \begin{bmatrix} \lambda_{22} & -\lambda_{12} & 0 \\ -\lambda_{21} & \lambda_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.21}$$

$$= \frac{1}{\det \lambda} \begin{bmatrix} \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} & 0 & 0 \\ 0 & \lambda_{11}\lambda_{12} - \lambda_{12}\lambda_{21} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.22}$$

$$= \mathbb{1})$$

But $\lambda^{-1} = \lambda^T =$

$$\begin{bmatrix} \lambda_{11} & \lambda_{21} & 0 \\ \lambda_{12} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.23}$$

$$\Rightarrow \quad \begin{aligned} \lambda_{11} &= \lambda_{22} \\ \lambda_{21} &= -\lambda_{12} \end{aligned} \tag{1.24}$$

$$\Rightarrow \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = \lambda_{11}^2 + \lambda_{12}^2 = 1 \tag{1.25}$$

Solve this by

$$\begin{aligned} \lambda_{11} &= \cos\theta \\ \lambda_{12} &= \sin\theta \end{aligned} \tag{1.26}$$

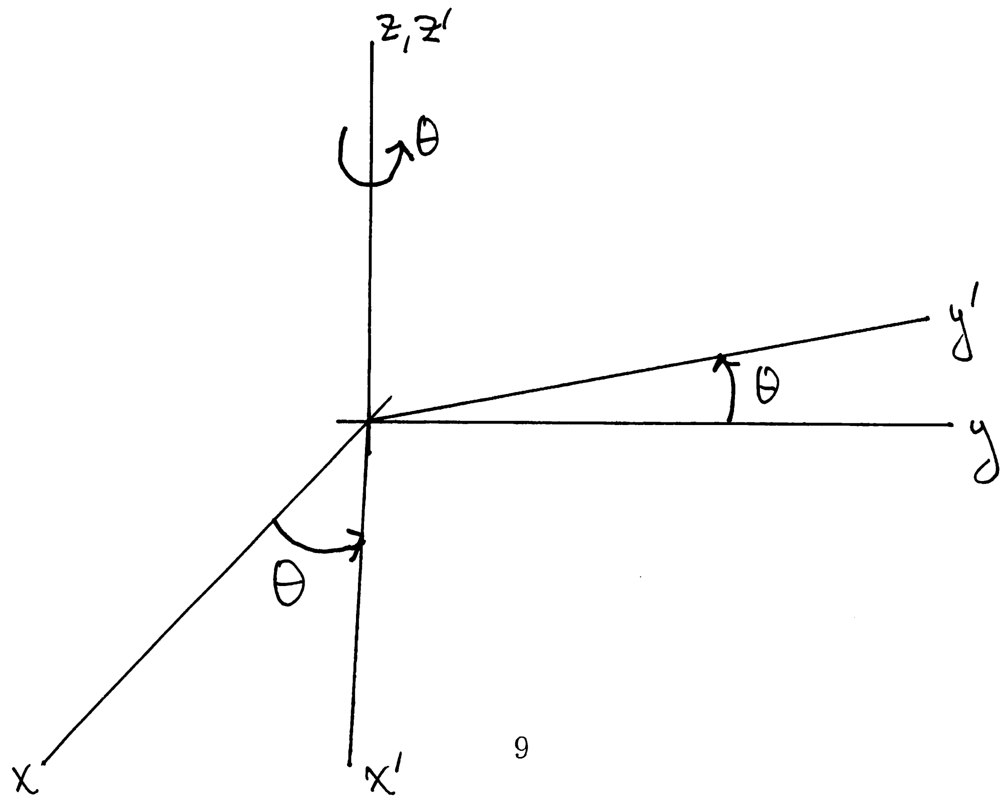
which implies that

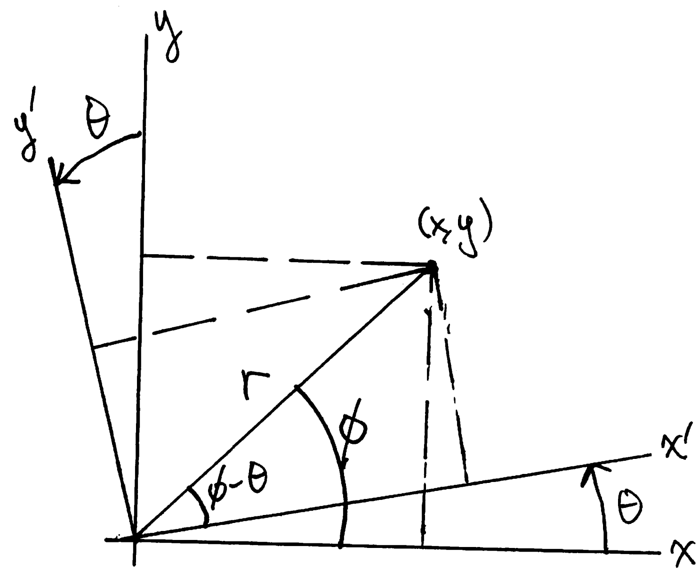
$$\lambda_{22} = \cos\theta \qquad ; \qquad \lambda_{21} = -\sin\theta \tag{1.27}$$

Hence

$$\lambda = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.28}$$

This describes a rotation about z-axis - rotation in the $x_1 - x_2$ plane:






$$\begin{array}{r|l} x = r\cos\phi & x' = r\cos(\phi - \theta) \\ y = r\sin\phi & y' = r\sin(\phi - \theta) \end{array} \tag{1.29}$$

$$\begin{aligned} x' &= r\cos\phi\cos\theta + r\sin\phi\sin\theta = x\cos\phi + y\sin\theta \\ y' &= r\sin\phi\cos\theta - r\cos\phi\sin\theta = y\cos\phi - x\sin\theta \\ z' &= z \end{aligned} \tag{1.30}$$

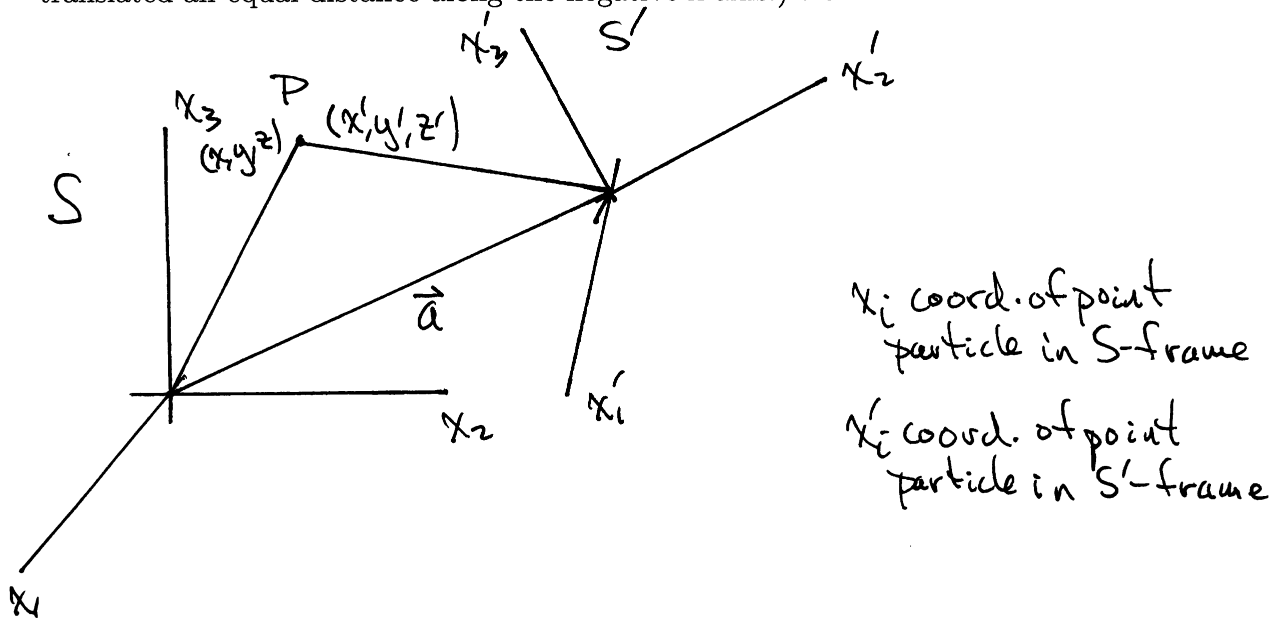
So

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.31}$$

$$= \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.32}$$

(Prove every $\lambda^{-1} = \lambda^T =$ rotation by infinitesimal?!)

Consider in more depth the transformations. First consider the transformations that do not involve relative motion (time) of the frames. This first type of transformation is often said not to involve a change of frame of reference (although we will consider all coordinate transformations whether they involve time or not as changes of frame and view this as 2 observers $S$, $S'$ and a transformation between their frames: this is called the passive viewpoint) but only a translation and/or a rotation of the coordinate system within the same frame of reference (this corresponds to the active viewpoint in which, for example, the system is translated from one point to another in the same frame of reference). (The points of view are just inverse operations to eachother. If the system is translated along the positive x-axis that is equivalent to using the coordinate system of another observer who is translated an equal distance along the negative x-axis.) i.e.




From the passive point of view $x_i$ is the coordinate of the point particle in the $S$-frame while $x'_i$ is the coordinate of the particle in the $S'$ frame.

$$x'_i = \lambda_{ij} x_j + a_i \tag{1.33}$$

(Einstein summation convention)

$$\lambda_{ij} = \cos(x'_i, x_j) \tag{1.34}$$

(direction cosines)

$$a_i = \text{constant translation vector} \tag{1.35}$$

Einstein summation convention: Repeated indices are summed over; hence the same letter can only occur twice in any equation since then it is summed over and is a dummy variable..

$$x'_i = \sum_{j=1}^{3} \lambda_{ij} x_j + a_i = \lambda_{ij} x_j + a_i \tag{1.36}$$

(Since both observers $S$ and $S'$ are using the same time $t$) Since this all occurs at the same time $t$ we have

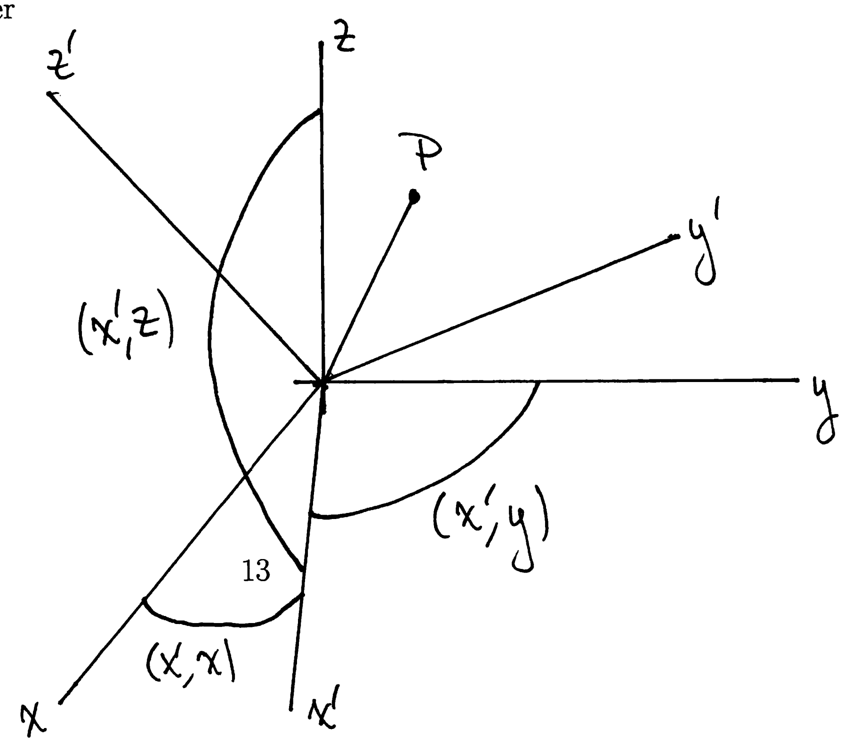
$$\dot{x}'_i = \lambda_{ij} \dot{x}_j \qquad \because a_i, \lambda_{ij} \text{ are constants} \tag{1.37}$$

$$\Rightarrow \ddot{x}'_i = \lambda_{ij} \ddot{x}_j = 0. \tag{1.38}$$

($\lambda =$ const or else $\ddot{x} \propto (\partial \lambda_{ij} / \partial x_k) \dot{x}_k \dot{x}_j \neq 0$)

Thus a free particle in uniform motion in $S$ is also in uniform motion in $S'$. We may use either set of coordinates in our frame of reference, the law of inertia remains the same. The position of the particle is a linear function of the time.

Let's focus on the transformations corresponding to spatial rotations. Consider the position of a particle as viewed from 2 observers rotated with respect to each other

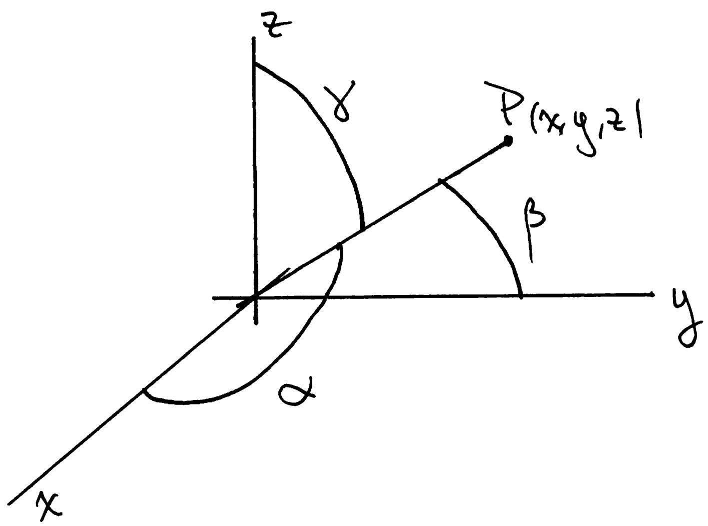


Now the direction cosines of the axes are the cosines of the angles between them

$$\lambda_{ij} \equiv \cos(x'_i, x_j) \tag{1.39}$$

where $(x'_i, x_j)$ denotes the angle between $x'_i$ and $x_j$ (see above).

For any line, denote the angles it makes with the $x, y, z$ axes $\alpha, \beta, \gamma$, the direction cosines of the line are $\cos\alpha, \cos\beta, \cos\gamma$

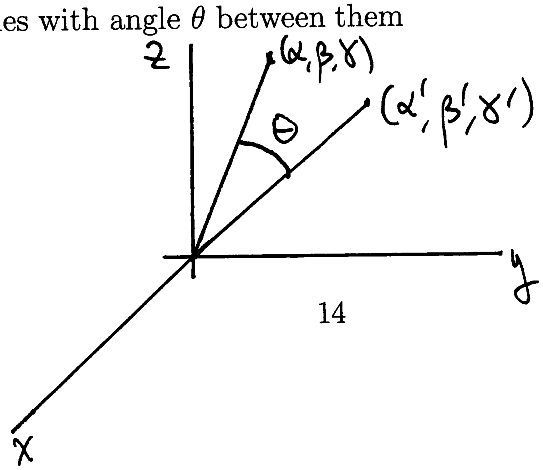



$$HW : \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(line has length $r \Rightarrow x^2 + y^2 + z^2 = r^2$

$$\begin{aligned} x &= r\cos\alpha \\ y &= r\cos\beta \quad \Rightarrow c^2\alpha + c^2\beta + c^2\gamma = 1 \\ z &= r\cos\gamma) \end{aligned} \tag{1.40}$$

Consider 2 lines with angle $\theta$ between them

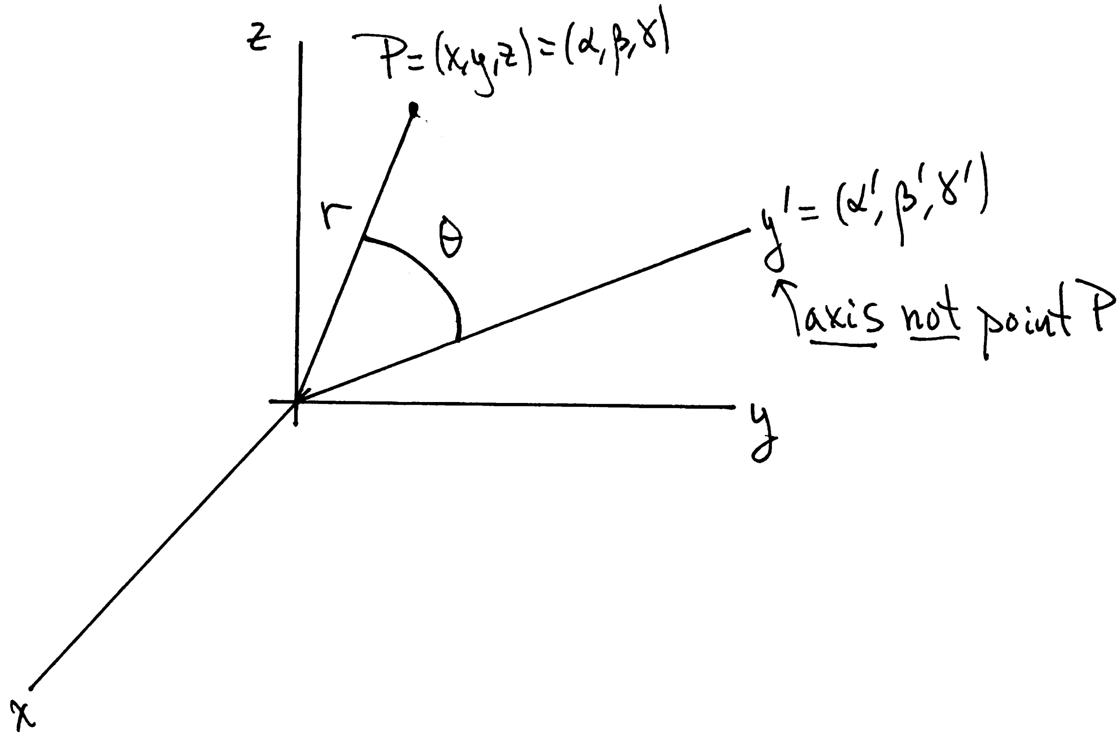


HW: $\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos r\cos\gamma'$

Then the point $P$ has coordinates in $S'$ related to those in $S$ by

$$x_i' = \sum_j \lambda_{ij}(x_i' x_j) x_j = \lambda_{ij} x_j \tag{1.41}$$

Consider the $y'$ coordinate of $P$ with $S$ coordinates $(x, y, z)$




$$\begin{aligned} x &= r\cos\alpha \\ y &= r\cos\beta \\ z &= r\cos\gamma \end{aligned} \tag{1.42}$$

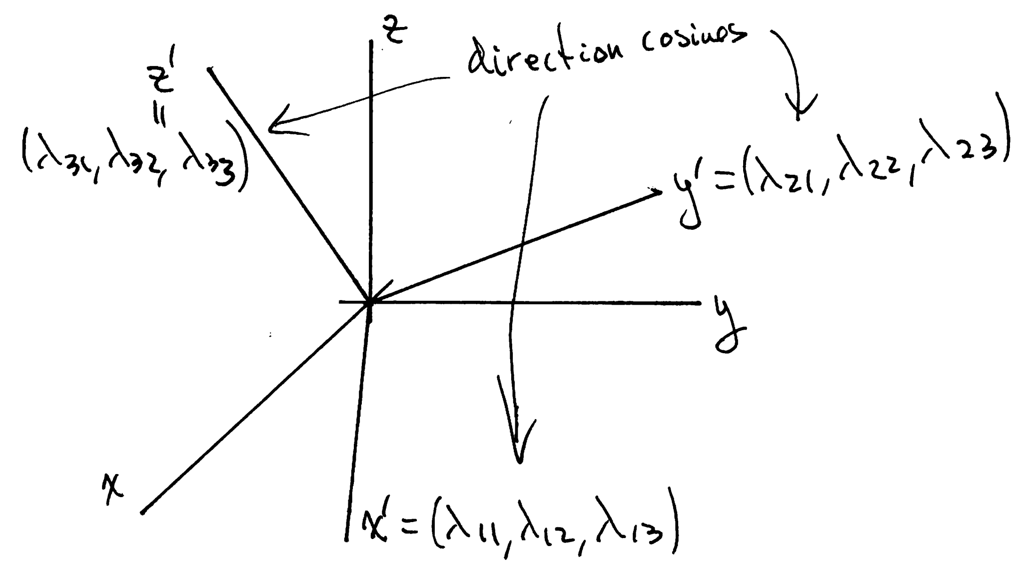
Now the $x_2' = y'$ coordinate of $P$ in the $S'$ frame is

$$\begin{aligned} y' &= r\cos\theta \\ &= r\cos\alpha\cos\alpha' + r\cos\beta\cos\beta' + r\cos\gamma\cos\gamma' \\ &= x\cos\alpha' + y\cos\beta' + z\cos\gamma' \end{aligned}$$

$$
\begin{aligned}
&= \lambda_{21}(x'_2, x_1)x_1 + \lambda_{22}(x'_2, x_2)x_2 + \lambda_{23}(x'_2, x_3)x_3 \\
x'_2 &= \lambda_{2j}x_j
\end{aligned} \tag{1.43}
$$

Similarly for $x'$ and $z' \Rightarrow x'_i = \lambda_{ij}x_j$. So rotations of frames are "described" by a $3 \times 3$ matrix of direction cosines of the angles between the axes.

Now consider the $x'_i$ axis, it has direction cosines in $S$ given by $(\lambda_{i1}, \lambda_{i2}, \lambda_{i3})$




Now the angle between $x'_i$ and $x'_j$ is $\frac{\pi}{2}$ so

$$
\cos\theta = \cos\frac{\pi}{2} = 0 \Rightarrow (\text{for } i \neq j) \;\; \lambda_{i1}\lambda_{j1} + \lambda_{i2}\lambda_{j2} + \lambda_{i3}\lambda_{j3} = \cos\frac{\pi}{2} = 0 \tag{1.44}
$$

$$
\Rightarrow \sum_{k=1}^{3} \lambda_{ik}\lambda_{jk} = 0 = \lambda_{ik}\lambda_{jk} \qquad (\text{for } i \neq j) \tag{1.45}
$$

using the Einstein summation convention.

Further

$$
\cos\alpha^2 + \cos^2\beta + \cos^2\gamma = 1 \tag{1.46}
$$

implies

$$x_{i1}x_{i1} + x_{i2}x_{i2} + x_{i3}x_{i3} = 1 \qquad \text{(no sum over } i) \tag{1.47}$$

that is

$$\sum_{k=1}^{3} x_{ik}x_{jk} = 1 \qquad (i = j) \tag{1.48}$$

So together we have

$$\begin{aligned} \sum_{k=1}^{3} \lambda_{ik}\lambda_{jk} &= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \\ &\equiv \delta_{ij} \end{aligned} \tag{1.49}$$

the Kronecker $\delta$-symbol (identity matrix). Using the Einstein summation convention this becomes

$$\lambda_{ik}\lambda_{jk} = \delta_{ij} \tag{1.50}$$

and yields the 6 orthogonality conditions. (Visa versa, the $x_i$ direction cosines in $S'$ frame

$$\lambda_{ki} = \cos(x'_k, x_i) \tag{1.51}$$

obey

$$\lambda_{ki}\lambda_{kj} = \delta_{ij}) \tag{1.52}$$

So $\lambda_{ik}$ can be viewed as a $3 \times 3$ matrix with $i$ labelling the 3 rows and $k$ labelling the 3 columns $\lambda_{ik} \rightarrow$ matrix $(\lambda)_{ik} = (\lambda_{ik})$ The orthogonality conditions can be expressed as matrix multiplication

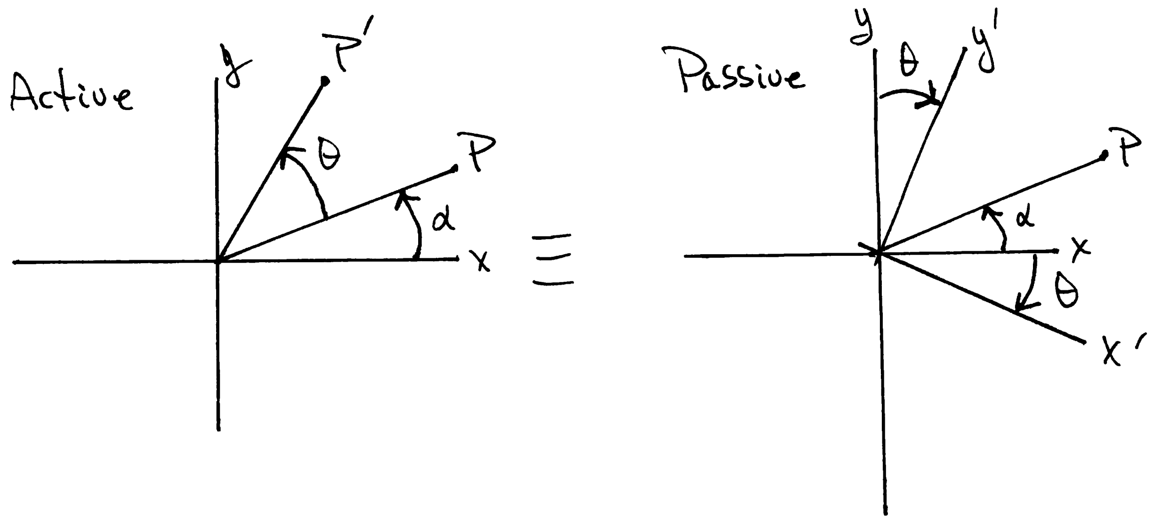
$$(\lambda)_{ik}(\lambda)_{jk} = (1)_{ij} \Rightarrow \lambda\lambda^T = 1 \Rightarrow \Lambda^T = \lambda^{-1} \tag{1.53}$$

6 conditions on 9 matrix elements implies that the rotation matrix $\lambda = 3 \times 3$ (real) orthogonal matrix. Hence the rotation matrix is a matrix with just 3 independent parameters.

Another useful way to characterize a rotation is using Euler's Theorem: Specify an arbitrary rotation in terms of a rotation through angle $\theta$ about a fixed axis whose orientation is given by the unit vector $\hat{\theta}$. Hence the vector $\vec{\theta} = \theta\hat{\theta}$ specifies the rotation $R_{ij} = R_{ij}(\vec{\theta})$ between $\vec{r}$ and $\vec{r}'$

$$x_i' = R_{ij}(\vec{\theta})x_j. \tag{1.54}$$

(Active viewpoint) Rotate vector in fixed frame $S$ is equivalent to rotating in opposite direction the axes with vector fixed (passive viewpoint)




(Active View)Rotate P CCW($\theta$) to $P'$ axes fixed.

(Passive View)Rotate axes $x$ CW($-\theta$) to $x'$ axes $P$-fixed.

Consider two observers whose Cartesian frames of reference are rotated relative to eachother with their origins in common. Or alternatively, suppose we rotate our system from points $\vec{r}$ to points $\vec{r}'$. The coordinates used by the two observers are related by the rotation transformation

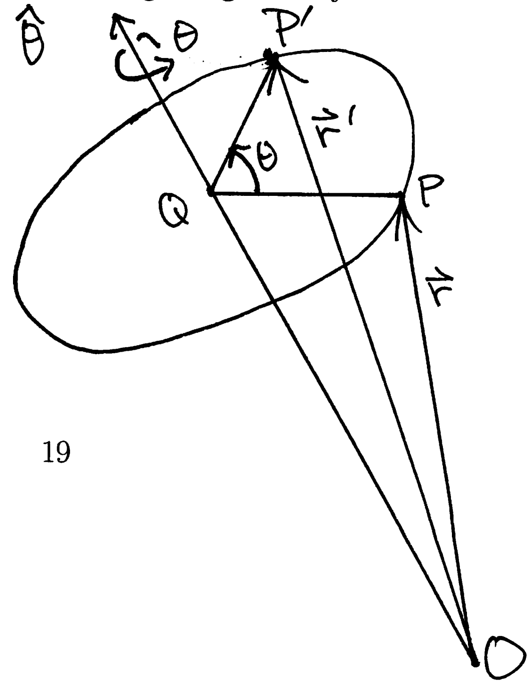
$$x_i' = R_{ij}(\vec{\theta})x_j. \tag{1.55}$$

Since rotations about a common origin leave the length of a vector unchanged we have

$$
\begin{aligned}
\vec{r}' \cdot \vec{r}' &= \vec{r} \cdot \vec{r} = x_j x_j = x_j \delta_{jk} x_k \\
&= (R_{ij} x_j)(R_{ik} x_k) = x_j (R^T_{ji} R_{ik}) x_k. \qquad (1.56)
\end{aligned}
$$

This implies that $R^T R = 1$ that is $R^T = R^{-1}$. Thus the rotation matrix $R$ is a real $3 \times 3$ orthogonal matrix. Further $\det(RR^T) = \det 1 = 1$, which implies that $\det R = \pm 1$. Since we are interested in rotations connected to the identity rotation, we will restrict ourselves to $\det R = +1$ rotations only, these are called *proper* rotations. $\det R = -1$ rotations can be obtained by making a proper rotation followed by a parity transformation, that is an inversion of the space axes. In general $R_{ij}$ has 9 independent matrix elements. However $R^T = R^{-1}$ reduces this to just 3 independent matrix elements. The choice of which 3 parameters specify the rotation in question is arbitrary. Again, according to Euler's Theorem (*Novi Comment. Petrop.* xx (1776), p. 189, §25, Whittaker: Analytical Mechanics p. 2, Goldstein: Chapter 4.6) an arbitrary rotation can be specified in terms of a rotation through angle $\theta$ about a fixed axis whose orientation is given by an unit vector denoted $\hat{\theta}$. Hence given the vector $\vec{\theta} \equiv \theta\hat{\theta}$, the rotation between $\vec{r}$ and $\vec{r}'$ is specified $R_{ij}(\vec{\theta})$.

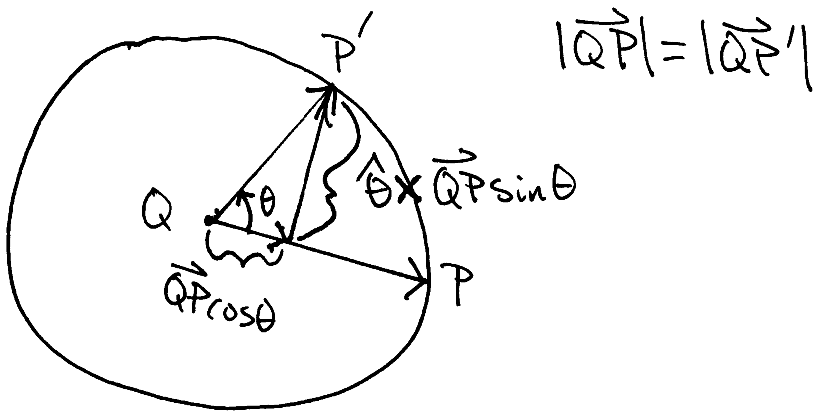
Given $\vec{\theta}$ we can find $R_{ij}(\vec{\theta})$ by considering the geometry of the rotation



So

$$\vec{r}\,' = \overrightarrow{OQ} + \overrightarrow{QP'} \tag{1.57}$$

Not to find $\overrightarrow{QP'}$ look down the $\hat{\theta}$-axis from above and note that $|\overrightarrow{QP}| = |\overrightarrow{QP'}|$




$$\vec{r}\,' = \overrightarrow{OQ} + \overrightarrow{QP}\cos\theta + \hat{\theta}\times\overrightarrow{QP}\sin\theta \tag{1.58}$$

But vector $\overrightarrow{OQ} = (\vec{r}\cdot\hat{\theta})\hat{\theta}$ and vector $\overrightarrow{QP} = \vec{r} - \overrightarrow{OQ} = \vec{r} - (\vec{r}\cdot\hat{\theta})\hat{\theta}$. So

$$\vec{r}\,' = (\vec{r}\cdot\hat{\theta})\hat{\theta} + (\vec{r}\cos\theta - (\vec{r}\cdot\hat{\theta})\hat{\theta}\cos\theta) + (\hat{\theta}\times\vec{r}\sin\theta - (\vec{r}\cdot\hat{\theta})\underbrace{\hat{\theta}\times\hat{\theta}}_{=0}\sin\theta) \tag{1.59}$$

Hence we find

$$\vec{r}\,' = \vec{r}\cos\theta + (\vec{r}\cdot\hat{\theta})\hat{\theta}(1-\cos\theta) + \hat{\theta}\times\vec{r}\sin\theta \tag{1.60}$$

In terms of components this yields

$$x_i' = [\delta_{ij}\cos\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta) + \epsilon_{ijk}\hat{\theta}_k\sin\theta]x_j = R_{ij}(\vec{\theta})x_j \tag{1.61}$$

Hence given $\vec{\theta}$ we have

$$R_{ij}(\vec{\theta}) = \delta_{ij}\cos\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta) + \epsilon_{ikj}\hat{\theta}_k\sin\theta \tag{1.62}$$

Note that $\mathrm{Tr}R = R_{ii} = 3\cos\theta + (1-\cos\theta) = 1 + 2\cos\theta$ and that $\epsilon_{kij}R_{ij} = \epsilon_{kij}\epsilon_{ilj}\hat{\theta}_l\sin\theta = -2\hat{\theta}_k\sin\theta$.

We can simplify the writing of this matrix by introducing the $3\times 3$ matrix generators of rotation matrices. Define the matrix

$$\begin{aligned}\Theta_{ij} &\equiv \epsilon_{ikj}\hat{\theta}_k = \begin{pmatrix} 0 & -\hat{\theta}_3 & \hat{\theta}_2 \\ \hat{\theta}_3 & 0 & -\hat{\theta}_1 \\ -\hat{\theta}_2 & \hat{\theta}_1 & 0 \end{pmatrix} \\ &\equiv \hat{\theta}\cdot\vec{I}_{ij}, \end{aligned} \tag{1.63}$$

where the 3 matrices $\vec{I}_{ij}$ are

$$(I_k)_{ij} \equiv \epsilon_{ikj}. \tag{1.64}$$

Writing these 3 matrice out we have

$$\begin{aligned} I_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = I_x \\ I_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = I_y \\ I_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_z. \end{aligned} \tag{1.65}$$

Since $\epsilon_{ijk}$ obeys the Jacobi identity

$$0 = \epsilon_{ijk}\epsilon_{mnk} + \epsilon_{jnk}\epsilon_{mik} + \epsilon_{nik}\epsilon_{mjk}, \tag{1.66}$$

we have that

$$[I_i, I_j] = \epsilon_{ijk} I_k. \tag{1.67}$$

In addition $\Theta_{ij}$ has simple multiplication properties

$$\left(\Theta^2\right)_{ij} = \hat{\theta}_i\hat{\theta}_j - \delta_{ij}$$

$$
\begin{aligned}
\left(\Theta^3\right)_{ij} &= -\Theta_{ij} \\
\left(\Theta^4\right)_{ij} &= -\left(\Theta^2\right)_{ij},
\end{aligned} \tag{1.68}
$$

and so on. So we find

$$
\begin{aligned}
\left(\Theta^{2n}\right)_{ij} &= (-1)^{n+1}\left(\Theta^2\right)_{ij} \\
\left(\Theta^{2n+1}\right)_{ij} &= (-1)^n\Theta_{ij}.
\end{aligned} \tag{1.69}
$$

Hence the exponential matrix yields

$$
\begin{aligned}
\left(e^{\theta\Theta}\right)_{ij} &= \delta_{ij} + \sum_{n=1}^{\infty}\frac{1}{n!}\theta^n\left(\Theta^n\right)_{ij} \\
&= \delta_{ij} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\theta^{2n+1}\left(\Theta^{2n+1}\right)_{ij} + \sum_{n=1}^{\infty}\frac{1}{(2n)!}\theta^{2n}\left(\Theta^{2n}\right)_{ij} \\
&= \delta_{ij} + \sum_{n=1}^{\infty}\frac{\theta^{2n}}{(2n)!}(-1)^{n+1}\left(\Theta^2\right)_{ij} + \sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}(-1)^n\left(\Theta\right)_{ij}.
\end{aligned} \tag{1.70}
$$

Using the identities $(-1)^{n+1} = -1(i^{2n})$ and $(-1)^n = -i(i)^{2n+1}$, the exponential becomes

$$
\begin{aligned}
\left(e^{\theta\Theta}\right)_{ij} &= \delta_{ij} - \left(\Theta^2\right)_{ij}\sum_{n=1}^{\infty}\frac{(i)^{2n}\theta^{2n}}{(2n)!} - \left(\Theta\right)_{ij}\sum_{n=0}^{\infty}\frac{i(i)^{2n+1}\theta^{2n+1}}{(2n+1)!} \\
&= \delta_{ij} + \left(\Theta\right)_{ij}\sin\theta - \left(\Theta^2\right)_{ij}(\cos\theta - 1).
\end{aligned} \tag{1.71}
$$

On the other hand recall that

$$
\begin{aligned}
R_{ij}(\vec{\theta}) &= \delta_{ij}\cos\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta) + \epsilon_{ikj}\hat{\theta}_k\sin\theta \\
&= \delta_{ij}\not{c}\mathrm{os}\,\theta + (\hat{\theta}_i\hat{\theta}_j - \delta_{ij})(1-\cos\theta) + \delta_{ij}(1-\not{c}\mathrm{os}\,\theta) + \left(\Theta\right)_{ij}\sin\theta \\
&= \delta_{ij} + \left(\Theta^2\right)_{ij}(1-\cos\theta) + \left(\Theta\right)_{ij}\sin\theta \\
&= \left(e^{\theta\Theta}\right)_{ij}.
\end{aligned} \tag{1.72}
$$

In addition we see that we have the properties

1) A rotation through $\theta$ can be built up by successive rotations about $\hat{\theta}$ since

$$\begin{aligned} R_{ij}(\theta\hat{\theta})R_{jk}(\theta'\hat{\theta}) &= \left(e^{\theta\Theta}\right)_{ij}\left(e^{\theta'\Theta}\right)_{jk} \\ &= \left(e^{(\theta+\theta')\Theta}\right)_{ik} \\ &= R_{ik}((\theta+\theta')\hat{\theta}) \end{aligned} \tag{1.73}$$

since $[\Theta,\Theta]=0$. Note that

$$R(\vec{\theta}) = \lim_{N\to\infty} \underbrace{[1+\frac{\theta}{N}\Theta]}_{=R(\vec{\theta}/N)}{}^N = e^{\theta\Theta} = \lim_{N\to\infty}[R(\frac{\theta}{N}\hat{\theta})]^N = R(\underbrace{(\frac{\theta}{N}+\frac{\theta}{N}+...+\frac{\theta}{N})}_{\text{N-terms}}\hat{\theta}). \tag{1.74}$$

2) Suppose $\vec{\theta}$ is infinitesimal $\vec{\theta}\equiv\omega\hat{\theta}\equiv\vec{\omega}$, with $\omega$ infinitesimal. Then

$$\begin{aligned} R_{ij}(\vec{\omega}) &= \left(e^{\omega\Theta}\right)_{ij} = \delta_{ij}+\omega\Theta_{ij} \\ &= \delta_{ij}+\omega_{ij}, \end{aligned} \tag{1.75}$$

with $\omega_{ij}\equiv\omega\Theta_{ij}=-\epsilon_{ijk}\omega_k$. Thus

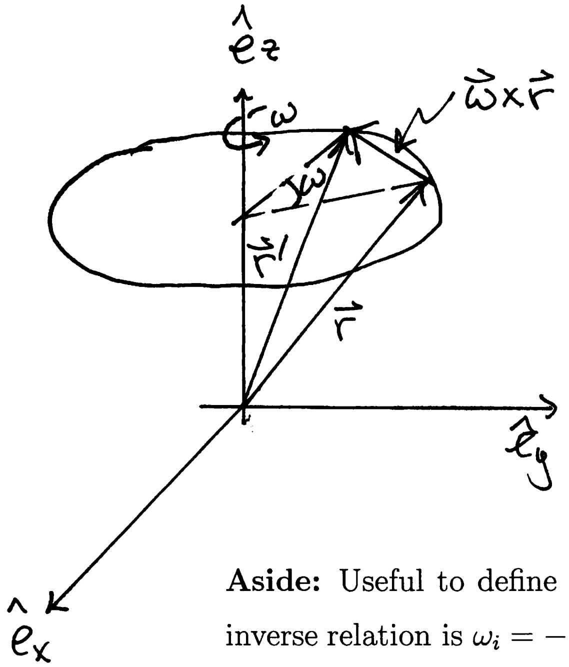
$$x'_i = R_{ij}(\vec{\omega})x_j = x_i+\omega_{ij}x_j \tag{1.76}$$

that is in vector notation

$$\vec{r}\,' = \vec{r}+\vec{\omega}\times\vec{r} \tag{1.77}$$

($\vec{\omega}$ is an angle not angular velocity here).

**Example:** Rotation about the $z$-axis has $\vec{\omega}=\omega\hat{e}_z \Rightarrow \omega_{12}=-\omega=-\omega_{21}$ and a rotation about the $z$-axis corresponds to a rotation in the $x_1-x_2$ plane. In general $\omega_{ij}$ corresponds to a rotation in the $x_i-x_j$ plane.



$$
\begin{aligned}
\vec{r}\,' &= \vec{r} + \vec{\omega} \times \vec{r} \\
x' &= x - \omega y \\
y' &= y + \omega x \\
z' &= z
\end{aligned}
\tag{1.78}
$$

**Aside:** Useful to define $\omega_{ij} \equiv -\epsilon_{ijk}\omega_k$ so $x_i' = x_i + \omega_{ij}x_j$. Then the inverse relation is $\omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$.

Finally we consider sequential rotations

$$\vec{r} \xrightarrow{R(\vec{\theta}_1)} \vec{r}\,' \xrightarrow{R(\vec{\theta}_2)} \vec{r}\,'' \tag{1.79}$$

$$R(\vec{\theta}_3) \tag{1.80}$$

Thus if

$$
\begin{aligned}
x_i'' &= R_{ij}(\vec{\theta}_2)x_j' \\
x_j' &= R_{jk}(\vec{\theta}_1)x_k \\
\Rightarrow x_i'' &= R_{ij}(\vec{\theta}_2)R_{jk}(\vec{\theta}_1)x_k \\
= R_{ik}(\vec{\theta}_3)x_k &\Rightarrow R_{ij}(\vec{\theta}_2)R_{jk}(\vec{\theta}_1) = R_{ik}(\vec{\theta}_3)
\end{aligned}
\tag{1.81}
$$

So the composition law for rotations is matrix multiplication. Since $R(\vec{\theta}_1)$ and $R(\vec{\theta}_2)$ are orthogonal with determinant $= 1$ so is the product

$$
\begin{aligned}
\det\left(R(\vec{\theta}_2)R(\vec{\theta}_1)\right) &= \det R(\vec{\theta}_2)\det R(\vec{\theta}_1) = 1 \\
(R(\vec{\theta}_2)R(\vec{\theta}_1))^T &= R(\vec{\theta}_1)^T R(\vec{\theta}_2)^T \\
&= R^{-1}(\vec{\theta}_1)R^{-1}(\vec{\theta}_2) = (R(\vec{\theta}_2)R(\vec{\theta}_1))^{-1}
\end{aligned}
\tag{1.82}
$$

Hence $R(\vec{\theta}_3) = R(\vec{\theta}_2)R(\vec{\theta}_1)$ is also a determinant $= 1$, orthogonal matrix, and hence a rotation. Since $R(\vec{0}) = \mathbb{1}$ and $R^{-1}(\vec{\theta}) = R(-\vec{\theta})$ and matrix multiplication is associaive we see that the set of all rotations forms a group—the group of $3 \times 3$ orthogonal matrices with determinant one. This group is denoted $SO(3)$; the special ($\det R = 1$), orthogonal ($R^{-1} = R^T$) group of $3 \times 3$ matrices. It is a group whose elements $R(\vec{\theta})$ depend continuously on 3 parameters given by $\vec{\theta}$.

Finally to complete the specification of the group multiplication law, we would like to specify $\vec{\theta}_3$ in terms of $\vec{\theta}_1$ and $\vec{\theta}_2$. Clearly this is somewhat messy. Since we can build up finite rotations from successive infinitesimal ones, it will suffice to consider the group product for the transformations

$$\vec{r} \xrightarrow{R^{-1}(\vec{\theta})} \vec{r}\,' \xrightarrow{(1+\omega)} \vec{r}\,'' \tag{1.83}$$

$$(1+\omega') \searrow \ \vec{r}\,''' \swarrow R(\vec{\theta}) \tag{1.84}$$

Thus

$$\begin{aligned} x_i''' &= R_{ij}(\vec{\theta})x_j'' \\ x_j'' &= (\delta_{jk} + \omega_{jk})x_k' \\ x_k' &= R_{kl}^{-1}(\vec{\theta})x_l \end{aligned} \tag{1.85}$$

$$\Rightarrow x_i''' = R_{ij}(\vec{\theta})(\delta_{jk} + \omega_{jk})R_{kl}^{-1}(\vec{\theta})x_l \tag{1.86}$$

equivalently

$$x_i''' = (\delta_{il} + \omega_{il}')x_l \tag{1.87}$$

$$\Rightarrow \begin{aligned} \delta_{il} + \omega_{il}' &= R_{ij}(\vec{\theta})(\delta_{jk} + \omega_{jk})R_{kl}^{-1}(\vec{\theta}) \\ &= \delta_{il} + (R(\vec{\theta})\omega R^{-1}(\vec{\theta}))_{il} \end{aligned} \tag{1.88}$$

Hence given $\vec{\theta}$ and $\vec{\omega}$ we have that the composite rotation resulting from the above sequence of transformations is

$$\omega_{ij}' = (R(\vec{\theta})\omega R^{-1}(\vec{\theta}))_{ij} \tag{1.89}$$

i.e.

$$R_{ij}(\vec{\omega}\,') = \delta_{ij} + \omega_{ij}' \tag{1.90}$$

so

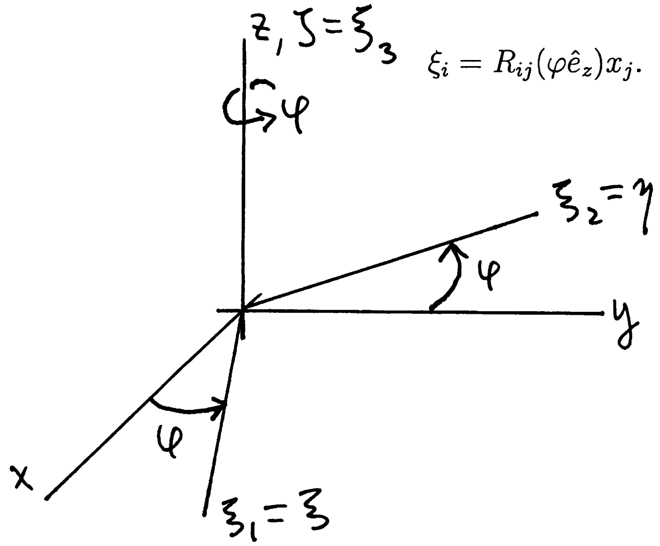
$$R(\vec{\omega}\ ')R(\vec{\theta}) = R(\vec{\theta})R(\vec{\omega}) \tag{1.91}$$

That is

$$R(\vec{\omega}\ ') = R(\vec{\theta})R(\vec{\omega})R^{-1}(\vec{\theta}). \tag{1.92}$$

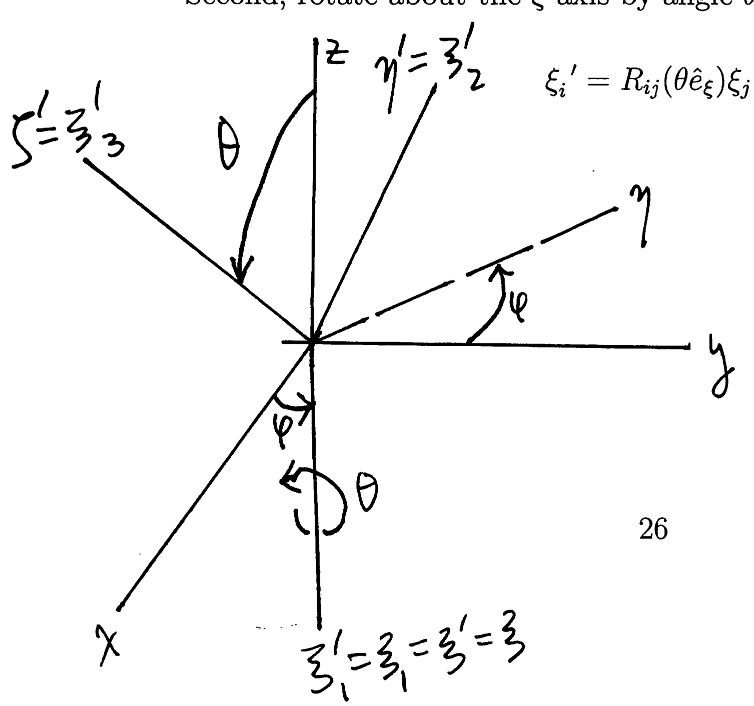
Of course we could specify our rotations by means of other parameterizations. For instance we could use the Euler angles to specify the orientation of the rotated frame. First rotate about the $z$-axis by angle $\varphi$

$$\xi_i = R_{ij}(\varphi\hat{e}_z)x_j. \tag{1.93}$$



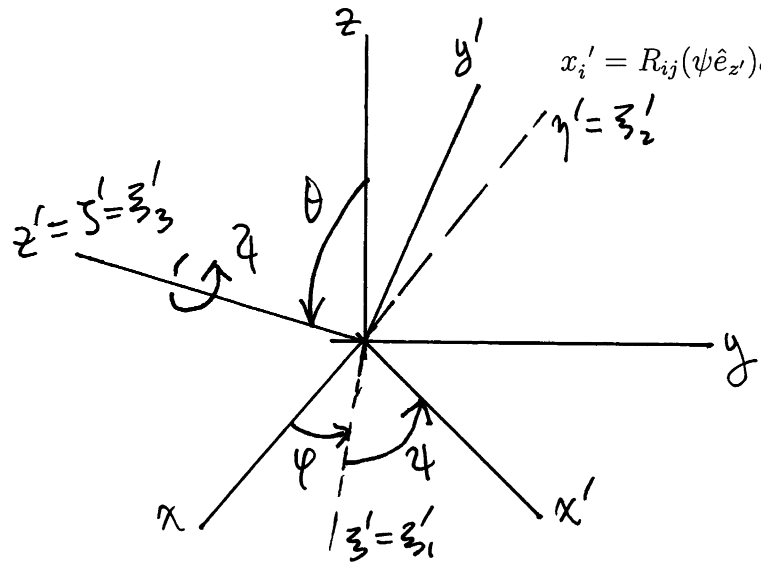

Second, rotate about the $\xi$-axis by angle $\theta$

$$\xi_i{}' = R_{ij}(\theta\hat{e}_\xi)\xi_j. \tag{1.94}$$



Finally rotate about the $\zeta'$-axis by angle $\psi$

$$x_i{}' = R_{ij}(\psi \hat{e}_{z'})\xi'_j. \tag{1.95}$$




The Euler angles $(\theta, \varphi, \psi)$ completely specify the rotation. Thus we can label $R$ by $R_{ij}(\theta, \varphi, \psi)$. So we have

$$x_i{}' = R_{ij}(\theta, \varphi, \psi)x_j, \tag{1.96}$$

where we made $R_{ij}(\theta, \varphi, \psi)$ by 3 successive rotations

$$R_{ij}(\theta, \varphi, \psi) = \left(R(\psi \hat{e}_{z'})R(\theta \hat{e}_{\xi})R(\varphi \hat{e}_z)\right)_{ij}. \tag{1.97}$$

But these are simple rotations about one of the coordinate axes. Hence

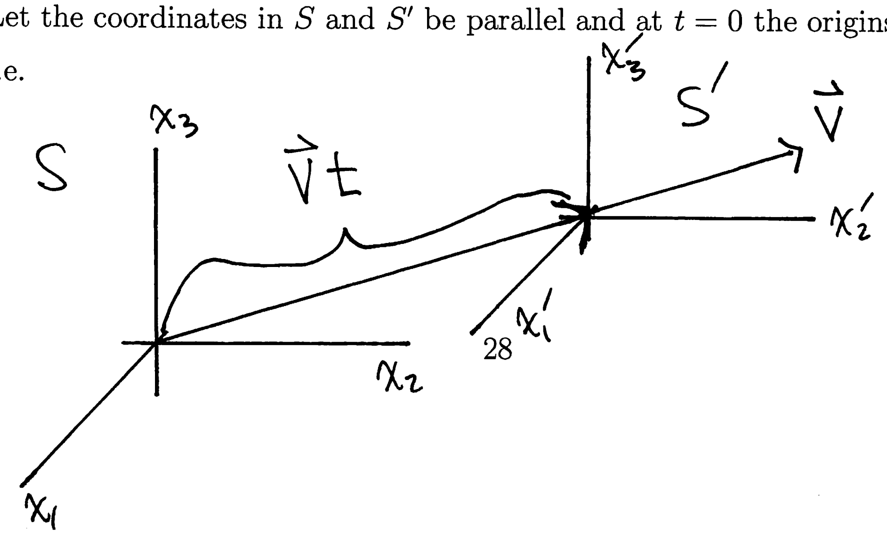
$$
\begin{aligned}
R(\varphi \hat{e}_z) &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R(\theta \hat{e}_\xi) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\
R(\psi \hat{e}_{z'}) &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\end{aligned}
\tag{1.98}
$$

Multiplying these matrices together we find

$$
R(\theta, \varphi, \psi) =
\begin{pmatrix}
\cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\
-\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\
\sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta
\end{pmatrix} .
\tag{1.99}
$$

The inverse transformation $R^{-1} = R^T$ gives the transformation in the active view.

The next transformation we can consider is to another frame of reference moving with constant velocity relative to the other. Call the new system $S'$. Let the coordinates in $S$ and $S'$ be parallel and at $t = 0$ the origins coincide i.e.



So $x'_i = x_i - V_i t$ where $\vec{V} = \text{const}$. Here we have assumed that time is absolute

$$t' = t + \tau \tag{1.100}$$

$\tau = \text{constant}$ (origin of time is irrelavant; time is homogeneous). Of course we know that Einstein realized that signals only travel at finite speeds so that distances and time intervals as seen by relatively moving frames must depend on simultaneity for the observer and so space and time coordinates in the two frames must be related; not just space coordinates. We will return to this modification of the relation between inertial frames later to see how Einstein saved the principle of relativity by modifying the space-time coordinate transformation law.

So Newton (Galilean) physics assumes time is absolute; the same for all inertial observers. From above we have

$$\dot{x}'_i = \dot{x}_i - V_i \tag{1.101}$$

the classical law of addition of velocities. Further $\ddot{x}'_i = \ddot{x}_i$, and if $\ddot{x}_i = 0$ so does $\ddot{x}'_i = 0$. The law of inertia is unchanged. So all frames of reference and coordinate systems that leave $\ddot{x}_i = 0$ valid for a free particle are the (infinite) set of inertial frames of reference.

These inertial frames differ by uniform motion relative to each other or the Cartesian coordinate system in the same frame differs by a rotation and/or a translation in space. These coordinate transformations and changes of frame leave the law of inertial unchanged. The law is covariant with respect to these transformations. In addition all the laws of mechanics are covariant with respect to these transformations as well as space and time having the same properties. This is the principle of relativity. All inertial frames are equivalent (there is no preferred or absolute frame) for the description of the laws of physics. As we have seen all inertial frames are related by Galilean transformations (rotations, translations and Galilean boosts)

$$\begin{aligned} x'_i &= \lambda_{ij}x_j + a_i - V_i t \\ t' &= t + \tau. \end{aligned} \tag{1.102}$$

Thus, for classical mechanics, all the laws of physics should be covariant under these Galilean transformations, i.e. the laws have the same form in any inertial frame. The mathematicians have catagorized quantities that have well defined transformation properties under these changes of frame tensors

Note that we can summarize the Galilean invariance by specifying that inertial frames are those in which space and time have the same properties. In classical mchanics by properites we mean that time is **absolute and homogeneous**, so that intervals of time are the same in every inertial frame

$$dt' = dt \tag{1.103}$$

And that space is **isotropic and homogeneous**, hence the separation nof points in space is what should be the same thus

$$ds'^2 = \sum_{i=1}^{3}(dx'_i)^2 = \sum_{i=1}^{3}(dx_i)^2 = ds^2 \tag{1.104}$$

Inertial frames are those frames which preserve these properties of space and time.

The most general transformations that do this are the (10) Galilean transformations

$$\begin{aligned} t' &= t + \tau \\ x'_i &= \lambda_{ij}x_j + a_i - V_i t \end{aligned} \tag{1.105}$$

Hence

$$
\begin{aligned}
dt' &= dt \\
dx'_i &= \lambda_{ij} dx_j \\
ds'^2 = \sum_{i=1}^{3} dx'_i dx'_i &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \lambda_{ij} dx_j \lambda_{ik} dx_k \\
= dx'_j(\delta_{jk}) dx'_k &= dx_j(\lambda_{ij}\lambda_{ik}) dx_k = dx_j(\delta_{jk}) dx_k = ds^2
\end{aligned}
\tag{1.106}
$$

That is

$$\lambda_{ij}\lambda_{ik} = \delta_{jk} \tag{1.107}$$

or in matrix notation $\lambda^T \lambda = \mathbb{1} \Rightarrow \lambda$ is a $3 \times 3$ orthogonal $(\lambda^{-1} = \lambda^T)$ matrix and

$$\det \lambda^T \lambda = \det \mathbb{1} = 1 \Rightarrow \det \lambda = \pm 1 \tag{1.108}$$

These are the rotation matrices (direction cosine matrices). As we will see they have $\det \lambda = 1$ and are orthogonal. We can imagine making consecutive rotations to yet another inertial frame, since the product of direction cosine matrices is again a direction cosine matrix; **i.e.**, if $\lambda_1^{-1} = \lambda_1^T; \lambda_2^{-1} = \lambda_2^T$; and $\det \lambda_i = 1$ then

$$\lambda = \lambda_1 \lambda_2 \tag{1.109}$$

has the same properties

$$\lambda^{-1} = (\lambda_1\lambda_2)^{-1} = \lambda_2^{-1}\lambda_1^{-1} = \lambda_2^T \lambda_1^T = (\lambda_1\lambda_2)^T = \lambda^T \tag{1.110}$$

and

$$\det \lambda = \det \lambda_1 \det \lambda_2 = 1. \tag{1.111}$$

The set of all such matrices (rotations) forms a **group** known as $SO(3)$ (3 angles of rotation).

Indeed two consecutive Galilean transformations is again a Galilean transformation hence the set of all Galilean transformations forms a group called the **Galilean group**. It has 10 parameters which specify which transformation in the group you are talking about (3 for $\vec{a}$, 3 for $\vec{V}$, 3 for $\lambda_{ij}$, plus 1 for $\tau$.) The group of transformations has an identity (no transformation at all), an inverse (just the opposite sign transformation parameters) and the composition law that is associative

$$\begin{aligned}
t' &= t + \tau \\
x'_i &= \lambda_{ij} x_j + a_i - V_i t \\
t'' &= t' + \hat{\tau} = t + (\tau + \hat{\tau}) \\
x''_i &= \hat{\lambda}_{ij} x'_j + \hat{a}_i - \hat{V}_i t' \\
&= \hat{\lambda}_{ij} [\lambda_{jk} x_k + a_j - V_j t] + \hat{a}_i - \hat{V}_i (t + \tau) \\
&= (\hat{\lambda}_{ij} \lambda_{jk}) x_k + (\hat{\lambda}_{ij} a_j + \hat{a}_i - \hat{V}_i \tau) - (\hat{\lambda}_{ij} V_j + \hat{V}_i) t
\end{aligned} \tag{1.112}$$

Hence the composite transformation is found to be

$$\begin{aligned}
t'' &\equiv t + \tilde{\tau} \\
x''_i &\equiv \tilde{\lambda}_{ij} x_j + \tilde{a}_i - \tilde{V}_i t
\end{aligned} \tag{1.113}$$

with

$$\begin{aligned}
\tilde{\tau} &= \tau + \hat{\tau} \\
\tilde{\lambda}_{ij} &= (\hat{\lambda}\lambda)_{ij} \\
\tilde{a}_i &= \hat{a}_i - \hat{V}_i \tau + \hat{\lambda}_{ij} a_j \\
\tilde{V}_i &= \hat{\lambda}_{ij} V_j + \hat{V}_i.
\end{aligned} \tag{1.114}$$

Hence the Galilean group is the symmetry group of classical mechanics. All the laws of mechanics are covariant under these group transformations. Since the laws of physics are to have the same appearance in each inertial frame, it is useful to express the laws in terms of quantities that have well defined transformation properties from frame to frame. **Cartesian Tensors** are defined to have such useful transformation properties. They are invariant under space and time translations but transform as products of space coordinates under space rotations.

In Euclidean 3 (or $N$) space a **vector** is defined as a quantity which transforms as $x_i$ under rotations

$$V_i' = \lambda_{ij} v_j \tag{1.115}$$

That is

$$\vec{V}' = \underline{\lambda} \vec{V} \tag{1.116}$$

(for **vector fields** $V_i'(\vec{x}\,') = \lambda_{ij} V_j(\vec{x})$)

In general an $m^{th}$ rank (Cartesian) **tensor** transforms as the product of the $m$ vectors

$$T'_{i_1 \cdots i_m} = \lambda_{i_1 j_1} \lambda_{i_2 j_2}, \ldots \lambda_{i_m j_m} T_{j_1 \ldots j_m} \tag{1.117}$$

where $i, j = 1, 2, \ldots, m$ (again for tensor fields

$$T'_{i_1 \cdots i_m}(\vec{x}\,') = \lambda_{i_1 j_1} \lambda_{i_2 j_2}, \ldots \lambda_{i_m j_m} T_{j_1 \ldots j_m}(\vec{x}).) \tag{1.118}$$

**ex.** 2nd Rank Tensor $T'_{ij} = \lambda_{ik} \lambda_{j\ell} T_{k\ell}$

Inertial tensor $I_{ij}$

Quadrupole moment tensor $Q_{ij}$

Rank 1 tensor = vector $p_i$

Rank 0 tensor = scalar m, time intervals dt

Example of Newton's Second Law

$$\frac{dp_i}{dt} = F_i \tag{1.119}$$

1st law $\Rightarrow$ covariance of laws of Nature $\equiv$ form invariance in inertial frame $S$: $\dot{p}_i = F_i \Leftrightarrow$ in inertial frame $S'$: $\dot{p}'_i = F'_i$. Form invariance guaranteed if law is expressible in terms of tensor quantities as are $\vec{p}$ and $\vec{F}$.

Examples of special tensors that are invariant or constant tensors

1.) 2nd rank Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{1.120}$$

Consider

$$\begin{aligned} \delta'_{ij} &= \lambda_{ik}\lambda_{jl}\delta_{kl} \\ &= \lambda_{ik}\lambda_{jk} = \lambda_{ik}\lambda^T_{kj} \\ &= \lambda_{ik}\lambda^{-1}_{kj} = \delta_{ij} \end{aligned} \tag{1.121}$$

hence, $\delta_{ij}$ is invariant under rotations

2.) Levi-Civita tensor, permutation tensor, the anti-symmetric tensor of rank 3

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{1.122}$$

i.e. 2 or more indices the same $\epsilon_{111} = 0 = \epsilon_{222} = \epsilon_{122}$. So we have for the elements of the Levi-Civita tensor

$$\begin{aligned} \epsilon_{123} &= +1 = \epsilon_{231} = \epsilon_{312} \\ \epsilon_{213} &= -1 = \epsilon_{132} = \epsilon_{321} \\ \epsilon_{iij} &= 0 \quad \text{(No sum on } i\text{)} \end{aligned} \tag{1.123}$$

Properties of the $\epsilon$ tensor

a)

$$\begin{aligned} \epsilon'_{ijk} &= \lambda_{il}\lambda_{jm}\lambda_{kn}\epsilon_{lmn} \\ &= (\det\lambda)\epsilon_{ijk} = \epsilon_{ijk} \end{aligned} \tag{1.124}$$

This can be seen from the general formula for the determinant of a matrix $M$

$$\begin{aligned} &M_{il}M_{jm}M_{km}\epsilon_{lmn} = \det M\epsilon_{ijk} \\ &= M_{i1}M_{j2}M_{k3} - M_{j1}M_{j3}M_{k2} \\ &+M_{i2}M_{j3}M_{k1} - M_{i2}M_{j1}M_{k3} \\ &+M_{i3}M_{j1}M_{k2} - M_{i3}M_{j2}M_{k1} \end{aligned} \tag{1.125}$$

$$= \begin{vmatrix} M_{i1}M_{i2}M_{i3} \\ M_{j1}M_{j2}M_{j3} \\ M_{k1}M_{k2}M_{k3} \end{vmatrix} \tag{1.126}$$

Permuting the rows implies

$$= \epsilon_{ijk} \qquad \underbrace{\det M} \tag{1.127}$$

$$= \begin{vmatrix} M_{11}M_{12}M_{13} \\ M_{21}M_{22}M_{23} \\ M_{31}M_{32}M_{33} \end{vmatrix}$$

b)

$$\epsilon_{abc}\epsilon_{ijk} = \begin{vmatrix} \delta_{ai}\delta_{aj}\delta_{ak} \\ \delta_{bi}\delta_{bj}\delta_{bk} \\ \delta_{ci}\delta_{cj}\delta_{ck} \end{vmatrix} \tag{1.128}$$

$$\begin{aligned} &= \delta_{ai}\delta_{bj}\delta_{ck} - \delta_{ai}\delta_{bk}\delta_{cj} \\ &+ \delta_{aj}\delta_{bk}\delta_{ci} - \delta_{aj}\delta_{bi}\delta_{ck} \\ &+ \delta_{ak}\delta_{bi}\delta_{cj} - \delta_{ak}\delta_{bj}\delta_{ci} \end{aligned} \tag{1.129}$$

This general formula yields the specific cases b.1)

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \begin{vmatrix} \delta_{il}\delta_{im} \\ \delta_{jl}\delta_{jm} \end{vmatrix} \tag{1.130}$$

b.2.)

$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il} \tag{1.131}$$

b.3.)

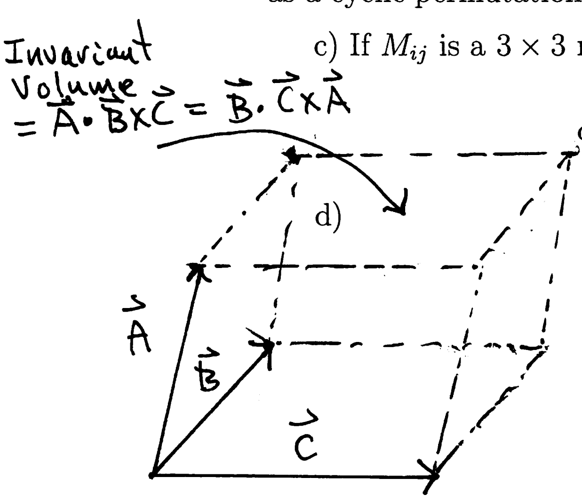
$$\epsilon_{ijk}\epsilon_{ijk} = 3! = 6. \tag{1.132}$$

b.4.) Jacobi Identity

$$\epsilon_{ijk}\epsilon_{mnk} + \epsilon_{jnk}\epsilon_{mik} + \epsilon_{nik}\epsilon_{mjk} = 0 \tag{1.133}$$

where in the Jacobi identity the $k$ index is summed over in each term, the position of the $m$ index is fixed in each term and finally the terms are obtained as a cyclic permutation fixed of (i,j,n)

c) If $M_{ij}$ is a $3 \times 3$ matrix




$$\det M = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}M_{il}M_{jm}M_{kn} \tag{1.134}$$

d)

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk}A_jB_k \tag{1.135}$$

$$\begin{aligned} \vec{A} \cdot \vec{B} \times \vec{C} &= \epsilon_{ijk}A_iB_jC_k \\ &= \epsilon_{jki}B_jC_kA_i \\ &= \vec{B} \cdot \vec{C} \times \vec{A} \end{aligned} \tag{1.136}$$

where the permutation of the indices of the permutation tensor was used $\epsilon_{ijk} = \epsilon_{jki}$. Additional cyclic permutations of the vectors yields further forms of the identity.

$$(\vec{\nabla} \times \vec{V})_i = \epsilon_{ijk}\partial_j V_k. \tag{1.137}$$

After this quite extensive discussion of Newton's First Law, we consider the Second Law of Newton.

II) A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

Newton further gave the definition of momentum of a particle as its mass times its velocity. Thus for a particle with coordinate $x_i$ in an inertial frame $S$, the momentum is

$$\vec{p} = m\vec{v} \tag{1.138}$$

or in components in that frame

$$p_i = m\dot{x}_i \tag{1.139}$$

where $m$ is the mass of the particle and is the same in all inertial frames (a scalar) while $\vec{v}$ and hence $\vec{p}$ are vectors.

Newton's second law states

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1.140}$$

or in components

$$F_i = \dot{p}_i \tag{1.141}$$

and for point particles (with constant mass)

$$F_i = m\ddot{x}_i \tag{1.142}$$

Since we know how the acceleration is defined in our inertial frame and we will use the 3rd law to define the mass more precisely; we can view this as a **law** relating the particle to its environment (force) and we need to define the Force law independently i.e. gravitational force, harmonic force, etc.. Note the force is a vector. (Alternatively we can view the 2nd law as a definition of force.)

Finally we come to the 3rd Law which is an observation about (central) forces that will allow us to more precisely define mass.

III) If two bodies exert forces on each other then the forces are in opposite directions and the ratio of the magnitudes of the accelerations are constant. This constant ratio is the inverse ratio of the masses of the bodies.

So mathematically the 3rd law states

$$\vec{F}_1 = -\vec{F}_2 \tag{1.143}$$

where $\vec{F}_1$ is the force on particle 1 and $\vec{F}_2$ is the force on particle 2. The 2nd law implies $m_1\vec{a}_1 = -m_2\vec{a}_2$, so that

$$\frac{m_2}{m_1} = \frac{|\vec{a}_1|}{|\vec{a}_2|}. \tag{1.144}$$

So if we choose an object as the standard mass, say $m_1$, then measure $\vec{a}_1$ and $\vec{a}_2$, we can define $m_2$ as the ratio.

Also we can check if forces are opposite in direction. In fact, not all forces obey the 3rd law. In particular forces that propagate at a finite speed (e.g. electromagnetic forces) do not obey the 3rd law since they have a velocity dependence and are not along the line joining the particles. However, we can base most of what we do on laws 1 and 3 and the mass of a particle. Newton's laws really define the useful concepts of force, mass, acceleration, and we can can then experimentally discover the force laws from this point of view.

(Aside: Another point of view is to consider force as a primitive concept measured by say comparing to a standard spring and using the 3rd law to define the mass. Then $\vec{F} = m\vec{a}$ is a law as well as the form of the force law itself. Another point of view is to assume the principle of equivalence of the inertial mass and the gravitational mass. We define mass by weighing each particle against a standard then $m_G\vec{g} = m_I\vec{a}$ and $\vec{a} = \vec{g}$ independent of mass, $\Rightarrow m_G = m_I$. The equivalence principle was tested by Eötvos' experiments to 1 part to $10^{11}$. From this point of view we then do experiments to discover the force laws and test the 3rd law.)

The point of all this is that certain primitive concepts are assumed in the statement of Newton's laws and we can view the laws from various points of view. In practice we will be concerned with finding the trajectory of a given particle $\vec{x}(t)$ when we are told its mass and the force acting on it. It is perhaps the more practical way of applying $\vec{F} = m\vec{a}$ to define the force a particle experiences. If a particle undergoes a certain trajectory then you know the force law it is experiencing. This is what Newton "discovered".

Finally, let's consider some consequences of the principle of Galilean relativity onthe form of the force law. If two frames are related by a Galilean transformation

$$\begin{aligned} t' &= t + \tau \\ x'_i &= \lambda_{ij}x_j + a_i - V_i t \end{aligned} \tag{1.145}$$

then $\ddot{x}'_i = \lambda_{ij}\ddot{x}_j$ and $m$ is invariant since it is the ratio of $|\vec{a}_2|/|\vec{a}_1|$. Hence $F'_i = m\ddot{x}'_i = \lambda_{ij}m\ddot{x}_j = \lambda_{ij}F_j$
i.e.

$$F'_i = \lambda_{ij}F_j \tag{1.146}$$

the force is a vector. So Newton's 2nd law is covariant if $F'_i = \lambda_{ij}F_j$ reversing the above argument

$$\begin{aligned} F_i' &= \lambda_{ij} F_j \\ = m a_i' &= \lambda_{ij} m a_j \end{aligned} \tag{1.147}$$

and so implies that $F_j = m a_j$. For rotations of the coordinate system

$$F_i' = \lambda_{ij} F_j \tag{1.148}$$

implies $F_i$ are the components of a vector that is a rank 1 tensor.

Further for Galiliean boosts to another frame of reference $\vec{x}\,' = \vec{x} - \vec{v}t \Rightarrow \vec{F}' = \vec{F}$; the force must not change. This is a strong restriction. It tells us that the Force depends upon the vector distance (i.e. the relative vector separation between two particles (and t). For example, suppose we have two particles with position coordinates $\vec{x}_a, \vec{x}_b$ and velocities $\vec{v}_a, \vec{v}_b$, etc. Then if

$$\vec{F}_{\substack{ab\\ba}} = \vec{F}_{\substack{ab\\ba}}(\vec{x}_a - \vec{x}_b, \vec{v}_a - \vec{v}_b, \ldots, t) \tag{1.149}$$

We have, since

$$\begin{aligned} \vec{x}_{\substack{a\\b}}{}' &= \vec{x}_{\substack{a\\b}} - \vec{v}t \\ \vec{v}_{\substack{a\\b}}{}' &= \vec{v}_{\substack{a\\b}} - \vec{v} \end{aligned} \tag{1.150}$$

so that

$$\begin{aligned} \vec{x}_a{}' - \vec{x}_b{}' &= \vec{x}_a - \vec{x}_b \\ \vec{v}_a{}' - \vec{v}_b{}' &= \vec{v}_a - \vec{v}_b \end{aligned} \tag{1.151}$$

and hence

$$\begin{aligned} \vec{F}_{\substack{ab\\ba}} &= \vec{F}_{\substack{ab\\ba}}(\vec{x}_a - \vec{x}_b, \vec{v}_a - \vec{v}_b, \ldots, t) \\ &= \vec{F}_{\substack{ab\\ba}}(\vec{x}_a{}' - \vec{x}_b{}', \vec{v}_a{}' - \vec{v}_b{}', \ldots, t) \\ &= \vec{F}_{\substack{ab\\ba}}{}'. \end{aligned} \tag{1.152}$$

Further if the force is to obey the 3rd law (the 3rd law excludes $\vec{F}_{ab} \propto (\vec{v}_a - \vec{v}_b)\tilde{f}_{ab}$), then

$$\vec{F}_{ab} = (\vec{x}_a - \vec{x}_b) f_{ab}(\vec{x}_a - \vec{x}_b, \vec{v}_a - \vec{v}_b, \ldots, t) \tag{1.153}$$

and $\vec{F}_{ab} = -\vec{F}_{ba}$ so $f_{ab}$ = scalar i.e. invariant under rotations. That is the forces are **central forces** they act along the direction of the line joining the particles.

Often we will ignore the motion of the other particles usually because they are constrained by forces not to move or that their mass is very large and we will only concentrate on the motion of a single particle in the effective force field produced by the other fixed particles.

Another important (single particle) force is that of an **irrotational force**. This is a force that depends upon $\vec{x}$ and $t$ only, not on $\dot{\vec{x}}$, etc., and has zero curl

$$\vec{\nabla} \times \vec{F}(\vec{x}, t) = 0. \tag{1.154}$$

If $\vec{\nabla} \times \vec{F}(\vec{x}, t) = 0$, then $\vec{F}(\vec{x}, t)$ at any time can be written in terms of a gradiant of a scalar potential energy $U(\vec{x}, t)$ :

$$\vec{F}(\vec{x}, t) = -\vec{\nabla} U(\vec{x}, t). \tag{1.155}$$

If we further demand invariance under time translations $t' = t + \tau$ then $\vec{F}(\vec{x}, t, +\tau) = \vec{F}(\vec{x}, t) \Rightarrow \vec{F}$ is time independent. In such a case $\vec{F}$ is called **conservative** and $\vec{F}(\vec{x}) = -\vec{\nabla} U(\vec{x})$. As we shall see shortly, conservative forces lead to energy conservation.